

The Abbaco Tradition (1300-1500): its Role in the Development of European Algebra

Albrecht Heeffer
Ghent University, Belgium *

November 30, 2008

Abstract

Abbaco algebra is a coherent tradition of arithmetical and algebraic problem solving, mostly based in the merchant cities of fourteenth and fifteenth-century Italy. This period is roughly situated between two important works dealing with algebra: the *Liber Abbaci* by Fibonacci (1202) [5] and the *Summa di Arithmetica et Geometria* by Lucca Pacioli (1492) [20]. Such continuous tradition of mathematical practice was hardly known before the first transcriptions of extant manuscripts by Gino Arrighi from the 1960's (most notably [1], [2], [3] and [4]) and the ground-breaking work by Warren van Egmond [28]. After some decades of manuscript study and the recent assessment of Jens Høyrup [10] we now have a better understanding of this tradition. In this paper we provide an overview of the basic characteristics of the abbaco tradition and discuss the role it played towards the new symbolic algebra as it emerged in sixteenth-century Europe. We argue that its influence on the sixteenth century has largely been ignored and that the new *ars analytica* from the French algebraists much depended on the general practice of abbaco problem solving.

1 Abbacus algebra: a brief characterization

With some exceptions, algebraic practice was completely absent from the scholarly tradition or university curriculum before the mid-sixteenth century.¹ It took until the late seventeenth century before algebra became taught at universities. Instead, algebra flourished within the vernacular tradition of the abbacus schools in Italian cities during the fourteenth and fifteenth century. We call this the abbacus or abbaco tradition, spelled as in the *Liber abbaci* of Fibonacci

*Post-doctoral fellow of the Research Foundation Flanders (FWO Vlaanderen). This paper was written while the author was visiting researcher at Kobe University, Japan. The author would like to thank Prof. Nobuo Miura.

¹One such exception is the *Quadrupartitum numerorum* of Jean de Murs (1343) [17], which provides evidence that algebra, as known from Fibonacci's *Liber abbaci*, was studied in scholarly circles in Paris. However, it is unlikely that it was ever taught within the *quadrivium*.

(1202) to distinguish it from the abacus as a calculating tool. The abbacists practised calculation with hindu-arabic numerals as opposed to calculation using material means such as the *tavola* or the abacus. The abbaco masters were hardly known before the first transcriptions of their manuscript treatises by Gino Arrighi during the 1960s and 1970s. It is only with Warren van Egmond's extensive catalogue of manuscripts that we have a fairly complete picture of the extent of texts from this tradition [27].

1.1 Social context

Abbacus masters earned a living from teaching commercial arithmetic to sons of merchants and artisans, renting rooms and occasionally surveying assignments [8]. Abbacus masters taught in so-called *bottega* in the mercantile centers of Northern Italy, such as Florence, Siena, Lucca and Parma. They were either employed by the city or worked on a private basis. Earning a good living, they belonged to the middle class. The profession often depended on family relations. The Calendri family is known to consist of five generations of abbaco masters. Based on the archive work of Ulivi [24] [25], we know that about twenty abbaco schools were active in Florence alone between 1340 and 1510. Around 1343 there were no less than 1200 students attending abbaco schools in Florence. Some famous names are known to have been taught by abbaco masters, most notably Dante Alighieri and Leonardo da Vinci.

1.2 Abbaco treatises

Many abbaco masters wrote long treatises on arithmetic and algebra in which they solved hundreds of problems. Such manuscripts were often illustrated and presented as gifts to patrons and important merchants. Van Egmond's catalogue lists about 250 extant abbaco manuscripts kept in libraries all over the world, many dealing with algebra. We do not know much about the role of these treatises but it is unlikely that they were used for teaching. The subjects treated in these manuscripts exceed those that were taught within the school program. They seem to have functioned as a way to display the masters knowledge of arithmetic, geometry and algebra. Treatises passed hands from master to apprentice and were often copied and appropriated. The earliest extant abbaco treatise probably is the *Columbia Algorism* (c. 1290, [29]), although it was not acknowledged as such when Kurt Vogel published the transcription. It is only through a recent study of a coin list contained in the manuscript that an earlier dating was established [23]. In 1978 van Egmond wrote an article on what he then considered "The Earliest Vernacular Treatment of Algebra: The *Libro di Ragioni* of Paolo Gherardi" [26]. However, it has since been established that Gherardi's algebra, written at Montpellier in 1328, is preceded by that of Maestro Jacopo da Firenze. Jacopo wrote the text *Tractatus algorismi*, also at Montpellier, in 1307. Some of the later abbaco manuscripts reach into the early sixteenth century but we can state that the tradition of vernacular manuscripts was effectively replaced by printed books.

