An Adaptive Logic Based on Jaśkowski's Approach to Paraconsistency

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1 Introduction

Discussive logics (also called discursive logics) were introduced in 1948 by Stanisław Jaśkowski and constitute the first family of formal paraconsistent logics.[1]

The basic mechanism behind discussive logics is as simple as ingenious. Where **L** is some modal logic and $\Gamma^\Diamond = \{\Diamond A \mid A \in \Gamma\}$, a discussive logic **DL**, associated with **L**, is obtained by specifying the language $\mathcal{L}^D$ of **DL** and by stipulating that, where A and the members of Γ are well-formed formulas of $\mathcal{L}^D$, $\Gamma \vdash_{\mathbf{DL}} A$ iff $\Gamma^\Diamond \vdash_{\mathbf{L}} \Diamond A$.

It is easily observed that, given an appropriate choice of $\mathcal{L}^D$ and of **L**, **DL** is paraconsistent. This is the case, for instance, if $\mathcal{L}^D$ is the language of Classical Logic (henceforth **CL**) and **L** is **S5** (in view of $\Diamond A, \Diamond{\sim}A \nvdash_{\mathbf{S5}} \Diamond B$). Where "$\wedge$" stands for the classical conjunction, discussive logics moreover do not allow to infer $A \wedge {\sim}A$ from A and ${\sim}A$ (in view of $\Diamond A, \Diamond{\sim}A \nvdash_{\mathbf{L}} \Diamond(A \wedge {\sim}A)$).

Especially from the perspective of interpreting discussions, discussive logics seem highly attractive. If two participants in a discussion contradict each other, we tend to interpret their statements in a modal way: "Someone accepts A; someone accepts ${\sim}A$". From this, neither "someone accepts B" nor "someone accepts both A and ${\sim}A$" follows. This is exactly what discussive logics allow for.

There is, however, a drawback. If $\mathcal{L}^D$ comprises the classical connectives, the above mechanism leads to a system that is as rich as **CL** for single-premise inferences, but that invalidates all interesting multiple-premise inferences of **CL** (Adjunction, *Modus Ponens*, *Modus Tollens*, ...).

This is why Jaśkowski dismissed the idea to formulate discussive logics in terms of the classical connectives (see [17, pp. 149–150]). Instead, he proposed

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[1] The English translation of Jaśkowski's 1948 paper dates from 1969, see [17].

to define a number of "discussive connectives" that satisfy some or all of the valid schemes and rules of the positive fragment of **CL**. For instance, the discussive logic $\mathbf{D_2}$ is obtained by defining a discussive conjunction "$\wedge_d$", a discussive implication "$\supset_d$" and a discussive equivalence "$\equiv_d$":

$$\begin{array}{rcl} A \wedge_d B & =_{df} & A \wedge \Diamond B \\ A \supset_d B & =_{df} & \Diamond A \supset B \\ A \equiv_d B & =_{df} & (A \supset_d B) \wedge_d (B \supset_d A) \end{array}$$

In addition to these three discussive connectives, the language of $\mathbf{D_2}$ contains the classical negation "$\sim$" and the classical disjunction "$\vee$". The definition of $\mathbf{D_2}$ is completed by stipulating that, where $\mathbf{S5^d}$ is **S5** extended with the above definitions and where A and the members of Γ contain at most the connectives "$\wedge_d$", "$\supset_d$", "$\equiv_d$", "$\sim$" and "$\vee$", $\Gamma \vdash_{\mathbf{D_2}} A$ iff $\Gamma^{\Diamond} \vdash_{\mathbf{S5^d}} \Diamond A$.[2]

It is easily observed that, in $\mathbf{D_2}$, "$\supset_d$" satisfies *Modus Ponens* (in view of $\Diamond A$, $\Diamond(A \supset_d B) \vdash_{\mathbf{S5^d}} \Diamond B$) and that "$\wedge_d$" satisfies Adjunction (in view of $\Diamond A$, $\Diamond B \vdash_{\mathbf{S5^d}} \Diamond(A \wedge_d B)$). However, this way out seems to be inadequate with respect to the intended application context. When interpreting a discussion, we typically assume that a statement made by some participant is accepted by *all* participants, unless and until this assumption turns out to be mistaken. This is justified in view of the fact that participants are expected to object when they do not agree with some of the statements made. Hence, even if we are willing to 'isolate' statements that contradict each other, we tend to conjoin them *unless and until* they are proven to be inconsistent. Thus, if in a discussion A is asserted by some participant and B by another, we assume that all participants accept "*A and B*", unless and until proven otherwise. However, the discussive conjunction does not capture this. This is easily seen from the fact that, in $\mathbf{D_2}$, $A \wedge_d {\sim}A$ follows from A and ${\sim}A$.

The aim of this paper is to propose a different solution to the problem. Instead of defining a series of discussive connectives, I shall design a discussive system in which the multiple-premise rules of **CL** are validated 'as much as possible'. The resulting logic will be called $\mathbf{D_2^r}$ and is meant as an alternative for $\mathbf{D_2}$. Unlike what is the case for $\mathbf{D_2}$, $\mathbf{D_2^r}$ will be defined both at the propositional and the predicative level.

Like $\mathbf{D_2}$, $\mathbf{D_2^r}$ incorporates the attractive properties of discussive logics (it is paraconsistent and does not allow for the derivation of contradictions). However, unlike $\mathbf{D_2}$, it is entirely formulated in the language of **CL**. As I shall show below, this alternative solution leads to a more natural interpretation of discussions than $\mathbf{D_2}$.

Another important difference between $\mathbf{D_2}$ and $\mathbf{D_2^r}$ is that the latter is non-monotonic. For instance, although $p \wedge q$ is $\mathbf{D_2^r}$-derivable from $\{p, {\sim}p, q\}$, it is not $\mathbf{D_2^r}$-derivable from $\{p, {\sim}p, q\} \cup \{{\sim}q\}$. As we shall see below, the non-monotonic character of $\mathbf{D_2^r}$ is related to the fact that it validates the multiple-premise rules of **CL** 'as much as possible'.

The techniques that led to $\mathbf{D_2^r}$ derive from the adaptive logic programme. The first adaptive logic was designed by Diderik Batens around 1980 (see [2])

[2] In the original version, the language of $\mathbf{D_2}$ does not include a discussive conjunction. The latter was introduced in a later paper (see [18] for a translation). Axiomatizations of $\mathbf{D_2}$ are presented in, for instance, [14], [19], and [15]; the axiomatization presented in [15] can also be found in [1, p. 117]

and was meant to interpret (possibly) inconsistent theories *as consistently as possible*.[3] Later the notion of an adaptive logic was generalized in different ways (for instance, to capture ampliative forms of reasoning) and a whole variety of adaptive logics was designed—see [5] for a survey. As we shall see below, the importance of adaptive logics is that they enable one to study in a formally exact way reasoning patterns that are non-monotonic and/or dynamic.[4]

I mentioned at the beginning of this section that a discussive logic **DL** is defined in terms of a modal logic **L**. In line with this, $\mathbf{D_2^r}$ will be defined in terms of a modal adaptive logic (based on **S5**). The latter is called $\mathbf{AJ^r}$.

I shall proceed as follows. After briefly explaining the basic ideas behind adaptive logics (Section 2), I present an intuitive characterization of the logic $\mathbf{AJ^r}$ (Section 3). The semantic and the (dynamic) proof theory of the propositional fragment of $\mathbf{AJ^r}$ is presented in Sections 4 and 5 and the extension to the predicative level in Section 6. In Section 7, I prove some central metatheoretical properties. The logic $\mathbf{D_2^r}$ is presented in Section 8. I end with some conclusions and open problems in Section 9.[5]

2 Some Basics of Adaptive Logics

One of the main characteristics of dynamic reasoning processes is that some presupposition is maintained 'as much as possible', that is, *unless and until* it is explicitly violated. When interpreting a discussion, for instance, one will normally rely on the presupposition that a statement made by some participant is accepted by all of them. However, when it is discovered that some participants contradict each other with respect to one of the statements, the presupposition will be abandoned for that particular statement. As a result, conclusions that were previously drawn may be rejected.

Until quite recently, the existing formal systems were not suitable for reasoning processes like these. In most logics, the violation of some presupposition is sanctioned with triviality. **CL**, for instance, presupposes consistency and turns any theory that violates this presupposition into the trivial one.

In order to deal with theories that violate one or more **CL**-presuppositions, a whole variety of non-classical logics was designed. Most of these are obtained by simply dropping some **CL**-presuppositions and by restricting the inference rules accordingly. Examples are Jaśkowski's $\mathbf{D_2}$, da Costa's **C**-systems, and Priest's **LP**. All of these drop the consistency presupposition, and restrict the rules of inference in such a way that *Ex Falso Quodlibet* is invalidated. Also these logics are inadequate for dynamic reasoning processes. Not only do they fail to capture the dynamics involved, they are usually too poor to make sense of actual reasoning processes (see [20] and [22] for examples from the history of the sciences that illustrate this).

