Estienne de la Roche's appropriation of Chuquet (1484)

Albrecht HEEFFER

Center for Logic and Philosophy of Science
Ghent University, Blandijnberg 2,
B-9000 Ghent, Belgium
albrecht.heeffer@ugent.be

1. Abstract

The importance of *Larismethique* of de La Roche, published in 1520, has been seriously underestimated. One reason for the neglect is related to the inscrutable way he is referred to. Buteo and Wallis called him Stephanus à Rupe de Lyon. Other obscure references, such as Gosselin calling him Villafrancus Gallus have been overlooked by many commentators. His influence can be determined in several works that do not credit him but use problems or definitions from the *Larismethique*. However, most damaging for its historical assessment was Aristide Marre's misrepresentation of the *Larismethique* as a grave case of plagiarism. Marre discovered that the printed work of 1520 by Estienne de la Roche contained large fragments that were literally copied from Chuquet's manuscript of the *Triparty*. Especially on the *Appendice*, which contains the solution to a large number of problems, Marre writes repeatedly that it is a literal copy of Chuquet. However, he fails to mention that the structure of the text of de la Roche, his solution methods and symbolism differs significantly from Chuquet. De la Roche introduces several improvements, especially with regards to the use of the second unknown. We provide an in-depth comparison of some problems solved by the so-called *regle de la quantite* by Chuquet with those of de la Roche. We further report on the surprising finding that Christoff Rudolff's solution to linear problems by means of the second unknown in his *Behend vnnd Hubsch Rechnung* of 1525 depends on Chuquet and de la Roche. As it is generally considered that algebra was introduced in Germany through Italy this provides a new light on the transmission of algebraic knowledge from France to the rest of Europe.

2. More than plagiarism: Estienne de la Roche

The *Larismethique nouellement composee par maistre Estienne de La Roche*, published in 1520 is after Pacioli's *Summa* (1489) and the work of Grammateus, the third printed book dealing with algebra and the first one in French. We do not know much about de la Roche. Tax registers from Lyon reveal that his father lived in the Rue Neuve in the 1480s and that Estienne owned more than one property in Villefranche, from which he derived his nickname (Flegg 1988, 61). De la Roche is described as a "master of argorisme" as he taught merchant arithmetic for 25 years at Lyon. He owned the manuscript of the *Triparty* after the death of Chuquet (1488). It is therefore considered that de la Roche was on friendly terms with Chuquet and possibly learned mathematics from him.

The importance of the *Larismethique* has been seriously underestimated. There are several reasons for this. Probably the most important one is Aristide Marre's misrepresentation of the *Larismethique* as a grave case of plagiarism. Marre discovered that the printed work by Estienne de la Roche, contained large fragments that were literally copied from Chuquet's manuscript (Marre 1880, introduction). Indeed, especially on the *Appendice*, which contains the solution to many problems, Marre (1881) writes repeatedly in footnotes "This part is reproduced word by word in the *Larismethique*" with due references. However, giving a transcription of the problem text only, Marre withholds that for many of the solutions to Chuquet's problems de la Roche uses different methods and an improved symbolism.[1] In general, the *Larismethique* is a much better structured text than the *Triparty* and one intended for a specific audience. Chuquet was a bachelor in medical educated in Paris within the scholarly tradition and well acquainted with Boethius and Eulcid (L'Huillier 1987). On the other hand, de la Roche was a reckoning master operating within the abbaco tradition. It becomes clear from the structure of the book that de la Roche had his own didactic program in mind. He produced a book for teaching and learning arithmetic and geometry which met the needs of the mercantile class. He rearranges Chuquet's manuscript using Pacioli's *Summa* as a model. He even adopts Pacioli's classification in books, distinctions and chapters. He moves problems from Chuquet's *Appendice* to relevant sections within the new structure. He adds introductory explanations to each section of the book, such as for the second unknown, discussed below. With the judgment of an experienced teacher, he omits sections and problems from the *Triparty* which are of less use to merchants and craftsmen and adds others which were not treated by Chuquet such as problems on exchange and barter. This didactical program of de la Roche was apparently not understood by Marre whose conclusion on the book is very harsh:[2]

> One can state, pure and simple that, [de la Roche] copied a mass of excerpts from the *Triparty*, that he omitted several important passages, especially on algebra, that he abridged and extended others for producing the *Arismetique*, a work much inferior to the *Triparty*.

Comparing the problem texts only, the denunciation of de la Roche would also apply to numerous others, including Chuquet's use of various problems from Barthelemy of Romans (Spiesser, 2003). As yet, there has not been a published transcription of Chuquet's solution to problems of his *Appendice*. The partial English translation by Flegg, Hayes and Moss (1985) only added more to the confusion than that it resolved the issue. We will further try to give a clear picture of the improvements made by de la Roche with regards to the second unknown.

Another reason for the neglect of de la Roche is related to the inscrutable way he is referred to. Buteo (1559, 118, 183, 184, 216) for example, calls him Stephanus à Rupe de Lyon, and so does Wallis (1685, 63). Together with other obscure references, such as Gosselin's naming of Villafrancus Gallus, instances mentioning de la Roche have been overlooked by his peers as well as commentators. Therefore his name and the title of his book is not very well represented in sixteenth-century works on algebra. However, he did

[1] For example, problem 70 from Chuquet's *Appendice* is solved by two unknowns by de la Roche (see below) while Chuquet solves it by false position.

[2] Marre 1881, 28; this and subsequent translations are mine.

have an influence which can be determined in several works that do not credit him but which use problems and definitions from the *Larismethique*.