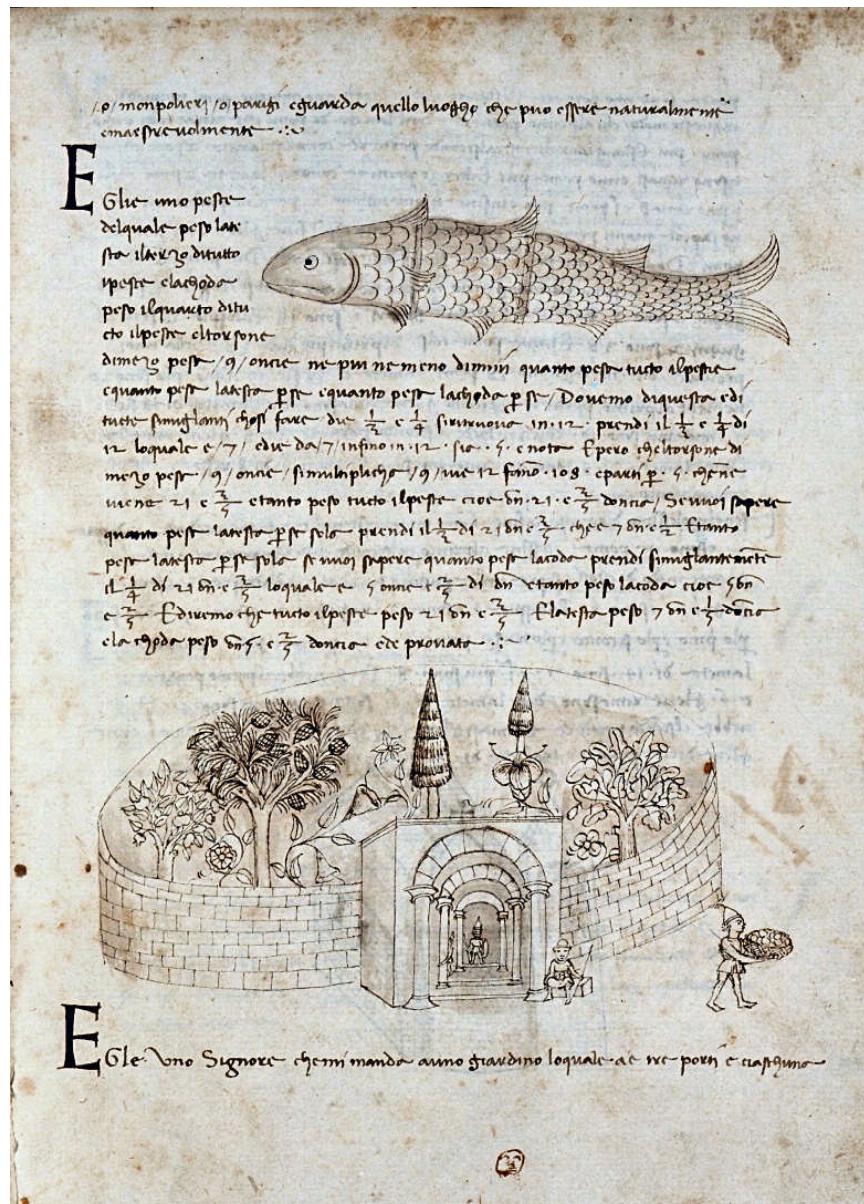


Figure 1: One typical page from an abbaco manuscript
 with two illustrated arithmetical problems (from Paolo
 Dagomari, *Trattato d'abbaco, d'astronomia e di segreti
 naturali e medicinali*, Plimpton ms 167, f. 66, Columbia
 University, The Digital Scriptorium)