[3]Logics that satisfy this property are referred to as *inconsistency-adaptive logics*.

[4]A reasoning pattern is called dynamic if the mere analysis of the premises may lead to the withdrawal of previously drawn conclusions. Not all dynamic reasoning patterns are non-monotonic. In [7], for instance, Batens shows that the pure logic of relevant implication can be characterized by a dynamic proof theory.

[5]Readers familiar with adaptive logics will see that I rely on insights and proof techniques from [2], [3], [4] and [9]. The proofs of Lemma 1 and Theorem 10 rely on proof techniques first presented in [3], the proof of Theorem 5 on a technique from [4].

Adaptive logics are designed along a completely different line. An adaptive logic **AL** that can handle violations of some particular presupposition **P** is obtained, not by dropping **P**, but by ensuring that **AL** 'adapts' itself to *specific* violations of **P**. Whenever **P** is violated, **AL** restricts the rules of inference that depend on **P** (in such a way that triviality is avoided). However, where this is not the case, the same inference rules are applicable in their full strength. As a result of this 'adaptation', adaptive logics do not invalidate a set of inference rules, but invalidate specific *applications* of such rules. Inconsistency-adaptive logics, for instance, 'localize' the specific inconsistencies that follow from a possibly inconsistent theory and adapt their inference rules to these. In inconsistent contexts, they invalidate the application of some inference rules of **CL**. In consistent contexts, however, the application of these same rules is validated.

The mechanism by means of which this contextual validation of inference rules is realized will become clear below. At the moment, it suffices to note that a rule may be validated with respect to some 'parts' of a theory, but invalidated with respect to other 'parts' of that same theory. For instance, where $\Gamma = \{p, {\sim}p, p \vee q, {\sim}r, r \vee s\}$, the inconsistency-adaptive logics **ACLuN1** and **ACLuN2** (see [3]) invalidate the application of Disjunctive Syllogism to ${\sim}p$ and $p \vee q$, but validate it with respect to ${\sim}r$ and $r \vee s$. This is why inconsistency-adaptive logics lead (in general) to a much richer consequence set than the paraconsistent logics mentioned above. Note also that, because of this contextual validation of inference rules, adaptive logics tend to be *non-monotonic.* For instance, in the above example, s is **ACLuN1**-derivable from Γ, but not from $\Gamma \cup \{r\}$.

Formally, an adaptive logic is defined in terms of three elements: a 'lower limit logic' (some monotonic logic), a set of 'abnormalities' (a set of formulas characterized by a possibly restricted logical form), and an 'adaptive strategy'. Two important restrictions on the first two elements are (i) that any abnormality should be verified in some model of the lower limit logic, and (ii) that extending the lower limit logic with the requirement that none of the abnormalities is logically possible should result in a monotonic logic (called the 'upper limit logic'). In view of these restrictions, the upper limit logic incorporates a presupposition that is absent in the lower limit logic, namely that all abnormalities are false. This is the presupposition that is defeasible in the *adaptive* logic: it is maintained 'as much as possible', but abandoned when necessary to avoid triviality. Thus, in an adaptive logic, abnormalities are supposed to be false, unless and until proven otherwise.[6]

An example may help to clarify the matter. Where $\exists A$ abbreviates A preceded by a sequence of existential quantifiers (in some preferred order) over the variables that occur free in A, the set of abnormalities of the inconsistency-adaptive logics **ACLuN1** and **ACLuN2** consists of all formulas of the form $\exists(A \wedge {\sim}A)$. The lower limit logic of **ACLuN1** and **ACLuN2** is the paraconsistent system **CLuN** (full positive **CL** plus the axiom $A \vee {\sim}A$). As ${\sim}(A \wedge {\sim}A)$ is not valid in **CLuN**, any formula of the form $\exists(A \wedge {\sim}A)$ is true in some **CLuN**-model. Extending **CLuN** with the requirement that no formula of the form $\exists(A \wedge {\sim}A)$ is true (for instance, by extending it with the axiom ${\sim}(A \wedge {\sim}A)$) results in **CL**, which is the upper limit logic of **ACLuN1** and **ACLuN2**. The

[6]Note that the term "abnormality" does not refer to the purported standard of reasoning, say **CL**. It refers to properties of the application context—to presuppositions that are considered desirable, but that may be overruled.

only difference between **ACLuN1** and **ACLuN2** concerns their adaptive strategy (see below).

In all currently available adaptive logics, the abnormalities are delineated by a certain logical form. For some adaptive logics, however, the set of abnormalities does not comprise all formulas of the form at issue, but only those that satisfy some restriction. For instance, in some inconsistency-adaptive logics, a formula of the form $\exists(A \wedge \sim A)$ only counts as an abnormality if A is a sentential letter. A similar restriction will be introduced for the adaptive logic presented below.

From what is said in the previous paragraphs, it may seem that abnormalities are assumed to be false, unless they are derivable (by the lower limit logic) from the set of premises. Although this holds true for some adaptive logics, the situation is usually a bit more complicated. This is related to the fact that, for most lower limit logics, a set of premises may entail a disjunction of abnormalities, without entailing any of its disjuncts. For instance, most inconsistency-adaptive logics are based on a paraconsistent logic according to which $(p \wedge \sim p) \vee (q \wedge \sim q)$ is entailed by $\{p \vee q, \sim p, \sim q\}$, without $p \wedge \sim p$ and $q \wedge \sim q$ being entailed by it.

In line with the conventions from [8], disjunctions of abnormalities will be called *Dab-formulas* and an expression of the form $Dab(\Delta)$ will refer to $\bigvee(\Delta)$, in which Δ is a (finite) subset of the set of abnormalities. The *Dab*-formulas that are derivable by the lower limit logic from the set of premises Γ are called the *Dab*-consequences of Γ. $Dab(\Delta)$ is called a *minimal Dab-consequence* of Γ if and only if there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is a *Dab*-consequence of Γ. If $Dab(\Delta)$ is a minimal *Dab*-consequence of Γ, it can be inferred from Γ that some member of Δ behaves abnormally, but it cannot be inferred which one. Hence, except for the case where Δ is a singleton for every minimal *Dab*-consequence of Γ, there are different ways to interpret abnormal theories 'as normally as possible'.

It is in view of this fact that an *adaptive strategy* is needed. Intuitively, the adaptive strategy specifies what it means that the abnormalities are supposed to be false *unless and until proven otherwise*. The two basic strategies are the *Reliability* strategy and the *Minimal Abnormality* strategy. The difference between the two is most easily explained after discussing the semantics.

Given some set of premises Γ, the semantics of an adaptive logic is obtained by selecting a subset of the models of the lower limit logic that verify Γ. Intuitively, those models are selected that, in view of the adaptive strategy, are as normal as possible. Note especially that the intended selection can only be defined by referring to some set of premises. Hence, it does not make sense to say that some model of the lower limit logic is an adaptive model. It only makes sense to say that it is an adaptive model of the set of premises at issue.

In order to see how the semantic selection is realized for the different adaptive strategies, we need some further definitions. Where M is a model of the lower limit logic, the *abnormal part* of M is denoted by $Ab(M)$ and consists of all the abnormalities that are verified by M. Where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal *Dab*-consequences of Γ, the set $U(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$ stands for the set of formulas that are *unreliable* with respect to Γ.

Whether or not some model of the lower limit logic is an adaptive model of Γ depends on its abnormal part and on the adaptive strategy. According to the Reliability strategy, for instance, a model M of the lower limit logic is an adaptive model of Γ iff $M \vDash \Gamma$ and $Ab(M) \subseteq U(\Gamma)$. So, on this strategy an abnormality is assumed to be false unless it is unreliable. The Minimal Abnormality strategy is a bit less cautious: it selects those models that verify (in a set-theoretical sense) a minimal number of abnormalities. Put in somewhat more precise terms, a model M of the lower limit logic is selected by the Minimal Abnormality strategy iff $M \vDash \Gamma$ and there is no model M' of the lower limit logic such that $M' \vDash \Gamma$ and $Ab(M') \subset Ab(M)$.

The proof theory of an adaptive logic is obtained by formulating a set of unconditional rules and a set of conditional rules. The former comprises the rules that are validated by the lower limit logic, the latter those that are validated by the upper limit logic, but not by the lower limit logic. In addition to these rules, the proof theory contains a 'marking definition' (see below).