3. The Regula quantitatis

That de la Roche made an important contribution to the emergence of symbolic algebra during the sixteenth century can best be argued by his treatment of the second unknown, sometimes called *Regula quantitatis* or Rule of Quantity. The importance of the use of multiple unknowns in the process leading to the concept of an equation cannot be overestimated. We have traced the use and the development of the second unknown in algebraic problem solving from the introduction of Arab algebra in Europe until the systematic treatment of a system of linear equations by Gosselin (1577) (Heeffer 2008). The first important step can be attributed to the Florentine abacus master Antonio de' Mazzinghi, who wrote an algebraic treatise around 1380. Luca Pacioli copied almost literally the solution method in his *Summa* of 1494, and Cardano used the second unknown both in his *Arithmetica* and the *Ars Magna*. A second thread of influence is to be distinguished through the *Triparty* by Chuquet (1484) and the printed works of de la Roche (1520) and Christoff Rudolff (1525). The Rule of Quantity finally culminates in the full recognition of a system of linear equation by Buteo and Gosselin. The importance of the use of letters to represent several unknowns goes much further than the introduction of a useful system of notation. It contributed to the development of the modern concept of unknown and that of a symbolic equation. These developments formed the basis on which Viète (1591) could build his theory of equations.

Before discussing the examples by Chuquet, de la Roche and Rudolff, it is appropriate to emphasize the difference between the rhetorical unknown and unknowns used in modern transliterations. Firstly, the method of using a second unknown is an exception in algebraic practice before 1560. In general, algebraic problem solving before the seventeenth century uses a single unknown. This unknown is easily identified in Latin text by its name *res* (or sometimes *radix*), *cosa* in Italian and *coss* or *ding* in German. The unknown should be interpreted as a single hypothetical value used within the analytic method. Modern interpretations such as an indeterminate value or a variable, referring to eighteenth century notions of function and continuity, are unwarranted. In solving problems by means of algebra, abbacus masters often use the term 'quantity' or 'share' or 'value' apart from the *cosa*. The rhetoric of abacus algebra requires that the quantities given in the problem text are formulated in terms of the hypothetical unknown. The problem solving process typically starts with 'suppose that the first value sought is one *cosa'*. These values or unknown quantities cannot be considered algebraic unknowns by themselves. The solution depends on the expression of all unknown quantities in terms of the *cosa*. Once a value has been determined for the *cosa*, the unknown quantities can then easily be determined.

However, several authors, even in recent publications, confuse the unknown quantities of a problem, with algebraic unknowns. As a result, they consider the rhetorical unknown as an auxiliary one. For example, in his commentary of Leonardo of Pisa's *Flos*, Picutti (1983) consistently uses the unknowns x, y, z for the sought quantities and regards the *cosa* in the linear problems solved by Leonardo to be an auxiliary unknown. The "method of auxiliary variable" as a characterization by Hughes (2001, 126) for a problem-solving method by ben-Ezra also follows that interpretation. We believe this to be a misrepresentation of the original text and problem-solving method.

The more sophisticated problems sometimes require a division into sub problems or subsequent reasoning steps. These derived problems are also formulated using an unknown but one which is different from the unknown in the main problem. For example, in the anonymous manuscript 2263 of the Biblioteca Riccardiana in Florence (c. 1365), the author solves the classic problem of finding three numbers in geometric proportion given their sum and the sum of their squares (Simi 1994, 39-40). He first uses the middle term as unknown, arriving at the value of 3. Then the problem of finding the two extremes is treated as a new problem, for which he selects the lower extreme as unknown. We will not consider such cases as the use of two unknowns, but the use of a single one at two subsequent occasions.

We have given some examples of what should not be comprehended as a second unknown, but let us turn to a positive definition. The best characterization of the use of several unknowns is operational. We will consider a problem solved by several unknowns if all of the following conditions apply in algebraic problem solving:

1) The reasoning process should involve more than one rhetorical unknown which is named or symbolized consistently throughout the text. One of the unknowns is usually the traditional *cosa*. The other can be named *quantità*, but can also be a name of an abstract entity representing a share or value of the problem.
2) The named entities should be used as unknowns in the sense that they are operated upon algebraically by arithmetical operators, by squaring or root extraction. If no operation is performed on the entity, it has no operational function as unknown in solving the problem.
3) The determination of the value of the unknowns should lead to the solution or partial solution of the problem. In some cases the value of the second unknown is not determined but its elimination contributes to the solution of the problem. This will also be considered as multiple unknowns.
4) The entities should be used together at some point of the reasoning process and connected by operators or by a substitution step. If the unknowns are not connected in this way the problem is considered to be solved by a single unknown.

In all the examples discussed below, these four conditions apply.

4. Chuquet's use of the second unknown

Without any previous introduction or explanation, Chuquet uses a second unknown in problem 71 of the Appendice.[3] (Paris, Fonds Français 1346, 1484, f. 168^r-169^r):

[3] Paris, Fonds Français 1346, 1484, f. 168^r-169^r: "Encore plus. Ilz sont troys qui ont chascun tant de deniers et en telle p*ro*porcion que le p*re*mier avoit 7 ds. des ault*re*s il auroit le quintuple de ce qui leur reste plus 1. Le second dit que sil avoit 9 ds. du p*re*mier et du tiers il aurait le sextuple de leur reste et 2 plus. Et le tiers dit que sil avoit 11 ds. des deux ault*re*s Il auroit le septuple de leur reste plus 3. Assavoir moult quantz deniers a ung chascun deulx", transcription mine.