One member of the Strozzi family of wealthy Florentine bankers, Carlo di Tommaso Strozzi (1587-1670), built a significant library which contained many abbaco manuscripts. Most of these are now part of the manuscript collection of the Biblioteca Nazionale Centrale in Florence. As abbaco treatises were collected by the wealthy they were copied by professional scribes and some of them are abundantly illustrated. Given that the *Trattato d'abbaco* by Maestro Benedetto da Firenze, after more than 500 years, is still extant in eighteen copies, professional scribes must have had their hands full with abbaco manuscripts. This has some consequences for the type of texts we are dealing with. Professional scribes produced nicely written texts but they were not always familiar with the subject. An abbaco text typically contains many omissions, errors and misinterpretations. The move towards first algebraic symbolism was seriously impeded by the scribal process. The scribes did not always understand the function of the symbolic representations. Such passages were often mutilated, abbreviated or completely omitted [12]. Although some symbols for the unknown, powers of the unknown and addition and multiplication co-appeared in multiple manuscripts, no single or dominating representations system emerged within the abbaco tradition. Only during the sixteenth century, profiting from the advent of book printing, did symbolic algebra benefit from the effect of typographical fixity. This led to two competing systems of representation, the Italian and German cossic traditions.

1.3 The emergence of the abbaco tradition

The seemingly evident narrative that the tradition was initiated with Fibonacci's book is currently challenged by Jens Høyrup [9][10]. Although the first abbaco manuscript dealing with algebra dates from 1307, there is evidence that the tradition existed at Fibonacci's time. Furthermore it seems that it originated from the Provence (south of France) and Catalan regions (north of Spain) [9]. If it did not originate with Fibonacci where did the abbacists knowledge of algebra come from? Evidently we have to look at the three Latin translations of al-Khwārizmī's *Algebra* extant in sixteen manuscripts [14]. These translations have been identified as from Robert of Chester (c. 1145) [16], Gerard of Cremona (c. 1150) [15] and Guglielmo de Lunis (c. 1250) [19], although there is still discussion whether the latter translation was Latin or Italian. What became available to the West was only the first part of al-Khwārizmī's treatise. However, there are several peculiarities of early Arabic algebra, clearly present in the Latin translations, partly in Fibonacci but not present in abbaco algebra.²

Firstly, there are some anomalies with the concept of an unknown in early Arabic algebra. It is generally accepted that the term *māl* refers to possession, or wealth or even a specific sum of money (in Robert's translation *substancia* and in Gerard's *census*). The *shay'* is translated as 'thing' ever since the first commentators wrote about it [7]. From the beginning, *shay'* was considered the

²We here summarize the basic arguments. For an extensive discussion of this subject see my [11].

unknown [6]. The difficulties of interpretation arise when we translate *māl* by ‘square’ and *shay* by ‘root’. Rosen [21] and Karpinski [18] both use ‘square’ for *māl* on most occasions. Karpinski even uses the symbolic x^2 . However, when the problem can be stated without the use of a square term, they both change the interpretation of the *māl*. To cut a long story short, as early Arabic algebra came into practice from different traditions, this is reflected in the ambiguity of the concept of an unknown and its relation to the square term of the unknown. The ambiguity is partly present in Fibonacci’s *Liber Abbaci*. For most of the algebra part, Fibonacci uses the *res* and *census* terminology of Gerard of Cremona. However, in the middle of chapter 15 he switches from *census* to *avere* for *māl* ([22], 578-601). As far as we have studied abbaco algebra, such anomaly does not show with respect to the *cosa*. Thus Arabic algebra must already have been well-digested before Jacopo da Firenze wrote his treatise in 1307.