The proof theory of adaptive logics is *dynamical* in a strong sense: conclusions accepted at some stage of the proof may at a later stage be rejected. The mechanism by which this is realized is quite simple. If a formula is added by the application of a conditional rule, a 'condition' (a set of abnormalities) is written to the right of the line. If a formula is added by the application of an unconditional rule, no condition is introduced, but the conditions (if any) that affect the premises of the application are conjoined for its conclusion. The idea is that the condition specifies the abnormalities that are, in that particular inference, *assumed to be false.* At each stage of the proof—with each formula added—the marking definition is applied: for each line that has a (non-empty) condition attached to it, it is checked whether the underlying assumption can be maintained or not. If it cannot, the line at issue is marked and the formula that occurs on it is no longer considered to be derived.

3 Intuitive Characterization of $\mathbf{AJ^r}$

As $\mathbf{AJ^r}$ is meant to function as the basis for a discussive logic, it will only be defined for sets of premises of the form $\Gamma^\Diamond$, where Γ is a set of non-modal wffs. Intuitively, and in line with the intended application context, each member of Γ can be taken to correspond to the conjunction of statements made by some participant in a discussion.

The idea behind $\mathbf{AJ^r}$ is to presuppose that a statement made by some participant in a debate is accepted by all of them—that $\Box A$ (everybody accepts A) is derivable from $\Diamond A$ (somebody accepts A) or, what comes to the same, that $\Diamond A \wedge \Diamond{\sim}A$ (somebody accepts A and somebody accepts ${\sim}A$) is *false*—unless and until proven otherwise. Let us call this the *unanimity presupposition.*

Where this presupposition is violated, $\mathbf{AJ^r}$ should behave like **S5**. This is motivated by the fact that, even if $\Diamond A \wedge \Diamond{\sim}A$ holds true for some statement A, it should still be possible to derive all **CL**-consequences from A and ${\sim}A$ separately. So, **S5** is chosen as the lower limit logic of $\mathbf{AJ^r}$.

The unanimity presupposition also enables us to define the set of abnormalities. At the propositional level, it consists of all formulas of the form $\Diamond A \wedge \Diamond{\sim}A$ in which A is a sentential letter—note that $\Diamond A \wedge \Diamond{\sim}A$ is **S5**-equivalent to ${\sim}(\Diamond A \supset \Box A)$. The reason why a formula of the form $\Diamond A \wedge \Diamond{\sim}A$ only counts as an abnormality if A is primitive will be explained in the next section.

Extending **S5** with the requirement that all abnormalities are false—for instance, by extending it with the axiom ${\sim}(\Diamond A \wedge \Diamond{\sim}A)$—gives us the logic **Triv** (see [16, p. 65]). This is the upper limit logic of $\mathbf{AJ^r}$.[7]

As was mentioned in the previous section, a set of premises may entail a disjunction of abnormalities without entailing any of its disjuncts. This is also the case here. Consider, for instance, $\Gamma^{\Diamond} = \{\Diamond(p \vee q), \Diamond{\sim}p, \Diamond{\sim}q\}$: neither $\Diamond p \wedge \Diamond{\sim}p$ nor $\Diamond q \wedge \Diamond{\sim}q$ is **S5**-derivable from $\Gamma^{\Diamond}$, but $(\Diamond p \wedge \Diamond{\sim}p) \vee (\Diamond q \wedge \Diamond{\sim}q)$ is.

This brings us to the choice of the adaptive strategy. As I mentioned above, the idea behind $\mathbf{AJ^r}$ is to assume that the statements made in a debate are shared *as much as possible*. Central in the decision on the adaptive strategy is how this "as much as possible" should be interpreted in the case of disjunctions of abnormalities. Suppose, for instance, that there are three participants in a debate (A, B, and C), and that they make the following statements:

A: $p \wedge r$

B: $q \wedge s$

C: ${\sim}p \vee {\sim}q$

Where Γ is the set of these three statements, $\Gamma^{\Diamond}$ entails $(\Diamond p \wedge \Diamond{\sim}p) \vee (\Diamond q \wedge \Diamond{\sim}q)$. So, we are able to infer that there is a disagreement with respect to p *or* a disagreement with respect to q. Now, what does it mean that the different statements are shared 'as much as possible'?

One possible interpretation would be that the differences in opinion are *minimized*—that the participants disagree with respect to p or with respect to q, *but not with respect to both.* This is the interpretation offered by the Minimal Abnormality Strategy. On this strategy, it is assumed that either all three participants accept p or all of them accept q. This seems intuitively unjustified. A realistic interpretation of the discussion requires that we also take into account situations in which, for instance, all three participants accept the statement made by C. In such a situation, A would accept both p and ${\sim}q$ and B would accept both q and ${\sim}p$. Hence, there would be a disagreement with respect to both p and q. This is what the Reliability Strategy allows for. On this strategy, it is assumed that the participants maximally agree with one another with respect to reliable statements, but not necessarily with respect to unreliable ones.

This is the strategy $\mathbf{AJ^r}$ will be based on. As mentioned above, the Reliability Strategy leads to a slightly poorer consequence set than the Minimal Abnormality Strategy. For instance, where $\Gamma^{\Diamond} = \{\Diamond p, \Diamond q, \Diamond({\sim}p \vee {\sim}q), \Diamond({\sim}p \vee r), \Diamond({\sim}q \vee r)\}$, $\Diamond r$ follows from $\Gamma^{\Diamond}$ according to the Minimal Abnormality Strategy, but not according to the Reliability Strategy. This is related to the fact that, also in this case, the Minimal Abnormality Strategy assumes that either all participants accept p or all of them accept q.

4 Semantics of $\mathbf{AJ^r}$

In this Section and the next one, I present the propositional fragment of $\mathbf{AJ^r}$. The extension to the predicative level will be discussed in Section 6.

[7] **Triv** is an analogue of **CL**: its language is modal, but A, $\Diamond A$ and $\Box A$ are logically equivalent.

Let $\mathcal{L}_p$ be the standard propositional language and $\mathcal{L}_p^\Diamond$ the propositional fragment of the standard modal language. The sets $\mathcal{W}_p$ and $\mathcal{W}_p^\Diamond$ will refer, respectively, to the wffs of $\mathcal{L}_p$ and of $\mathcal{L}_p^\Diamond$. In this section and the next one, an expression of the form $\Gamma^\Diamond$ will always refer to the set $\{\Diamond A \mid A \in \Gamma\}$ in which Γ is a subset of $\mathcal{W}_p$.

As explained above, the lower limit logic of $\mathbf{AJ^r}$ is **S5** and its upper limit logic is **Triv**. For the propositional case, one may consider any of the usual formulations of **S5** and of **Triv** (for instance, the ones from [16]).

Semantically, $\mathbf{AJ^r}$ is obtained from the **S5**-models of $\Gamma^\Diamond$ by the Reliability Strategy. The idea is that any $\Gamma^\Diamond$ defines a set of unreliable formulas and that those models are selected in which an abnormality is verified only if it is unreliable. If the set of unreliable formulas defined by $\Gamma^\Diamond$ is empty, the $\mathbf{AJ^r}$-models of $\Gamma^\Diamond$ are its **Triv**-models.

At the propositional level, the set of abnormalities Ω consists of all formulas of the form $\Diamond A \wedge \Diamond{\sim}A$ in which A is a sentential letter. Let $Dab(\Delta)$ refer to $\bigvee(\Delta)$, in which Δ is a (finite) subset of Ω and let Dab-formulas and (minimal) Dab-consequences be defined as in Section 2. For the sake of generality, $A \vee Dab(\emptyset)$ will denote A.

I first define the *abnormal part* of an **S5**-model of $\Gamma^\Diamond$:

Definition 1 *For any* **S5**-*model* $\mathcal{M}$ *of* $\Gamma^\Diamond$, $Ab(\mathcal{M}) = \{A \mid A \in \Omega; v_{\mathcal{M}}(A) = 1\}$.

The set of formulas that are *unreliable* with respect to $\Gamma^\Diamond$ is defined as:

Definition 2 $U(\Gamma^\Diamond) = \bigcup\{\Delta \mid Dab(\Delta)$ *is a minimal Dab-consequence of* $\Gamma^\Diamond\}$.

Given these definitions, the semantic selection can be defined. An **S5**-model of $\Gamma^\Diamond$ is *reliable* iff all abnormalities verified by it are unreliable:

Definition 3 *An* **S5**-*model* $\mathcal{M}$ *of* $\Gamma^\Diamond$ *is* reliable *iff* $Ab(\mathcal{M}) \subseteq U(\Gamma^\Diamond)$.

The $\mathbf{AJ^r}$-models of $\Gamma^\Diamond$ are the reliable models of $\Gamma^\Diamond$ and the semantic consequence relation is defined with respect to these:

Definition 4 *For any* $A \in \mathcal{W}_p^\Diamond$, $\Gamma^\Diamond \vDash_{\mathbf{AJ^r}} A$ *iff all reliable models of* $\Gamma^\Diamond$ *verify* A.