Chuquet’s direct source is most likely the very similar problem from the *Le Compendy De la practique des nombres*, by Barthelemy of Romans (Spieser 2003, 99-100, 317-20). Barthelemy’s problem originates from Fibonacci (Boncompagni 200, Sigler 301, Spiesser 658). In modern notation:

$$
\begin{aligned}
a+7 &= 5(b+c-7)+1 \\
b+9 &= 6(a+c-9)+2 \qquad (1.1) \\
c+11 &= 7(a+b-11)+3
\end{aligned}
$$

The solution method by Fibonacci is purely arithmetical. Barthelemy gives two solutions, the first corresponding with Fibonacci, the second using the rule of false position. The arithmetical solution depends on the reformulation of the problem using *s*, as the sum of the three shares, as follows:

$$
\begin{aligned}
a+6 &= \frac{5}{6}\ s-1 \\
b+7 &= \frac{6}{7}\ s-2 \\
c+8 &= \frac{7}{8}\ s-3
\end{aligned}
$$

From this, one can determine the value of the sum and hence the values of the three shares. Chuquet’s solves the problem in two ways, each method uses algebra. The first solution is based on a combination of the rule of double false position with algebra (“par la Regle de deux positions et par la Regle des premiers tout ensemble”).[4]

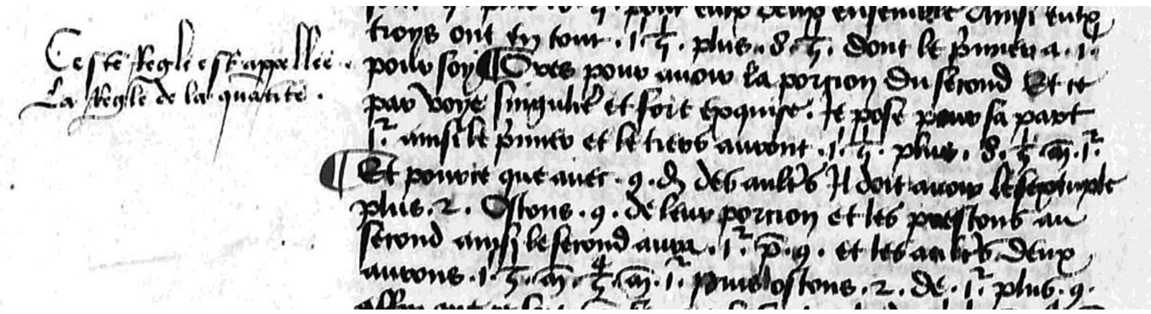

Figure 1: The first use of the second unknown in the *Triparty* by Chuquet (1484, f. 169*v*)

Chuquet introduces 1^2 for the second unknown without any introduction or previous explanation. At this point, a marginal annotation reads “This rule is called the rule of quantity” (“Ceste Regle est appelléé La Regle de la quantité”). This comment is most likely from the hand of de la Roche who explains the method in his arithmetic published in 1520.

[4] A marginal notation, probably by de la Roche, gives “par la regle de deux positions et par la regle de la chose ensemble”, f. 168^{r}.

Chuquet then turns to a method using “la Regle des premiers tant seulement” (f. 169^{v}), which is his expression for the algebraic method. Here, he introduces the second unknown, without even using a name for the new entity. To differentiate the second unknown from the first 1^1, he uses the notation 1^2, which could be read as 1*y*. The base number acts as the coefficient, so 6^2 signifies 6*y*. This is really puzzling as he previously used that notation for the second power of the unknown. Chuquet most certainly was aware of the possible confusion, but apparently had no other means to his disposal. Chuquet’s method can be summarized as follows:

- use the first unknown for the first share (1)
- use the second unknown for the second share
- express the value of the second share in the first unknown (2)
- use the second unknown for the third share
- express the value of the third share in the first unknown (3)
- add (1), (2) and (3) together and determine the value of the unknown
- the value of the other shares can be determined from (2) and (3)

In the *Appendice*, Chuquet solves five problems by this method.[5]

5. de la Roche and La Regle de la Quantite

De la Roche first mentions *la regle de la quantite* in the beginning of distinction six together with *la regle de la chose*. He properly introduces the second unknown in a separate chapter titled *Le neufiesme chapitre de la regle de la quantite annexee avec le dict primier canon, et de leur application,* in the sixth distinction of the first part :[6]

> This rule of quantity is added and follows from the first canon of the rule of the thing as a realization and perfection of this rule. The motivation is that it happens many times, for several reasons of this canon, that one has to pose two, three or more times that the first position is 1 ρ. After that, one has to pose a second or third time, etc. It is therefore necessary that the second, third or fourth position should be a number different from ρ. Because when the numbers for the second, third and fourth positions are the same and

[5] We have not been able to systematically inspect the complete manuscript, but problems 71, 75, 77, 78 and 79 seem to be the only ones solved by use of the second unknown.

[6] de la Roche 1538, f. 42^{v} : “Ceste regle de la quantite est annexee et inferee au primier canon de la regle de la chose com*m*e acomplissement et perfection dicelluy pource que souventes foys advient en plusiers raisons de ce cano*n* quil fault poser deux, trois ou plusiers fois desquelles tousiours la premiere position est 1 ρ. Sil vient apres quil faille poser une aultre fois ou deux ou troys etc. Il est besoing que la seconde, tierce ou quatre positions soyent aultres differences de no*m*bre que ρ. Car qui seroit la seconde tierce ou quatre positio*n*s de 1 ρ., le nombres et ρ de la premiere positions seroyent mesmes et intrinq*ue*s indiffere*n*ment avec les no*m*bres et ρ des autres positions qui seroit co*n*fusion. Car on ne scavroit distinguer de quelle positions seroyent lesdictz nombres et ρ. Et pour ce est necessaire de trouver aultre difference de nombre pour distincter et differer la premiere position des aultres. Et par ainsi en la seconde, tierce ou quatre positions etc”, translation mine.

indistinguishable from the numbers for ρ, or the other positions, this would lead to confusion. In that case we are unable to determine which positions stand for the numbers or for ρ. Therefore it is necessary to make another distinction between the numbers to distinguish and differentiate the first position from the others. And the same for the second, third and fourth positions, etc.