Secondly, Arabic algebra shows ambiguities with respect to the operations on equations, in particular the *al-jabr* operation (*restaurare* in the Latin translations). Often the *al-jabr* operation is interpreted as “adding a positive term to both parts of the equation in order to eliminate the negative term in one of the parts”. A detailed analysis of the source texts provides us with a different picture. we believe that the *al-jabr* operation in early Arabic algebra can be characterized as follows:

1. An operation aiming at the restoration of a defected quantity to its original completeness.
2. The restored quantity could initially have been a simple number in the sense of al-Karkhī, but for Abū Kāmil it also applies to polynomials.
3. The operation is probably derived from or to be interpreted in a geometrical sense.
4. The operation is not performed on an equation but on the affected part of one of two coequal polynomials.
5. The addition of the defected part to the coequal polynomial is not a part of but a consequence of the operation.

While this interpretation can be demonstrated for the works of al-Khwārizmī, al-Karkhī and Abū Kāmil and the Latin translations, the *al-jabr* operation gets a different meaning within the abbaco tradition. All abbaco treatises use a single verb for the two operations together. The earliest ones use *ristore* or *ristorare* derived from the Latin *restaurare*. Later texts also use *ragguagliare* or *disfare e debiti*. The shift in meaning had thus taken place before the first extant abbaco treatise.

A third important difference between abbaco and Arabic algebra lies in the normalization of the “equations”. In *all* abbaco texts on algebra, the first step of the rules prescribes normalization by dividing the quadratic term by its coefficient. Even when the coefficient is one, the step is often included as dividing by one. The Arabic texts and their Latin translations provide the rules

for the non-normalized equations. Again this points to a practice which was generally in use before the first abbaco text.

There are still some other arguments which we here summarize:

1. Abbaco treatises are free from arabisms. They do not seem to have been derived directly from Arabic sources.
2. Geometric demonstrations for the rules of algebra, known from Arabic sources, are very rare in abbaco algebra. The few ones that we know of are from the fifteenth century.
3. There is little or no reference to Fibonacci's *Liber abbaci*. The few ones that mention Fibonacci are mostly from the fifteenth century. If the abbaco tradition depended on Fibonacci we would expect a much closer connection with the contents of the *Liber abbaci*.

All these arguments support the thesis for a tradition of algebraic practice which was already in place when Fibonacci wrote his famous treatise.

2 Abbaco problem solving

Abbacus algebra is all about problem solving. Most of the folios of these sometimes hefty collections deal with arithmetical and algebraic solutions to a large number of problems. In these treatises the introduction – if there is one – explains the rules of algebra, exceptionally with a geometrical demonstration. The earliest treatises within the abbaco tradition already expand on the six rules of Arabic algebra, but later *maestri d'abbaco* extend the list to several more types, resulting in the rather preposterous list of 198 equation types of Maestro Dardi [28] accompanied by problems to illustrate each of them. Later treatises occasionally discuss addition and multiplication of polynomials as an introduction to algebra. But that is as far as it goes for the theory. The bulk of the text is pure problem solving. There is a remarkable consistency in the structure, style and rhetoric of abbaco texts during the two centuries of their existence. Practically every text dealing with algebra follows the same rigid structure which can be divided into six parts:

1. *problem enunciation*: in a first section the problem text is provided and a question is posed. Usually problems are set in a practical context.
2. *choice of the rhetorical unknown*: every solution start with the sentence “pose that <some unknown quantity of the problem> equals <some quantity of> the *cosa*” (the rhetorical unknown). Often a clever choice of the unknown or a power of the unknown is an important step in the solution of the problem. Most abbaco texts deal with a single unknown, though there are some exceptions [13]. However, a straightforward translation of all unknown quantities of the problem into symbolic form is a practice which is established in Europe only during the eighteen century.

3. *manipulation of polynomials*: using the unknown, the problem text is formulated in terms of coequal polynomials and manipulated in such a way that these are kept equal.
4. *reduction to a canonical form*: the purpose of manipulating the polynomials is to reduce them to a form in which a standard rule applies. This marks the end of the analytical part of the reasoning.
5. *applying a rule*: usually the rule is reformulated and literally applied. Typically it includes the normalization of the equation by dividing it by the coefficient of the square term even if this amounts to dividing by one.
6. *numerical test*: often, but not always, the validity of the solution is checked by a numerical test using the root of the equation. This test is always performed on the problem enunciation and not on the equation.