To see what the $\mathbf{AJ^r}$-semantics comes to, consider again the example from the previous section where $\Gamma^\Diamond = \{\Diamond(p \wedge r), \Diamond(q \wedge s), \Diamond({\sim}p \vee {\sim}q)\}$. The only minimal Dab-consequence that follows from $\Gamma^\Diamond$ is $(\Diamond p \wedge \Diamond{\sim}p) \vee (\Diamond q \wedge \Diamond{\sim}q)$. Hence, in this case, $U(\Gamma^\Diamond) = \{\Diamond p \wedge \Diamond{\sim}p, \Diamond q \wedge \Diamond{\sim}q\}$. In view of Definition 3 and the fact that all **S5**-models of $\Gamma^\Diamond$ verify $(\Diamond p \wedge \Diamond{\sim}p) \vee (\Diamond q \wedge \Diamond{\sim}q)$, some reliable models of $\Gamma^\Diamond$ verify $\Diamond p \wedge \Diamond{\sim}p$ (and hence, falsify $\Box p$), others verify $\Diamond q \wedge \Diamond{\sim}q$ (and hence, falsify $\Box q$). As a consequence, neither $\Box p$ nor $\Box q$ is $\mathbf{AJ^r}$-derivable from $\Gamma^\Diamond$ (in view of Definition 4).

Consider now the abnormality $\Diamond r \wedge \Diamond{\sim}r$. As is easily observed, some **S5**-models of $\Gamma^\Diamond$ verify $\Diamond r \wedge \Diamond{\sim}r$. However, as $\Diamond r \wedge \Diamond{\sim}r \notin U(\Gamma^\Diamond)$, all *reliable* models of $\Gamma^\Diamond$ falsify $\Diamond r \wedge \Diamond{\sim}r$, and hence, verify $\Box r$ (in view of the fact that all **S5**-models of $\Gamma^\Diamond$ verify $\Diamond r$). For analogous reasons, all reliable models of $\Gamma^\Diamond$ falsify $\Diamond s \wedge \Diamond{\sim}s$, and hence verify $\Box s$. But then, all reliable models verify $\Box(r \wedge s)$. Hence, in view of Definition 4, $\Box r$, $\Box s$, and $\Box(r \wedge s)$ are $\mathbf{AJ^r}$-consequences of $\Gamma^\Diamond$.

All this is exactly as it should be. From $\Gamma^{\Diamond}$, it follows that there is a disagreement with respect to p or with respect to q. Hence, it should neither be possible to derive that everybody accepts p nor that everybody accepts q. However, as there is no explicit disagreement with respect to either r or s, it should be possible to derive that the statements r and s are accepted by all participants, and hence, that also $r \wedge s$ is accepted by all of them.

Remark that if the set of abnormalities would comprise *all* formulas of the form $\Diamond A \wedge \Diamond{\sim}A$, the intended selection would not be possible. This is illustrated by the following example. Consider $\Gamma^{\Diamond} = \{\Diamond p, \Diamond{\sim}p, \Diamond q\}$. From an intuitive point of view, there is a disagreement with respect to p but not with respect to q. Hence, we want all selected models to falsify $\Diamond q \wedge \Diamond{\sim}q$, and accordingly, to verify $\Box q$. However, if all formulas of the form $\Diamond A \wedge \Diamond{\sim}A$ would count as abnormalities, one of the minimal disjunctions of abnormalities would be $(\Diamond({\sim}p \wedge q) \wedge \Diamond{\sim}({\sim}p \wedge q)) \vee (\Diamond q \wedge \Diamond{\sim}q)$. Hence, $\Diamond q \wedge \Diamond{\sim}q$ would be unreliable, and **S5**-models of $\Gamma^{\Diamond}$ that verify $\Diamond q \wedge \Diamond{\sim}q$ would be reliable. As a consequence, $\Box q$ would be *falsified* in some reliable models of $\Gamma^{\Diamond}$ and $\Box q$ would not be an $\mathbf{AJ^r}$-consequence of $\Gamma^{\Diamond}$.

5 Dynamic Proof Theory of $\mathbf{AJ^r}$

In view of the results on other adaptive logics, the design of the proof theory for $\mathbf{AJ^r}$ is straightforward. As is usual for adaptive logics, lines of a proof have five elements: (i) a line number, (ii) the formula A that is derived, (iii) the line numbers of the formulas from which A is derived, (iv) the rule by which A is derived, and (v) a condition . The condition contains those abnormalities that, in the inference at issue, are supposed to be false.

Following the generic proof format for adaptive logics from [8], I introduce a premise rule PREM, an unconditional rule RU and a conditional rule RC. As is usual, both RU and RC refer to derivability by the lower limit logic—in this case **S5**. In addition to the inference rules, there is a marking definition (that specifies which lines in the proof have to be marked).

The premise rule requires little explanation. Note only that, as it should be, premises are introduced on the empty condition:

PREM At any stage of a proof, and for any $A \in \Gamma^{\Diamond}$, one may add to the proof a line consisting of

(i) an appropriate line number,
(ii) A,
(iii) a dash,
(iv) "PREM", and
(v) "$\emptyset$".

The unconditional rule RU allows one to add all **S5**-consequences of formulas that already occur in the proof. As it name implies, the rule RU does not lead to the introduction of any new condition. However, if the formulas to which RU is applied occur themselves on a non-empty condition, then these conditions are conjoined for the conclusion:

RU For any $B \in \mathcal{W}_p^{\Diamond}$, if $A_1, \ldots, A_n \vdash_{\mathbf{S5}} B$, and $A_1, \ldots, A_n$ $(n \geq 0)$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of

(i) an appropriate line number,
(ii) B,
(iii) the line numbers of the A_i,
(iv) "RU", and
(v) $\Delta_1 \cup \ldots \cup \Delta_n$.

The conditional rule RC requires a bit more explanation. As will be proven in the next section, there is a special relation between derivability by the lower limit logic of $\mathbf{AJ^r}$ and its upper limit logic, namely $A_1, \ldots, A_n \vdash_{\mathbf{Triv}} B$ iff, for some $\Delta \subset \Omega$, $A_1, \ldots A_n \vdash_{\mathbf{S5}} B \vee Dab(\Delta)$. Thus, it is warranted that, whenever B is **Triv**-derivable from $A_1, \ldots, A_n$, there is a (possibly empty) set of abnormalities Δ such that B is **S5**-derivable from $A_1, \ldots, A_n$ or one of the members of Δ holds true. The following examples illustrate this relation:

(1) $\Diamond p \vdash_{\mathbf{S5}} \Box p \vee (\Diamond p \wedge \Diamond{\sim}p)$
(2) $\Diamond(p \wedge q) \vdash_{\mathbf{S5}} \Box(p \wedge q) \vee ((\Diamond p \wedge \Diamond{\sim}p) \vee (\Diamond q \wedge \Diamond{\sim}q))$
(3) $\Diamond(p \vee q),\ \Diamond{\sim}q \vdash_{\mathbf{S5}} \Diamond p \vee (\Diamond q \wedge \Diamond{\sim}q)$
(4) $\Diamond(p \vee q),\ \Diamond{\sim}q \vdash_{\mathbf{S5}} \Box p \vee ((\Diamond p \wedge \Diamond{\sim}p) \vee (\Diamond q \wedge \Diamond{\sim}q))$

The rule RC is based on this relation. Whenever $B \vee Dab(\Delta)$ is **S5**-derivable from some $A_1, \ldots, A_n$ that occur in the proof, RC allows one to derive B *on the assumption* that all members of Δ are false. Technically, this is realized by introducing Δ as a new condition. Evidently, if some of the A_i occur themselves on a non-empty condition, then these conditions are conjoined to Δ:

RC For any $B \in \mathcal{W}_p^{\Diamond}$, if $A_1, \ldots, A_n \vdash_{\mathbf{S5}} B \vee Dab(\Delta)$, and $A_1, \ldots, A_n$ ($n \geq 0$) occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of

(i) an appropriate line number,
(ii) B,
(iii) the line numbers of the A_i,
(iv) "RC", and
(v) $\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n$.

The following conditional rule is (obviously) derivable from RU and RC and leads to proofs that are more interesting from a heuristic point of view; $\omega(A)$ stands for the set $\{\Diamond B \wedge \Diamond{\sim}B \in \Omega \mid B \text{ occurs in } A\}$:

RD For any $A \in \mathcal{W}_p^{\Diamond}$, if $\Diamond A$ occurs in the proof on the condition Δ, then one may add to the proof a line consisting of

(i) an appropriate line number,
(ii) $\Box A$,
(iii) the line number of $\Diamond A$,
(iv) "RD", and
(v) $\omega(A) \cup \Delta$.