In the Rule of Quantity, de la Roche sees a perfection of algebra itself. The use of several unknowns allows for an easy solution to several problems which might otherwise be more difficult or even impossible to solve. Interestingly, he insists on the use of clearly distinguishable notation for the other unknowns, obviously referring to the ambiguities in Chuquet's use of 1^2. After this introduction, de la Roche gives six examples of the rule of quantity applied to the typical linear problems, though he removes the practical context. Then he presents five indeterminate problems under the heading "questions which have multiple responses", without use of the second unknown. Finally he solves five problems using the second unknown under the heading "other inventions on numbers". At the end of the book there is a chapter on applications in which four more problems are given (f. 149^v-150^r). In total, there are twenty problems solved by the *regle de la quantite*.

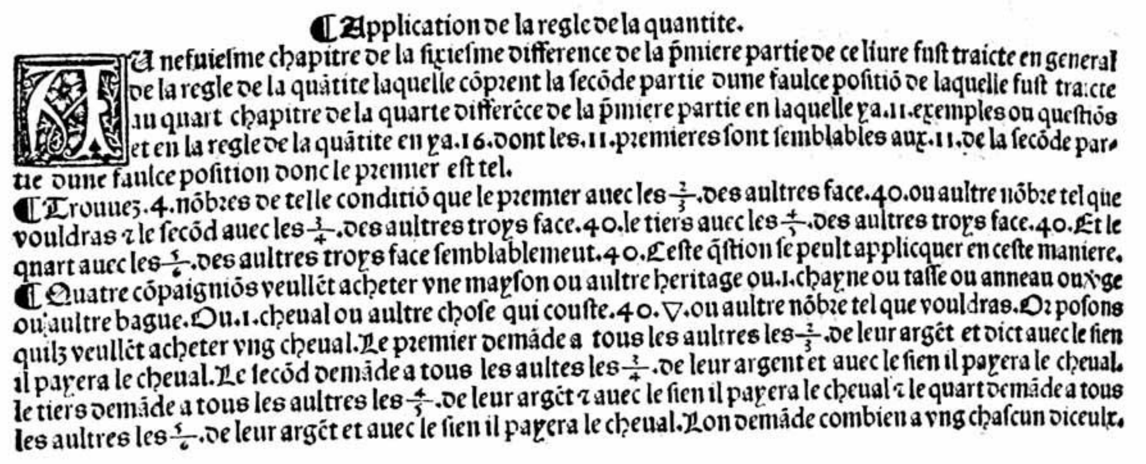
¶ Application de la regle de la quantite.
Au nefuiesme chapitre de la sixiesme difference de la pmiere partie de ce liure fust traicte en general
de la regle de la quãtite laquelle cõprent la secõde partie dune faulce positiõ de laquelle fust traicte
au quart chapitre de la quarte differẽce de la pmiere partie en laquelle ya.11.exemples ou questiõs
et en la regle de la quãtite en ya.16.dont les.11.premieres sont semblables aux.11.de la secõde par-
tie dune faulce position donc le premier est tel.
¶ Trouuez.4.nõbres de telle conditiõ que le premier auec les $\frac{2}{3}$.des aultres face.40.ou aultre nõbre tel que
vouldras & le secõd auec les $\frac{3}{4}$.des aultres troys face.40.le tiers auec les $\frac{4}{5}$.des aultres troys face.40.Et le
quart auec les $\frac{5}{6}$.des aultres troys face semblablement.40.Ceste q̃stion se peult applicquer en ceste maniere.
¶ Quatre cõpaigniõs veullẽt acheter vne mayson ou aultre heritage ou.1.chayne ou tasse ou anneau ou vge
ou aultre bague.Ou.1.cheual ou aultre chose qui couste.40.▽.ou aultre nõbre tel que vouldras.Or posons
quilz veullẽt acheter vng cheual.Le premier demãde a tous les aultres les $\frac{2}{3}$.de leur argẽt et dict auec le sien
il payera le cheual.Le secõd demãde a tous les aultes les $\frac{3}{4}$.de leur argent et auec le sien il payera le cheual.
le tiers demãde a tous les aultres les $\frac{4}{5}$.de leur argẽt & auec le sien il payera le cheual & le quart demãde a tous
les aultres les $\frac{5}{6}$.de leur argẽt et auec le sien il payera le cheual.Lon demãde combien a vng chascun diceulx.

Figure 2: de la Roche (1538, f. 149v) applies the second unknown.

In a section on the application of the Rule of Quantity, de la Roche solves some difficult linear problems with the second unknown. The first problem is formulated in two ways. Once as properties of four numbers and once in a practical context of four companions buying a house.

We will look at a typical solution of one problem also treated by Chuquet as problem 70 (*Appendice*, f. 168^r). Marre (1881, 432-3) points out that de la Roche uses the text from Chuquet "word by word", but does not mention that Chuquet solves the problem in a different way. As we have discussed above, Chuquet uses a combination of double false position and algebra. In meta-description the problem can be formulated as follows:

$$\begin{aligned} a+12 &= 2(b+c-12)+6 \\ b+13 &= 4(a+c-13)+2 \qquad (1.2) \\ c+11 &= 3(a+b-11)+3 \end{aligned}$$

Estienne de La Roche solves the problem twice. Once under the heading *Other Inventions on Numbers* (1538, f. 43v) and once in the final chapter (1520, f. 217^r; 1538, f. 150^r).[7] He proceeds in a way very similar to Chuquet. We discuss here the first solution.