The lack of symbolism in abbaco algebra is compensated by the rigid rhetorical structure. Each problem is dealt with in the same way. Every rule is reformulated and applied as it were for the first time. Repetition, cadence and structure facilitates the understanding and memorization of the problem solving procedure. Only in very rare cases are problems and solutions generalized or is there a transfer of results from one problem to another.

2.1 Example

As an illustration we show here one simple example which an application of one of the rules for solving quadratic problems taken from the earliest extant treatise dealing with algebra, the *Tractatus algorismi*, a treatise by Jacopo da Firenze from the early fourteenth century [10] (fol. 39^v – 40^r). We first show the original Italian text, highlighting the six sections distinguished above, and then provide an literal English translation with a symbolic rendering.

2.1.1 [RAA313]

- (E) Uno fa doi viaggi, et al primo viagio guadagna 12. Et al secondo viagio guadagna a quella medesema ragione che fece nel primo. Et quando che conpiuti li soi viaggi et egli se trovò tra guadagnati et capitale 54. Vo sapere con quanti se mosse.
- (C) Poni che se movesse con una chosa, et nel primo viaggio guadagni 12.
- (P) Dunqua conpiuto el primo viaggio si trovò una cosa et 12. Adunqua manifestamente vedi che de ogni una cosa nel primo viaggio fa una chosa e 12. Quanto serrà a quella medesema ragione nel secondo viaggio? Convienti multiprichare una cosa et 12 via [fol. 40^r] una cosa et 12, che fa uno censo et 24 cose e 144 numeri, li quali sicondo che dice la regola si vole partire in una cosa, et dè ne venire 54. Et però multipricha 54 via una cosa. Fa 54 cose, le quali se oglagliano a uno censo et 24 cose e 144 numeri. Ristora ciascheuna parte, cioè de cavare 24 cose de ciascheuna parte.

- (F) Et arai che 30 cose sonno oguali a uno censo et 144 numeri.
- (R) Parti in uno censo, vene quello medesemo. Dimezza le cose, remanghono 15. Multipricha per se medesemo, fanno 225. Traine li numeri, che sonno 144, resta 81. Trova la sua radice, che è 9. Trailo del dimezzamento delle cose, cioè de 15. Resta 6, et cotanto vale la chosa. Et noi dicemmo che se movesse con una chosa. Dunqua vedi manifestamente che se mosse con 6.
- (T) Et se la voi provare, fa così. Tu di che nel primo viaggio guadagnio 12 et con 6 se mosse a 18. Siché nel primo viaggio se trovò 18. E però di così, se de 6 io fo 18, que farò de 18 a quella medesema ragione? Multipricha 18 via 18. Fa 324. Parti in 6, che ne vene 54, et sta bene. Et così se fanno le simili ragioni.

2.1.2 [Rule 6]

When the x are equal to the x^2 and to the number, we shall divide the x^2 and next halve the x and multiply by itself and remove the number and the root of which remains increased with the halving of the x is worth the x . Or indeed the halving of the x less the root of that which remains.

$$dx = ax^2 + e \rightarrow bx = x^2 + c \quad (R05)$$

$$x = \frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

2.1.3 English translation

Somebody makes two business trips, and on the first trip he wins 12. And on the second trip he wins in that same rate as he did on the first. And when he had completed his trips he is found with 54, profit and capital together. I want to know with how much he started with.

$$\frac{x}{x+12} = \frac{x+12}{54} \quad (1)$$

Posit that he started with one x , and on the first trip he wins 12. Hence, when the first trip was completed, he is found with one x and 12. Hence you manifestly see that for each one x on the first trip he makes an x and 12. How much will it make in that same ratio on the second trip? It suits you to multiply x and 12 times a x and 12, which makes one x^2 and $24x$ and 144 numbers, which, according to what the rule says, one shall divide in a x , and 54 shall result from it. And therefore multiply 54 times a x . It makes $54x$, which is equal to one x^2 and $24x$ and 144 numbers. Restore each part, that is, you shall remove $24x$ from each part. And you will get that $30x$ are equal to one x^2 and 144 numbers. Divide in one x^2 , the same results. Halve the x , 15 remain. Multiply by itself, it makes 225. Detract from it the numbers, which are 144, 81 is left. Find its root, which is 9. Detract it from the halving of the x , that is, from 15, 6 is left,

and so much is worth the thing. And we said that he started with x . Hence you manifestly see that he started with 6.