Let us now turn to the marking definition. Given a dynamic proof, $Dab(\Delta)$ will be said to be a minimal Dab-formula *at stage s* of a proof iff, at that stage, $Dab(\Delta)$ occurs in the proof on the empty condition and, for any $\Delta' \subset \Delta$, $Dab(\Delta')$ does not occur in the proof on the empty condition. The set $U_s(\Gamma^\Diamond)$ comprises the unreliable formulas at stage s:

Definition 5 $U_s(\Gamma^\Diamond) = \bigcup\{\Delta \mid Dab(\Delta)$ *is a minimal Dab-formula at stage* $s\}$.

The marking of lines is governed by the following definition:

Definition 6 *Line* i *is* marked *at stage* s *iff, where* Δ *is its condition,* $\Delta \cap U_s(\Gamma^\Diamond) \neq \emptyset$.

If, at stage s of a proof from $\Gamma^\Diamond$, a formula A occurs on a line that is not marked, then A is said to be derived from $\Gamma^\Diamond$ *at that stage of the proof.* Whenever a line is added to the proof, lines that were previously marked may be unmarked and vice versa. However, the notion of *final derivability* is independent of the way in which the proof proceeds:

Definition 7 A *is* finally derived *in a proof from* $\Gamma^\Diamond$ *iff* A *is derived on a line that is not marked, and any extension of the proof in which* A *is marked may be further extended in such a way that* A *is unmarked.*

As may be expected, the derivability relation is defined with respect to final derivability:

Definition 8 *For any* $A \in \mathcal{W}_p^\Diamond$, $\Gamma^\Diamond \vdash_{\mathbf{AJ^r}} A$ *(*A *is finally derivable from* $\Gamma^\Diamond$*) iff* A *is finally derived in an* **AJ^r**-*proof from* $\Gamma^\Diamond$.

To end this section, I present a simple example of an **AJ^r**-proof from $\Gamma^\Diamond = \{\Diamond(p\wedge r), \Diamond(q\wedge(\sim r\vee s)), \Diamond(\sim p\vee\sim q)\}$. Let us begin by introducing all premises:

1	$\Diamond(p \wedge r)$	–	PREM	$\emptyset$
2	$\Diamond(q \wedge (\sim r \vee s))$	–	PREM	$\emptyset$
3	$\Diamond(\sim p \vee \sim q)$	–	PREM	$\emptyset$

From these, we can derive each of the following by means of RU:

4	$\Diamond p$	1	RU	$\emptyset$
5	$\Diamond r$	1	RU	$\emptyset$
6	$\Diamond q$	2	RU	$\emptyset$
7	$\Diamond(\sim r \vee s)$	2	RU	$\emptyset$

From 4, we can derive, by means of RD, that everybody accepts p, on the condition that "somebody accepts p and somebody accepts $\sim p$" is false:

8	$\Box p$	4	RD	$\{\Diamond p \wedge \Diamond\sim p\}$

By an analogous reasoning, we can also add each of the following:

9	$\Box r$	5	RD	$\{\Diamond r \wedge \Diamond\sim r\}$
10	$\Box q$	6	RD	$\{\Diamond q \wedge \Diamond\sim q\}$
11	$\Box(\sim r \vee s)$	7	RD	$\{\Diamond r \wedge \Diamond\sim r,\ \Diamond s \wedge \Diamond\sim s\}$

Suppose, however, that, at this stage of the proof, we realize that the following Dab-formula is derivable from 3, 4 and 6:

12 $(\lozenge p \wedge \lozenge{\sim}p) \vee (\lozenge q \wedge \lozenge{\sim}q)$	3, 4, 6	RU	$\emptyset$

As soon as the formula on line 12 is added to the proof, the conditions of lines 8 and 10 are no longer satisfied. Hence, the marking definition forces us to *mark* these lines. This is how the proof looks like at stage 12 (lines 1 to 7 are as before):

...			
8 $\Box p$	4	RD	$\{\lozenge p \wedge \lozenge{\sim}p\}$ $\checkmark^{12}$
9 $\Box r$	5	RD	$\{\lozenge r \wedge \lozenge{\sim}r\}$
10 $\Box q$	6	RD	$\{\lozenge q \wedge \lozenge{\sim}q\}$ $\checkmark^{12}$
11 $\Box({\sim}r \vee s)$	7	RD	$\{\lozenge r \wedge \lozenge{\sim}r,\ \lozenge s \wedge \lozenge{\sim}s\}$
12 $(\lozenge p \wedge \lozenge{\sim}p) \vee (\lozenge q \wedge \lozenge{\sim}q)$	3, 4, 6	RU	$\emptyset$

At this stage of the proof, the formulas that occur on lines 8 and 10 are no longer considered to be derived in the proof. Note, however, that the formulas on lines 9 and 11 are still derivable, despite the fact that, at the level of the premises, r as well as ${\sim}r \vee s$ are conjoined to statements about which there is a disagreement. This is as it should be: it is not because one of the statements of a participant turns out to be problematic, that *all* statements made by this participant should be considered as problematic. (Remember that each of the premises stands for a *conjunction* of statements made by some participant in the debate.)

For this simple propositional example, it is easily observed that the formulas on lines 9 and 11 are finally derivable from $\Gamma^{\lozenge}$. So are the following formulas:

13 $\Box s$	9, 11	RU	$\{\lozenge r \wedge \lozenge{\sim}r, \lozenge s \wedge \lozenge{\sim}s\}$
14 $\lozenge(p \wedge s)$	4, 13	RU	$\{\lozenge r \wedge \lozenge{\sim}r, \lozenge s \wedge \lozenge{\sim}s\}$

As the formula on line 14 indicates, statements about which there is a disagreement can be conjoined to statements that are accepted by all participants. Also this is as it should be: if it is derivable that all participants accept s and that somebody accepts p, it should also be derivable that somebody accepts $p \wedge s$.

6 The Predicative Version of $\mathbf{AJ^r}$

In order to generalize $\mathbf{AJ^r}$ to the predicative level, I first need to present the predicative version of its lower limit logic. In view of the metatheory, I shall rely on a somewhat unusual version of **S5** that was first presented in [9]. What is peculiar about it is that the **S5**-models are characterized in terms of **CL**-models.

Let $\mathcal{L}$ be the standard language of **CL**, and let $\mathcal{C}$, $\mathcal{V}$, $\mathcal{F}$, $\mathcal{F}^p$ and $\mathcal{W}$ be the sets of individual constants, individual variables, formulas, primitive formulas and wffs (closed formulas). A formula is considered as primitive if it is a sentential letter or a primitive predicative formula. I shall use small Greek letters for metavariables of individual variables and individual constants, small letters from the beginning of the Latin alphabet for individual constants and small letters from the end of the Latin alphabet for individual variables. From now on, $\Gamma^{\lozenge}$ will refer to the set $\{\lozenge A \mid A \in \Gamma\}$ in which Γ is a subset of $\mathcal{W}$.

Let $M = \langle D, v\rangle$ be a standard **CL**-model, with D the domain and v the assignment function. To simplify matters, quantifiers are handled in terms of the pseudo-language $\mathcal{L}^+$. This is obtained by adding, to the set of constants $\mathcal{C}$, a set of pseudo-constants $\mathcal{O}$ that has at least the cardinality of the largest model one wants to consider and by requiring that the assignment function $v : \mathcal{C} \cup \mathcal{O} \longrightarrow D$ is surjective. $\mathcal{W}^+$ will refer to the set of wffs of $\mathcal{L}^+$.

Where $\mathcal{L}^\Diamond$ is the standard modal language, $\mathcal{L}^{\Diamond +}$ is obtained from the latter by introducing the set of pseudo-constants $\mathcal{O}$ next to the set of constants $\mathcal{C}$. Let $\mathcal{W}^\Diamond$ and $\mathcal{W}^{\Diamond +}$ refer to the wffs of $\mathcal{L}^\Diamond$ and $\mathcal{L}^{\Diamond +}$.

An **S5**-model is a couple $\mathcal{M} = \langle \Sigma_\Delta, M_0 \rangle$, where Δ is a set of wffs of $\mathcal{L}$, Σ_Δ is the set of *all* **CL**-models of Δ, and $M_0 \in \Sigma_\Delta$.