Nr	**Symbolic representation**	**Meta description**	**Original text**
1	$a+12=x+12$	choice of first unknown	pose sue le premier soit 1 ρ adiouster avec 12 fait 1 ρ p 12
2	$a+12-6=x-6$	subtract 6 from (1)	dont fault oster 6 r reste 1 ρ p 6
3	$\frac{a+12-6}{2}=\frac{x}{2}+3$	divide (2) by 2	dont la ½ gest ½ ρ p 3
4	$b+c-12=\frac{x}{2}+3$	substitute (3) in (1.2)(a)	sont les autres deux no*m*bres
5	$a+b+c=\frac{3}{2}x+15$	add *x* + 12 to (4)	ainsi tous troys sont 1 ρ ½ p 15.
6	$b=y$	choice of second unknown	Apres pose le second no*m*bre 1 q*uan*tite
7	$b+13=y+13$	add 13 to (7)	adioustee a 13 font 13 p 1 q*uan*tite.
8	$b+11=y+11$	subtract 2 from (7)	Do*n*t fa[u]lt lever 2 et reste 11 p 1 q*uan*tite
9	$b+11=4(a+c-13)$	from (1.2)(b)	
10	$y+11=4(a+c-13)$	substitute (8) in (9)	
11	$\frac{1}{4}y+2\frac{3}{4}=a+c-13$	divide (10) by 4	dont le ¼ qui est 2 ¾ p ¼ q*uan*tite
12	$\frac{1}{4}y+2\frac{3}{4}=\frac{3}{2}x+2-y$	substitute (5) and (6) in (11)	sont egaulz a 1 ρ ½ p 2 m 1 q*uan*tite qui sont les aultres 2 no*m*bres quantite les 13
13	$1\frac{1}{4}y=1\frac{1}{2}x-\frac{3}{4}$	add y – 2 ¾ to (12)	en sont levez ores egaliz
14	$y=1\frac{1}{5}x-\frac{3}{5}$	divide (10) by 5/4	et p*ar*tiz par les q*uan*titez et trouvereras 1 ρ1/5 m 3/5 pour le second no*m*bre.
15	$c+11=z+11$	reuse of the second unknown	puis pose le tiers nombre 1 q*uan*tite adioustee avec 11 font 11 p 1 q*uan*tite.
16	$c+8=z+8$	subtract 3 from	Do*n*t fault lever 3 et reste*n*t

[7] Marre only mentions the second solution.

		(15)	8 p 1 q*uan*tite
17	$z+8=3(a+b-11)$	substitute (16) in (c)	qui sont les aultres deux no*m*bres quant les 11 en sont leves.
18	$\frac{1}{3}z+2\frac{2}{3}=a+b-11$	substitutes (15) in (16)	dont le 1/3 ques est 2 2/3 p 1/3 q*uan*tite
19	$\frac{1}{3}z+2\frac{2}{3}=1\frac{1}{2}x+4-z$	substitute (5) and (16) in (18)	sont egault a 1 ρ ½ p 4 – 1 q*uan*tite
20	$\frac{4}{3}z=1\frac{1}{2}x+\frac{4}{3}$	add z – 2 2/3 to (20)	or*es* egalis
21	$z=1\frac{1}{8}x+1$	divide (20) by 4/3	et partis co*m*me dessus trouveras 1 ρ 1/8 p 1 pour le tiers no*m*bre.
22	$x+y+z=3\frac{13}{40}x+\frac{2}{5}$	substitute (14) and (21) in (5)	Ores adiouste tous les troys no*m*bre ensemble
23	$3\frac{13}{40}x+\frac{2}{5}=1\frac{1}{2}x+15$	equal (20) and (5)	et auras 3 p 13/40 p 2/5 egaulz a 1 ρ ½ p 15
24	$\frac{73}{40}x=\frac{73}{5}$	from (23)	puis egaliz
25	$a=8$	divide (24) by 40/73	et partiz le nombre par la ρ et auras 8 pour le premier nombre.
23	$b=9, c=10$		Et par consequent 9 pour le second et 10 pour le tiers.

By using a new symbol for the second unknown, de la Roche resolves one of the ambiguities from Chuquet's manuscript, the use of the same symbol as for the square of the unknown. The second ambiguity, the use of the same unknown for the second and third unknown quantities, is not eliminated. Only because the second and third unknown are not related during the derivation, he can use the entity *quantite* twice.

6. Christoff Rudolff's Coss

The first printed book entirely devoted to algebra was published in 1525 in Strassburg under the title *Behend vnnd Hubsch Rechnung durch die kunstreichen regeln Algebre so gemeincklich die Coss genennt werden*. We do not know much about its author Christoff Rudolff. As he mentions that he used the examples from the book for teaching his students algebra, he may have been a professor in Vienna. Rudolff gives a short introduction on arithmetic, followed by the basic operations on polynomials. The rest of the book is filled with the algebraic solution to over four hundred problems, divided into eight sections corresponding with the eight equation types. Rudolff explicitly uses the word *equation*.

The *Regula Quantitatis* is not mentioned in the introduction but used in the examples of the first rule (on linear equations). The rule is introduced as follows:[8]

> This rule teaches us how to approach numerous examples, by using also other positions or statements in addition to the root (as is usual). So, when you pose or allow 1 *x* for one thing, then you cannot do the same for another thing within the same process (to avoid confusion or misunderstanding). It goes as follows. When, after using 1 *x*, you find a thing for which you cannot pose or allow the position (as explained before). Suppose that this thing is 1 quantity, and proceed as the problem asks for, to the point that you find a value for it … Is there yet another thing. Take the previous position, give this thing 1 quantity, and proceed as instructed above.