And if you want to verify it, do as such. You say that on the first trip he won 12, and with 6 he started, one has 18. So that on the first trip he found himself with 18. And therefore say as such, if from 6 I make 18, what will I make from 18 in that same ratio. Multiply 18 times 18, it makes 324. Divide in 6, and 54 results from it, and it goes well. And thus similar computations are done.

3 Towards the new algebra

Abbaco treatises on algebra form a very coherent tradition of algebraic problem solving. For the production of their texts abbaco masters relied heavily on their tradition. They copied literally from previous manuscripts, abbreviated solutions, in the process changing the order of the problems and the problem values to conceal their sources. Often problem values or parameters were tweaked to increase the complexity of the problem solutions. In the process of almost three hundred years of text production the practice of algebraic problem solving within the abbaco tradition paved the road to the emergence of symbolic algebra during the first half of the sixteenth century. The tradition made it clear that all arithmetical problems could be solved by algebra. The expansion of the number concept, in particular the full acceptance of irrational numbers, and occasional occurrences of negative solutions contributed to the process. The preparation of a symbolic algebra was realized, not only through the acceptance of ‘symbolic’ solutions appearing during the fifteenth century, but also through the epistemic validity attributed to the operations of algebra. For example, the validity of the rules of signs is derived from correctly expanding the binomial product $(a - b)(a - b)$, which we consider as a form of symbolic reasoning.

References

- [1] Arrighi, Gino (ed.) (1964) *Paolo Dell'Abaco, Trattato d'aritmetica*. Pisa: Domus Galilaeana.
- [2] Arrighi, Gino (ed.) (1967) *Antonio de' Mazzinghi. Trattato di Fioretti, secondo la lezione del codice L.IV.21 (sec. XV) della Biblioteca degli Intronati di Siena*, Pisa: Domus Galilaeana.
- [3] Arrighi, Gino (ed.) (1973) *Libro d'abaco, Dal Codice 1754 (sec. XIV) della Biblioteca Statale di Lucca*, Lucca: Cassa di Risparmio di Lucca, 1973.
- [4] Arrighi, Gino (ed.) (1987) *Paolo Gherardi, Opera mathematica: Libro di ragioni Liber habaci. Codici Magliabechiani Classe XI, nn. 87 e 88 (sec. XIV) della Biblioteca Nazionale di Firenze*. Lucca: Pacini-Fazzi.