Let $v_M : \mathcal{W}^+ \longrightarrow \{0, 1\}$ be the valuation function determined by the **CL**-model M. The valuation function $v_\mathcal{M} : \mathcal{W}^{\Diamond +} \times \Sigma_\Delta \longrightarrow \{0, 1\}$ determined by the model $\mathcal{M} = \langle \Sigma_\Delta, M_0 \rangle$ is defined by:

C1 where $A \in \mathcal{F}^p$, $v_\mathcal{M}(A, M_i) = v_{M_i}(A)$
C2 $v_\mathcal{M}(\sim A, M_i) = 1$ iff $v_\mathcal{M}(A, M_i) = 0$
C3 $v_\mathcal{M}(A \wedge B, M_i) = 1$ iff $v_\mathcal{M}(A, M_i) = v_\mathcal{M}(B, M_i) = 1$
C4 $v_\mathcal{M}((\forall \alpha) A(\alpha), M_i) = 1$ iff $v_\mathcal{M}(A(\beta), M_i) = 1$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$
C5 $v_\mathcal{M}(\Box A, M_i) = 1$ iff $v_\mathcal{M}(A, M_j) = 1$ for all $M_j \in \Sigma_\Delta$.

The other logical constants are defined as usual. A model $\mathcal{M}$ verifies $A \in \mathcal{W}^\Diamond$ iff $v_\mathcal{M}(A, M_0) = 1$. A is valid iff it is verified by all models. From now on, the term "**S5**-model" will always refer to a model as defined here.

The following two theorems illustrate some interesting properties of the **S5**-semantics.

Theorem 1 *Where A is a wff of $\mathcal{L}^+$, $v_\mathcal{M}(A, M_i) = v_{M_i}(A)$.*

Proof. By an obvious induction on the complexity of A, where the complexity of a formula that contains defined logical symbols is identified with the complexity of the formula it abbreviates. The basis of the induction is provided by C1; the cases of the induction step are obvious in view of C2–C4. ■

Let the *fully modal formulas* of $\mathcal{L}^{\Diamond +}$ be the formulas of the form $\Box B$ and $\Diamond B$ as well as all formulas obtained from these by the formation rules of $\mathcal{L}^{\Diamond +}$, and let the *fully modal wffs* of $\mathcal{L}^{\Diamond +}$ be the closed fully modal formulas of $\mathcal{L}^{\Diamond +}$.

Theorem 2 *If $\mathcal{M} = \langle \Sigma_\Delta, M_0 \rangle$ is an* **S5***-model, and $M_1, M_2 \in \Sigma_\Delta$, then, for all fully modal wffs A, $v_\mathcal{M}(A, M_1) = v_\mathcal{M}(A, M_2)$.*

Proof. By an obvious induction on the complexity of A. The complexity of formulas that contain defined logical symbols is handled as in the proof for Theorem 1. The basis of the induction is provided by C5; the cases of the induction step are obvious in view of C2–C4. ■

The above semantics is rather peculiar. Each of the **S5**-models corresponds to a standard worlds-model, but not *vice versa.* Nevertheless, it is proven in [9] that the semantics is adequate with respect to a predicative version of **S5** (with the Barcan Formula, but without necessity of identity). Here is an axiomatization (the α and β should be interpreted in such a way that all main formulas are wffs):

A1	$A \supset (B \supset A)$
A2	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3	$((A \supset B) \supset A) \supset A$
A4	$\sim\sim A \supset A$
A5	$(A \supset B) \supset ((A \supset \sim B) \supset \sim A)$
A6	$(\forall\alpha)A(\alpha) \supset A(\beta)$
A7	$\alpha = \alpha$
A8	$\alpha = \beta \supset (A \supset B)$ where B is obtained by replacing in A an occurrence of α that occurs outside the scope of a modality by β
A9	$\Box A \supset A$
A10	$\Box(A \supset B) \supset (\Box A \supset \Box B)$
A11	$\Diamond A \supset \Box\Diamond A$
A12	$(\forall\alpha)\Box A(\alpha) \supset \Box(\forall\alpha)A(\alpha)$

MP	From A and $A \supset B$ to derive B
NEC	From $\vdash A$ to derive $\vdash \Box A$
R$\forall$	From $\vdash A \supset B(\beta)$, to derive $\vdash A \supset (\forall\alpha)B(\alpha)$, provided β does not occur in either A or $B(\alpha)$

D$\vee$	$A \vee B = \sim A \supset B$
D$\wedge$	$A \wedge B = \sim(\sim A \vee \sim B)$
D$\equiv$	$A \equiv B = (A \supset B) \wedge (B \supset A)$
D$\exists$	$(\exists\alpha)A = \sim(\forall\alpha)\sim A$
D$\Diamond$	$\Diamond A = \sim\Box\sim A$

Definition 9 *An* **S5***-proof is a list of wffs of $\mathcal{L}^\Diamond$ in which each member is an axiom, or a premise, or obtained by an application of a definition or of MP from previous members, or obtained by an application of NEC or R$\forall$ from previous members that do not have any premise in their paths.*

Definition 10 $\Gamma \vdash_{\mathbf{S5}} A$ *iff there is an* **S5***-proof of A in which only members of Γ occur as premises.*

The proof of the following theorem is standard:

Theorem 3 *If $A_1, \ldots, A_n \vdash_{\mathbf{S5}} A$, then $A_1, \ldots, A_{n-1} \vdash_{\mathbf{S5}} A_n \supset A$. (Deduction Theorem)*

For the Soundness and Completeness proof, I refer to [9]:

Theorem 4 $\Gamma \vDash_{\mathbf{S5}} A$ *iff* $\Gamma \vdash_{\mathbf{S5}} A$.

Note that the semantics for the predicative version of **Triv** is easily obtained from that for **S5**. The **Triv**-models are the **S5**-models $\mathcal{M} = \langle \Sigma_\Delta, M_0 \rangle$, such that, for some maximal consistent subset $\Theta \subset \mathcal{W}$, $\Sigma_\Delta = \Sigma_\Theta$. An axiomatization for **Triv** is obtained by adding the axiom $\sim(\Diamond A \wedge \Diamond\sim A)$ to the axiomatization for **S5**.

Given the predicative version of the lower limit logic of $\mathbf{AJ^r}$, the generalization to the predicative level is absolutely straightforward. The only further change concerns the set of abnormalities. This is most easily illustrated by an example. Consider, for instance, $\Gamma^\Diamond = \{\Diamond(\forall x)(Px \vee Qx), \Diamond(\exists x)(\sim Px \wedge \sim Qx)\}$. From this, no quantifier-free *Dab*-formula is **S5**-derivable, but $(\exists x)(\Diamond Px \wedge$

$\Diamond{\sim}Px) \vee (\exists x)(\Diamond Qx \wedge \Diamond{\sim}Qx)$ is. This gives us the desired generalization. Where $\exists A$ is defined as before, the set of abnormalities Ω consists of all formulas of the form $\exists(\Diamond A \wedge \Diamond{\sim}A)$, in which $A \in \mathcal{F}^p$. From now on, Ω will refer to this set.

The semantics and the proof theory are obtained in exactly the same way as for the propositional case, except that all references to "$\Gamma^{\Diamond}$", "Ω", "$Dab(\Delta)$", "**S5**" and "$\mathbf{AJ^r}$" should be interpreted at the predicative level, and that all occurrences of $\mathcal{W}_p^{\Diamond}$ should be replaced by $\mathcal{W}^{\Diamond}$.

7 Some Metatheory

In this Section, I show some basic properties of the semantics and the proof theory of $\mathbf{AJ^r}$. All proofs are for the predicative version. A first series of properties concerns the adequacy of the semantic selection.

If some Γ does not have **S5**-models, its set of consequences is trivial. Hence, it is important to check whether any Γ that has **S5**-models also has $\mathbf{AJ^r}$-models. This property is called Reassurance and was first discussed in [24]. Where **AL** is an adaptive logic and **L** is its lower limit logic, Reassurance holds in **AL** iff there are **AL**-models for any set of premises that has **L**-models. In [4], Batens discusses the related property Strong Reassurance: for every **L**-model M, there is an **AL**-model M' such that $Ab(M') \subseteq Ab(M)$.[8] I now show that both properties hold for $\mathbf{AJ^r}$.

We first need some definitions. Let $\Phi^{\circ}(\Gamma^{\Diamond})$ be the set of all sets that contain one disjunct out of each minimal Dab-consequence of $\Gamma^{\Diamond}$. Let $\Phi^{\star}(\Gamma^{\Diamond})$ contain, for any $\phi \in \Phi^{\circ}(\Gamma^{\Diamond})$, the set $Cn_{\mathbf{S5}}(\phi) \cap \Omega$. Finally, let $\Phi(\Gamma^{\Diamond})$ contain those members of $\Phi^{\circ}(\Gamma)$ that are not proper supersets of other members of $\Phi^{\circ}(\Gamma^{\Diamond})$. It is easily observed that, for every $\phi \in \Phi(\Gamma^{\Diamond})$, if A is true in an **S5**-model $\mathcal{M}$, for every $A \in \phi$, then every minimal Dab-consequence of $\Gamma^{\Diamond}$ is true in $\mathcal{M}$.