The procedure explains the way problems are solved with the second unknown by Chuquet and de la Roche, including the reuse of the second unknown for other unknown quantities in the problem. Example 9 goes as follows:[9]

> Give two numbers, which together make 20. Dividing the smaller one by 8 and the larger one by 3. Adding the quotients of both gives 5.

Rudolff gives two solutions, one using a single unknown and the second using the *Regula quantitates*. Rudolff holds the rule in great admiration, calling it the perfection of the Coss ("ein volkomenheyt der Coss. Ja warlich ein sölche volkomenheit, on welche sie nit vil mer gilt dan ein pfifferling"). Taking the first number as *x* and the second as 1 q ("das ist ein quantitet und bedeutet 1 q. auch ein ungezelete zal / als die noch ist verborgen / gleych so wol als 1 *x*"), he arrives at 1 $q = 20 - 1\ x$. Adding

$$\frac{x}{8} \text{ and } \frac{20-x}{3} \text{ gives } \frac{160-5x}{24} = 5 \text{ and } x \text{ is 8.}$$

He then adds that it is also possible to choose 1*x* for larger and 1 q. for the smaller number. The value of *x* would then be 12.[10] Problems 188 to 217 in the section concerning

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[8] Unfortunately, I have not been able to consult the original edition of the book. Some parts have been published in secondary sources. This quote is from Rudolff (1525, f. βvi[v], and cited by Treutlein (1879, 84): "Dise regl lernt wie man sich halten sol bey etlichen exempeln, so über den gesetzten radix (wie dan*n* der brauch ist) auch andere position oder satzungen erfordern. Dann so 1 *x* einem ding gesetzt oder zugeben ist mag er in dem selbigen procesz (confusion oder irsal zu vermeiden) keinem andern ding zugestelt werden. Laut also. Wan*n* nach setzung 1 *x* ein ding vorhanden ist welchem du (ausz vorgethaner underweysung) mit der position nit magst zukomme*n*. Setz dasselbig ding sei 1 quantitet, und procedir nach laut der auffgab, so lang bisz zwo ordnung der zalen einander gleich werden … Ist weiter etwas vorhanden. Nim war der vorigen satzungen, gib dem selbigen ding 1 quantitet, und procedir nach vorgemelter instruction" (translation mine). Treutlein discusses problems 188 and 191 from Rudolff and point at Pacioli's *Summa* as an earlier source for the *Regula quantitatis*. This quoted part has been ommitted in Stifel's edition (Rudolff 1553, f. 186[r]).

[9] Rudolff 1553, 185v: "Gib zwo zalen / welche zsamen 20 machen: wenn ich die kleyner dividir durch 8. Die grosser ducr 3. Thu die quotient zu samen / das 5 werden".

the first rule (linear problems) are all solved using the *regula quantitatis*. Problem 192 is a typical linear problem of four men buying a horse, formulated as follows:[11]

> Four companions bought a horse for 11 fl. Each of them needed something from the other three companions. Namely, the first ½ of their money, the next 2/3, the third ¾, and the fourth 4/5. So that each of them could buy the horse. How much had each?

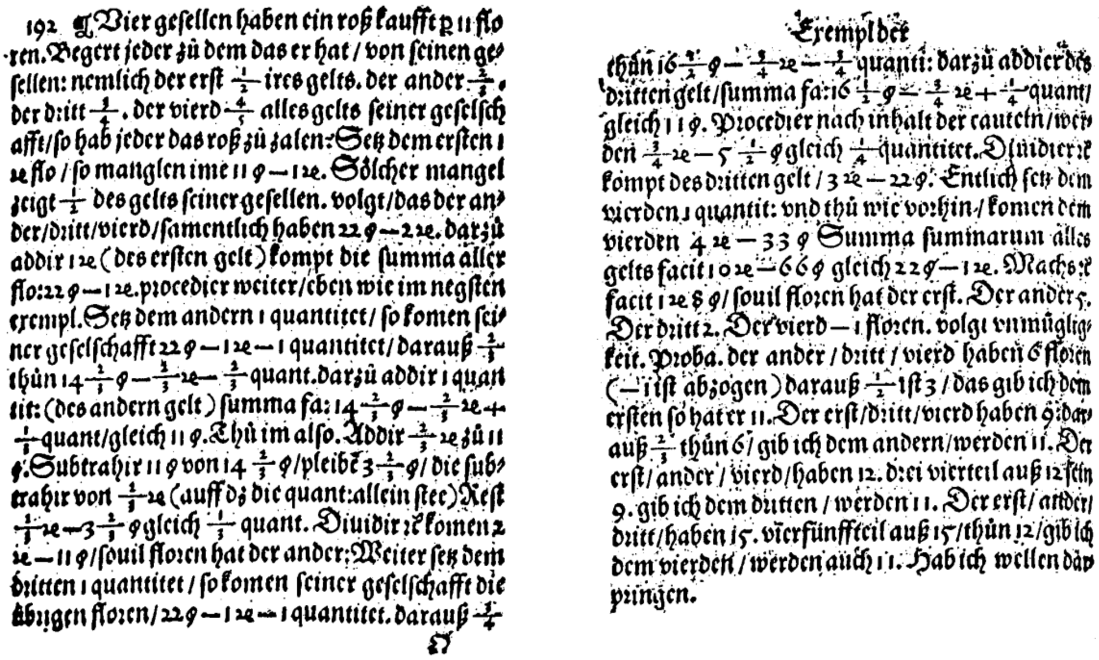

Figure 3: A solution to the "men buy a horse" problem from Rudolff (1525 f. Qi[r]-Qi[v]) using the *regula quantitatis*. (facsimili from Kaunzner 1996, 128-9)

This translates into modern notation as follows:

$$
\begin{aligned}
a+\frac{1}{2}(b+c+d)&=11\\
b+\frac{2}{3}(a+c+d)&=11\\
c+\frac{3}{4}(a+b+d)&=11\\
d+\frac{4}{5}(a+b+c)&=11
\end{aligned}
\qquad (1.3)
$$

[10] This observation is important for the distinction of double solutions to the quadratic equation.