- [5] Boncompagni, Baldassarre (1857, 1862) *Scritti di Leonardo Pisano, matematico del secolo decimoterzo* (2 vols.), Vol I.: *Liber Abbaci di Leonardo Pisano*, Vol II.: *Leonardi Pisani Practica geometriae ed opuscoli*. Tipografia delle Scienze Matematiche e Fisiche, Rome.
- [6] Colebrooke, Henry Thomas (1817) *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhaskara*. John Murray, London (reprint by Sandig Reprint Verlag Vaduz, Lichtenstein, 2001).
- [7] Cossali, Pietro (1797-9) *Origine, trasporto in Italia, primi progressi in essa dell'algebra. Storia critica di nuove disquisizioni analitiche e metafisiche arricchita* (2 vols.), Parmense.
- [8] Goldthwaite, Richard (1972-73) "Schools and Teachers of Commercial Arithmetic in Renaissance Florence", *Journal of European Economic History*, 1, pp. 418-433.
- [9] Høyrup, Jens (2006) "Jacopo da Firenze and the beginning of Italian vernacular algebra", *Historia Mathematica*, 33, (1), February 2006, pp. 4-42.
- [10] Høyrup, Jens (2007) *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*, Science Networks Historical Studies, 34, Basel: Birkhauser.
- [11] Heeffer, Albrecht (2008) "A Conceptual Analysis of Early Arabic Algebra" in T. Street, S. Rahman and H. Tahiri (eds.) *The Unity of Science in the Arabic Tradition: Science, Logic, Epistemology and their Interactions*, Dordrecht: Kluwer Academic Publishers, pp. 89-129.
- [12] Heeffer, Albrecht (2008) "Text production reproduction and appropriation within the abbaco tradition: a case study", Sources and Commentaries in Exact Sciences, 9, pp. 101-145.
- [13] Heeffer, Albrecht (2009) "The Regula Quantitatis: from the second unknown to the symbolic equation", monograph, forthcoming.
- [14] Hughes, Barnabas (1982) "The Medieval Latin Translations of Al-Khwārizmī's al-jabr", *Manuscripta*, 26, pp. 31-37.
- [15] Hughes, Barnabas (1986) "Gerard of Cremona's Translation of Al-Khwārizmī's al-Jabr: A Critical Edition", *Mediaeval Studies*, 48, pp. 211-263.
- [16] Hughes, Barnabas (1989) *Robert of Chester's Latin Translation of Al-Khwārizmī's al-jabr*, Steiner, Stuttgart.
- [17] L'Huillier, Ghislaine (1990) *Le quadripartitum numerorum de Jean de Murs, introduction et édition critique, Mémoires et documents publiés par la société de l'Ecole des chartes*, 32, Genève: Droz.

- [18] Karpinski, Louis Charles (1915) *Robert of Chester's Latin Translation of the Algebra of Al-Khwarizmi*, Macmillan Company, New York (Reprinted by University of Michigan Press, Ann Arbor, 1933).
- [19] Kaunzner, Wolfgang (1972) “Über einige algebraische Abschnitte aus der Wiener Handschrift nr. 5277”. *Österreichische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Denkschriften*, 116 Band 4. Abhandlunch. Vienna, pp. 115-188.
- [20] Pacioli, Luca (1494) *Summa de Arithmetica Geometria Proportioni et Proportionalita*. Venice: Paganino de Paganini.
- [21] Rosen, Frederic (1831) *The Algebra of Mohammed Ben Musa*, The Oriental Translation Fund, London (Reprinted by Olms Verlag, 1986).
- [22] Sigler, Laurence (2002) *Fibonacci's Liber Abaci. A Translation into Modern English of Leonardo Pisano's Book of Calculation*, Springer, Heidelberg.
- [23] Travaini, Lucia (2003) *Monete, Mercanti e Matematica. Le monete medievali nei trattati di aritmetica e nei libri di mercatura*, Roma: Jouvence.
- [24] Ulivi, Elisabetta (2002) “Benedetto da Firenze (1429-1479) un maestro dabaco del XV secolo. Con documenti inediti e con unAppendice su abacisti e scuole dabaco a Firenze nei secoli XIII-XVI”, *Bollettino di storia delle scienze matematiche* A. 22, n. 1 (2002) (full issue).
- [25] Ulivi, Elisabetta (2006) Raffaello Canacci, Piermaria Bonini e gli abacisti della famiglia Grassini, *Bollettino di storia delle scienze matematiche* A. 24, n. 2 (2004) (stampa 2006), pp. 125-212.
- [26] van Egmond, Warren (1978), “The Earliest Vernacular Treatment of Algebra: *The Libro di Ragioni* of Paolo Gerardi”, *Physis*, 20, pp. 155-189.
- [27] van Egmond, Warren (1980) *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abacus Manuscripts and Printed Books to 1600*, Monografia, 4, Florence: Istituto e Museo di Storia della Scienza.
- [28] van Egmond, Warren (1983) “The algebra of Master Dardi of Pisa”, *Historia Mathematica*, 10, pp. 399-421.
- [29] Vogel, Kurt (1977) *Ein italienisches Rechenbuch aus dem 14. Jahrhundert* (Columbia X 511 A13). (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C, Quellentexte und Übersetzungen, Nr. 33). München: Deutsches Museum.