Lemma 1 *If $\Gamma^{\Diamond}$ has* **S5***-models, then, for any $\phi \in \Phi(\Gamma^{\Diamond})$, $\Gamma^{\Diamond}$ has an* **S5***-model $\mathcal{M}$ such that $Ab(\mathcal{M}) = \phi$.*

Proof. Suppose that $\Gamma^{\Diamond}$ has **S5**-models. Consider a denumerable $\mathcal{O}' \subset \mathcal{O}$ and let $\mathcal{L}^{\Diamond'}$ be obtained from $\mathcal{L}^{\Diamond}$ by extending $\mathcal{C}$ to $\mathcal{C} \cup \mathcal{O}'$. Let $B_1, B_2, \ldots$ be a list of all wffs of $\mathcal{L}^{\Diamond'}$ such that, if $B_i = (\exists\alpha)C(\alpha)$ then $B_{i+1} = C(\beta)$ for some $\beta \in \mathcal{O}'$ that does not occur in $B_1, \ldots, B_i$. Select some $\phi \in \Phi(\Gamma^{\Diamond})$. In view of the definition of $\Phi(\Gamma^{\Diamond})$, it is easily observed that, for every $\Sigma \subset \Omega$, $\phi \vdash_{\mathbf{S5}} Dab(\Sigma)$ if $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} Dab(\Sigma)$. Define:

$$\Delta_0 = Cn_{\mathbf{S5}}(\Gamma^{\Diamond} \cup \phi)$$

$$\Delta_{i+1} = Cn_{\mathbf{S5}}(\Delta_i \cup \{B_{i+1}\})$$

if there is no Σ such that $Dab(\Sigma) \in Cn_{\mathbf{S5}}(\Delta_i \cup \{B_{i+1}\})$ and $\phi \nvdash_{\mathbf{S5}} Dab(\Sigma)$, and

$$\Delta_{i+1} = Cn_{\mathbf{S5}}(\Delta_i \cup \{{\sim}B_{i+1}\})$$

otherwise. Finally,

$$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \ldots$$

[8] As explained in [4], adaptive logics for which Strong Reassurance fails lead to counter-intuitive results.

Consider a function $h : \mathcal{C}\cup\mathcal{O} \longrightarrow \mathcal{C}\cup\mathcal{O}'$ such that $h(\alpha) = \alpha$ for all $\alpha \in \mathcal{C}\cup\mathcal{O}'$, and extend it to wffs by defining $h(A)$ as the result of replacing in A any $\alpha \in \mathcal{C} \cup \mathcal{O}$ by $h(\alpha)$. Finally, define $\Delta^\star = \{A \mid h(A) \in \Delta\}$. It is easily seen that $\Delta \subset \Delta^\star$ and that $\Delta^\star$ is closed under **S5**-derivability.

Define $\Theta = \{A \mid A \text{ is a wff of } \mathcal{L};\ \Box A \in \Delta^\star\}$, and $\Lambda = \{A \mid A \text{ is a wff of } \mathcal{L}^+;\ A \in \Delta^\star\}$. Let M_0 be the unique **CL**-model—see (ii) below—defined by Λ. Let $\mathcal{M} = \langle \Sigma_\Theta, M_0\rangle$.

Each of the following is easily proved:

(i) $\Theta \subseteq \Lambda$ (if $\Box C \in \Delta^\star$, then $C \in \Delta^\star$),

(ii) Λ defines a unique **CL**-model (for any wffs A and B of $\mathcal{L}^+$, $\sim A \in \Lambda$ iff $A \notin \Lambda$; $A \wedge B \in \Lambda$ iff $A, B \in \Lambda$; …; $(\exists\alpha)A(\alpha) \in \Lambda$ iff $A(\beta) \in \Lambda$ for some $\beta \in \mathcal{C} \cup \mathcal{O}$),

(iii) $\mathcal{M}$ is an **S5**-model of $\Delta^\star$ and hence of Δ, and

(iv) $\mathcal{M}$ verifies $\Gamma^\Diamond$.

I now prove that $Ab(\mathcal{M}) = \phi$. As ϕ is an arbitrary element of $\Phi(\Gamma^\Diamond)$ this proves that the theorem holds true.

In view of the construction of Δ, it suffices to show that, if there is a Σ such that $Dab(\Sigma) \in Cn_{\mathbf{S5}}(\Delta_i \cup \{B_{i+1}\})$, and $\phi \nvdash_{\mathbf{S5}} Dab(\Sigma)$, then there is no Σ' such that $Dab(\Sigma') \in Cn_{\mathbf{S5}}(\Delta_i \cup \{\sim B_{i+1}\})$, and $\phi \nvdash_{\mathbf{S5}} Dab(\Sigma')$. To see that the latter holds true, suppose that $\phi \nvdash_{\mathbf{S5}} Dab(\Sigma)$, $\phi \nvdash_{\mathbf{S5}} Dab(\Sigma')$, $\Delta_i \cup \{B_{i+1}\} \vdash_{\mathbf{S5}} Dab(\Sigma)$, and $\Delta_i \cup \{\sim B_{i+1}\} \vdash_{\mathbf{S5}} Dab(\Sigma')$. Hence, by the Deduction Theorem, the fact that $B_{i+1} \vee \sim B_{i+1}$ is an **S5**-theorem, Dilemma, and the fact that $Dab(\Sigma) \vee Dab(\Sigma') = Dab(\Sigma \cup \Sigma')$, it follows that $\Delta_i \vdash_{\mathbf{S5}} Dab(\Sigma \cup \Sigma')$. But then, $\phi \vdash_{\mathbf{S5}} Dab(\Sigma \cup \Sigma')$ by the definition of Δ. As ϕ is a set of formulas of the form $\exists(\Diamond A \wedge \Diamond{\sim}A)$, it follows that either $\phi \vdash_{\mathbf{S5}} Dab(\Sigma)$ or $\phi \vdash_{\mathbf{S5}} Dab(\Sigma')$, which contradicts the supposition. Hence, $Ab(\mathcal{M}) = \phi$. ■

Theorem 5 *If $\mathcal{M}$ is an **S5**-model of $\Gamma^\Diamond$, then there is an $\mathbf{AJ^r}$-model $\mathcal{M}'$ of $\Gamma^\Diamond$ such that $Ab(\mathcal{M}') \subseteq Ab(\mathcal{M})$. (Strong Reassurance)*

Proof. Consider some **S5**-model $\mathcal{M}$ of $\Gamma^\Diamond$. Where $D_1, D_2, \ldots$ is a list of all members of Ω, define:

$$\Delta_0 = \emptyset$$

$$\Delta_{i+1} = \Delta_i \cup \{\sim D_{i+1}\}$$

if there is an **S5**-model $\mathcal{M}'$ of $\Gamma^\Diamond \cup \Delta_i \cup \{\sim D_{i+1}\}$ such that $Ab(\mathcal{M}') \subseteq Ab(\mathcal{M})$, and

$$\Delta_{i+1} = \Delta_i$$

otherwise. Finally,

$$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \ldots$$

If some **S5**-model verifies $\Gamma^\Diamond$, then some **S5**-model verifies $\Gamma^\Diamond \cup \Delta$ (by the definition of Δ). So, I only have to prove (i) that $Ab(\mathcal{M}') \subseteq Ab(\mathcal{M})$, for each **S5**-model $\mathcal{M}'$ of $\Gamma^\Diamond \cup \Delta$, and (ii) that each such model is reliable. As $\mathcal{M}$ is an arbitrary model of $\Gamma^\Diamond$, this warrants that the theorem holds true.

(i) Consider any **S5**-model $\mathcal{M}'$ of $\Gamma^\Diamond \cup \Delta$, and suppose that there is a member of Ω, say D_j, such that $D_j \in Ab(\mathcal{M}') - Ab(\mathcal{M})$. As $D_j \notin Ab(\mathcal{M})$, $\mathcal{M}$ verifies $\Gamma^\Diamond \cup \Delta_{j-1} \cup \{\sim D_j\}$. Hence, $\sim D_j \in \Delta_j \subseteq \Delta$. But then, as $\mathcal{M}'$ is an **S5**-model of $\Gamma^\Diamond \cup \Delta$, $D_j \notin Ab(\mathcal{M}')$. This contradicts the supposition.

(ii) Suppose that $\mathcal{M}'$ verifies $\Gamma^\Diamond \cup \Delta$, but that it is not reliable. Hence, there is a member of Ω, say D_j, such that $D_j \in Ab(\mathcal{M}') - U(\Gamma^\Diamond)$. But then, for every minimal *Dab*-consequence $Dab(\Sigma)$ of $\Gamma^\Diamond$, $D_j \notin \Sigma$. Hence, in view of Lemma 1, there is an **S5**–model $\mathcal{M}''$ of $\Gamma^\Diamond$ such that $D_j \notin Ab(\mathcal{M}'')$ and $Ab(\mathcal{M}'') \subset Ab(\mathcal{M}')$.