[11] Rudolff 1525, f. Qi[r]-Qi[v], translation mine.

Rudolff uses the first unknown x for the money of the first person. The other three thus have 22 – 2x. Add to this x and you have the sum of all four as 22 – x. Then he introduces the second unknown for the second person's money ("Setz dem andern 1 quantitet, so komen seiner geschelschafft 22 Ø – 1x – 1 quantitet"). The sum of the three others then is 11 – 1x – 1y. Using this value in the second expression of (1.3) this gives

$$14\frac{2}{3}-\frac{2}{3}x+\frac{1}{3}y=11 \text{ or } \frac{2}{3}x-3\frac{2}{3}=\frac{1}{3}y$$

and the second persons money is y = 2x – 11.

Rudolff then uses the second unknown for the share of the third one ("Weiter setz dem dritten 1 quantitet"), which we also represent as y. Using the same reasoning, the share of the three others is 22 – 1x – 1y. Using the third expression in (1.3) and simplifying this leads to

$$\frac{3}{4}x-5\frac{1}{2}=\frac{1}{4}y \text{ or } y = 3x - 22.$$

Again for the fourth which gives 4x – 33. Adding the four shares together gives 10x – 66 = 22 – x leading to the interesting solution (8, 5, 2, -1). Rudolff explicitly denies the possibility of a negative value ("Der vierd – 1 floren, volgt unmugligkeit") but proceeds with a *proba* proving the validity of the solution. Stifel will later reproduce the problem, slightly adapting its solution (Stifel 1553, ff. 310^v – 312^r). He uses basically the same method but changes the second unknown to 1B, 1C and 1D respectively and uses 1A for the sum of the three others. He gives an interpretation of the negative value as follows:[12]

> The fourth has 0 – 1 fl., meaning that he has no money at all. He thus owes a debt of 1 fl. to the one who sells the horse. While the others have to pay 11 fl. for the horse, the fourth will have to pay 12 fl.

The interpretation by Stifel of the fourth man's share as a debt is denying the existence of negative solutions in the same spirit as Rudolff. It is therefore doubtful to consider this as an instance of the acceptance of negative solutions.[13]

Allegedly, Rudolff borrowed heavily from the Latin manuscript Vienna 5277 as was known by his peers (Cantor, II, 424; Kaunzner 1972, 115). Stifel refers to these allegations in his introduction to the 1553 edition (Stifel 1553). He does not really counter these

[12] Rudolff 1553, f. 311^v, translation mine.

[13] Similar linear problems by Fibonacci from the *Flos* and the *Liber Abaci* have recently been discussed both by Sesiano (1985) and Gericke (1996) in publications on negative solutions. Fibonacci also gives the interpretation that the man started with a debt to the other persons of 3 besants. Sesiano and Gericke have no trouble interpreting the text as an instance of a solution with a negative number.

claims, instead pitying those who demean oneself by such criticism. Even if it were true, he did not hurt anyone by doing so.[14] However, Rudolff must have kept some other sources secret. The Vienna codex contains practically no linear problems, while they are prominent in the *Coss*. No other known manuscripts from Vienna and München around 1500 use more than one unknown. Rudolff mentions that he was given the problem by Johann Seckerwitz from Breslau, and that he solved it using the *regula quantitatis* (this passage is deleted in Stifel's edition, Eneström, 1907, 204).[15]

Rudolff's solution compares very well with that of Chuquet and de la Roche as well as the example of Pacioli not discussed here.[16] Clearly, one of those must have been the source for the method and the name *regula quantitatis*. Treutlein and Tropfke assume that Rudolff learned the rule from Pacioli, but they do not cover the rule in relation to Chuquet.[17] While the exact influence will be hard to demonstrate there are strong arguments in favor of Chuquet or de la Roche rather than Pacioli. This would throw a new light on the German cossist tradition because a line of transmission from France to Germany has not previously been established. Let us look at the arguments:

1) Rudolff's use of the word *quantitet* favors de la Roche rather than Pacioli's who cites *quantita sorda* as the original name "used by the ancients".
2) Rudolff explicitly uses the term *Regula Quantitatis*. The French term is used by de la Roche in the heading of chapter 9 and in "Application de la regle de la quantite". Pacioli, on the other hand, does not make any reference to a rule. He considers the *quantita sorda* more as a generalization of the *cosa*.
3) de la Roche's reference to 'perfection' in "Ceste regle de la quantite est annexee et inferee au primier canon de la regle de la chose com*m*e acomplissement et perfection dicelluy", is echoed in Rudolff's description "ein volkommenheyt der Coss". We do not find such a qualification in the *Summa*.
4) Both Rudolff ("confusion oder irsal zu vermeiden") and de la Roche ("le nombres et ρ de la premiere positions seroyent mesmes et intrinq*ue*s indifferen*n*ment avec les no*m*bres et ρ des autres positions qui seroit co*n*fusion") argue for the use of *quantite* or *q* for the sake of avoiding confusing. The use of the foreign word

[14] Stifel 1553, f. A3: "Ob denn gleich Christoff Rudolff seine Exempla nicht alle selbs hatte gedichtet, sondern etzliche in der Librey zu Wien abgeschriben, und uns die selbige durch den truck mitgeteylet, wem hat er damit schaden gethon? Niemands.".

[15] Rudolff 1525, Rii[v], example 210: "Mit diesem exempl ward ich zu Breslaw von meine*m* guten freund Johansen Seckerwitz, Rechenmeister daselbs, durch die coss zu mache*n* ersucht, war zur selbigen zeyt der regl quantitatis nit bericht". Johann Seckerwitz published several arithmetic books. In 1519 *Rechpuchleyn* at Breslaw; and *Regel oder Satzbüchlein... auff der Linien*, Rękopis, and *Rechbuchleyn*, in 1535 at Wrocław, republished in 1547.

[16] Pacioli 1494, f. 192[r]: "Tre hano d*enari*. Dici el primo a li 2 altri: dateme la ½ di vostri che haro 50. Dici el secondo alialtri 2, datime el 1/3 di vostri che haro 50. Dici el 3° alialtri 2, datime el ¼ di vostri che haro 50. Dimando che a per uno"

[17] Tropfke (1933, II, 52): "Die Kenntnis der Neuerung Paciolis kam auch zu den deutschen Cossisten", and also III, 46.

confusion in German is significant. Pacioli makes no reference to possible confusions over *cosa* and *quantita*.

Convinced by such arguments, we embarked on a detailed comparison of linear problems solved with the second unknown in Chuquet (1484), de la Roche (1520) and Rudolff (1525, 1553). The result, summarized in the Appendix, is quite surprising. The relationship between Rudolff and Chuquet is closer than between Rudolff and de la Roche. Furthermore, several consecutive problems from Rudolff's *Coss* can be found in the same order in Chuquet's manuscript. It seems as if Rudolff went through Chuquet's manuscript and copied the most interesting cases while changing the values. Such close correlation raises more questions than it answers. The provenance of the manuscript was traced by Marre. After de la Roche it came into the hands of the Italian Leonardo de Villa (sold for 80 solidis), only to return to France, in the library of Jean-Baptiste Colbert, long after the publication of Rudolff.[18] With additional solid evidence, such influence of the early French algebraist on Rudolff would require a revision of the history of the German cossist tradition.

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[18] See Marre (1880) and Flegg, Hay and Moss (1985, 17).

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7. Appendix B: Linear problems from Chuquet and Rudolff

Roche	Chuquet (1484)		Rudolff (1525)	
107r	(3, 2; 4, 3; 5, 5)	CTA056		
	(7, 2; 9, 6).	CTA057	(3, 3; 3, 3)	RBH1134
	(20, 2; 30, 3).	CTA058	(2, 2; 3, 3)	RBH1135
	(1, 1; 1, 2).	CTA059		
	(2, 1; 3, 4).	CTA060		
	II-(3, 2; 4, 3; 5, 5).	CTA061		
	II-(2, 2; 2, 3; 2, 4).	CTA062	II-(2, 2; 3, 3; 4, 4).	RBH1136
	I-(7, 5; 9, 6; 11, 7).	CTA063	I-(200, 3; 300, 3; 376, 4) I-(2, 2; 3, 3; 4, 4)	RBH1137 RBH1138
	III-(7, 5, 1; 9, 6, 1; 11, 6, 1)	CTA064		
	IV-(7, 5, 1; 5, 6, 1)	CTA065		
	IV-(7, 3, 3; 9, 4, 1)	CTA066		
	IV-(7, 5, 1; 5, 7, 2)	CTA067		
	IV-(7, 5, 2; 9, 7, 5)	CTA068		
	IV-(7, 5, 1; 5, 6, 12)	CTA069		
150r	III-(12, 2, 6; 13, 4, 2; 11, 3, 3)	CTA070		
	III-(7, 5, 1; 9, 6, 2; 11, 7, 3)	**CTA071**	III-(14, 2, 1; 13, 3, 1; 12, 4, 1)	**RBH1197**
	IV-(7, 5, -1; 5, 7, -1)	CTA072		
	IV-(7, 5, -1; 5, 7, -3)	CTA073		
	IV-(7, 5, 4; 5, 7, -6)	CTA074		
	III-(7, 2, -1; 5, 9, -2; 3, 4, -3)	**CTA075**	III-(10, 1, -3; 10, 2, -3; 10, 3, -3)	**RBH1198**
	IV-(7, 5, 4; 5, 7, -6)	CTA076		
	(7, 5; 9, 6; 11, 6)	**CTA077**	(1, 2; 4, 3; 8, 4)	**RBH1199**
	(3, 100; 4, 106; 5, 145; 6, 170)	**CTA078**	(2, 20; 3, 13; 4, 8; 5, 22),	**RBH1200**
	(7, 2; 8, 3; 9, 4; 10, 5; 11, 6).	**CTA079**		
	(7/12, 14; 9/20, 12)	CTA080		
		CTA081		
	II-(½, 2/3) 30	CTA082		
	II-(1/3, 1/4, 1/5) 20	CTA083		

(a, b; c, d) the general form for two people "If I had had *a* from you, I would have *b* times .."

I-(a, b; c, d; ...) denotes the case for more people where 'you' means all the others.

II-(a, b; c, d; ...) denotes the same where 'you' means just the next person, taken cyclically.

III-(a1, b1, c1; a2, b2, c2; …)

IV-(a1, b1, c1; a2, b2, c2; …) same as III but cyclically

$$x + a_1 = b_1(y + z - a_1) + c_1$$
$$y + a_2 = b_2(x + z - a_2) + c_2$$
$$z + a_3 = b_3(x + y - a_3) + c_3$$