It follows that $\mathcal{M}''$ is an **S5**-model of $\Gamma^\Diamond \cup \Delta$. If it were not, then, as $\mathcal{M}''$ is an **S5**-model of $\Gamma^\Diamond$, there is a $\sim D_k \in \Delta$ such that $\mathcal{M}'$ verifies $\sim D_k$ and $\mathcal{M}''$ falsifies $\sim D_k$. But this is impossible in view of $Ab(\mathcal{M}'') \subset Ab(\mathcal{M}')$.

As $\mathcal{M}''$ is an **S5**-model of $\Gamma^\Diamond \cup \Delta$, it is an **S5**-model of $\Gamma^\Diamond \cup \Delta_{j-1}$. Moreover, as $D_j \notin Ab(\mathcal{M}'')$, $\mathcal{M}''$ verifies $\sim D_j$. Hence, $\mathcal{M}''$ is an **S5**-model of $\Gamma^\Diamond \cup \Delta_{j-1} \cup \{\sim D_j\}$. As $Ab(\mathcal{M}'') \subset Ab(\mathcal{M}')$ and, by (i), $Ab(\mathcal{M}') \subseteq Ab(\mathcal{M})$, $Ab(\mathcal{M}'') \subset Ab(\mathcal{M})$. It follows that $\Delta_j = \Delta_{j-1} \cup \{\sim D_j\}$, and hence that $\sim D_j \in \Delta$. But then $D_j \notin Ab(\mathcal{M}')$. So, the supposition of (ii) leads to a contradiction. ■

Corollary 1 *If $\Gamma^\Diamond$ has* **S5***-models, then it also has* $\mathbf{AJ^r}$*-models. (Reassurance)*

A second series of properties concerns the adequacy of the dynamical proof theory. As was mentioned in the previous section, the proof theory of $\mathbf{AJ^r}$ is based on a specific relation between derivability by the upper limit logic **Triv** and derivability by the lower limit logic **S5**. I now prove the lemmas and theorems that warrant this.

As before, $\omega(A)$ stands for the set of abnormalities that can be obtained from the primitive formulas that occur in A. At the predicative level, this gives us: $\omega(A) = \{\exists(\Diamond B \wedge \Diamond{\sim}B) \in \Omega \mid B \text{ occurs in } A\}$. Let $\sigma^\circ(A)$ stand for the set $\{B \in \mathcal{F}^p \mid B \text{ occurs in } A\}$ and let $\sigma(A)$ stand for the set of *closed* formulas that can be obtained from the members of $\sigma^\circ(A)$ by systematically replacing all members of $\mathcal{V}$ (if any) by members of $\mathcal{C} \cup \mathcal{O}$.

Lemma 2 *For any* **S5***-model $\mathcal{M}$ and any $A \in \mathcal{W}^\Diamond$, if there is no B, such that $B \in \omega(A)$ and $\mathcal{M}$ verifies B, then there is a* **Triv***-model $\mathcal{M}'$, such that $\mathcal{M}$ verifies A iff $\mathcal{M}'$ verifies A.*

Proof. Suppose that the antecedent holds true for some **S5**-model $\mathcal{M} = \langle \Sigma_\Delta, M_0 \rangle$ and some A. It follows that, for every $C \in \sigma(A)$, and for every $M_i, M_j \in \Sigma_\Delta$, $v_{M_i}(C) = v_{M_j}(C)$, and hence, that M_i verifies A iff M_j verifies A. Let Δ' be the set of all members of $\mathcal{W}$ that are verified by M', for some $M' \in \Sigma_\Delta$. It is easily observed that $\mathcal{M}' = \langle \Sigma_{\Delta'}, M'_0 \rangle$ is a **Triv**-model (Δ' is a maximal consistent subset of $\mathcal{W}$) and that it verifies A iff $\mathcal{M}$ verifies A. ■

Theorem 6 *For any $A \in \mathcal{W}^\Diamond$, if $\vDash_{\mathbf{Triv}} A$, then there is a $\Delta \subset \Omega$ such that $\vDash_{\mathbf{S5}} A \vee Dab(\Delta)$.*

Proof. If $\vDash_{\mathbf{S5}} A$, the theorem obviously holds. So, suppose that $\vDash_{\mathbf{Triv}} A$ and that $\nvDash_{\mathbf{S5}} A$ and let Δ be $\omega(A)$. As $\omega(A)$ is finite, $A \vee Dab(\Delta)$ is a wff. To prove that $A \vee Dab(\Delta)$ is **S5**-valid, consider an arbitrary **S5**-model $\mathcal{M}$. If $\mathcal{M}$ verifies B, for some $B \in \omega(A)$, then $\mathcal{M}$ verifies $Dab(\Delta)$. If $\mathcal{M}$ falsifies B, for all $B \in \omega(A)$, then $\mathcal{M}$ verifies A in view of Lemma 2 and the fact that, by the supposition, all **Triv**-models verify A. ■

Theorem 7 *For any* $\{A_1, \ldots, A_n, B\} \subset \mathcal{W}^{\Diamond}$, *if* $A_1, \ldots, A_n \vDash_{\mathbf{Triv}} B$, *then there is a* $\Delta \subset \Omega$ *such that* $A_1, \ldots, A_n \vDash_{\mathbf{S5}} B \vee Dab(\Delta)$.

Proof. Suppose that the antecedent holds true. It follows that $\vDash_{\mathbf{Triv}} (A_1 \wedge \ldots \wedge A_n) \supset B$, and hence, by Theorem 6, that there is a $\Delta \subset \Omega$ such that $\vDash_{\mathbf{S5}} ((A_1 \wedge \ldots \wedge A_n) \supset B) \vee Dab(\Delta)$. But then, $A_1, \ldots, A_n \vDash_{\mathbf{S5}} B \vee Dab(\Delta)$. ■

I now prove that the semantics of $\mathbf{AJ^r}$ is sound and complete with respect to the dynamic proof theory.

Lemma 3 *If, in an* $\mathbf{AJ^r}$*-proof from* $\Gamma^{\Diamond}$, A *occurs as the second element and* Δ *occurs as the fifth element of line* i, *then* $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} A \vee Dab(\Delta)$.

Proof. The proof proceeds by induction on the number of the line at which A occurs. The lemma obviously holds if $i = 1$, for then, in view of the generic rules, $A \in \Gamma^{\Diamond}$ or $\vdash_{\mathbf{S5}} A \vee Dab(\Delta)$. Suppose that the lemma holds for all lines that precede i.

Case 1: The third element of line i is empty. Analogous to the case where $i = 1$.

Case 2: The third element of line i is not empty. Suppose that the third element of i is $j_1, \ldots, j_n$ $(n \geq 1)$ and that $B_1, \ldots, B_n$ are the second elements of lines $j_1, \ldots, j_n$. RU and RC warrant that $B_1, \ldots, B_n \vdash_{\mathbf{S5}} A \vee Dab(\Delta)$, and hence, that $\vdash_{\mathbf{S5}} ((B_1 \wedge \ldots \wedge B_n) \supset A) \vee Dab(\Delta)$. As the fifth elements of lines $j_1, \ldots, j_n$ are subsets of Δ, the supposition warrants that $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} B_i \vee Dab(\Delta)$ for every B_i, and hence, that $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} (B_1 \wedge \ldots \wedge B_n) \vee Dab(\Delta)$. But then, $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} A \vee Dab(\Delta)$. ■

Theorem 8 $\Gamma^{\Diamond} \vdash_{\mathbf{AJ^r}} A$ *iff there is a* $\Delta \subset \Omega$ *such that* $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} A \vee Dab(\Delta)$, *and* $\Delta \cap U(\Gamma^{\Diamond}) = \emptyset$.

Proof. For the left-right direction, suppose that $\Gamma^{\Diamond} \vdash_{\mathbf{AJ^r}} A$. In that case, A is finally derived at some line j of an $\mathbf{AJ^r}$-proof from $\Gamma^{\Diamond}$. Hence, where Δ is the fifth element of line j, $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} A \vee Dab(\Delta)$ in view of Lemma 3. Suppose now that $\Delta \cap U(\Gamma^{\Diamond}) \neq \emptyset$. In that case, $\Delta \cap \Delta' \neq \emptyset$, for some minimal *Dab*-consequence $Dab(\Delta')$ of $\Gamma^{\Diamond}$. As **S5** is compact, there is an extension of the proof in which $Dab(\Delta')$ occurs unconditionally. But then, line j is marked in that extension, and will remain marked in any further extension. This contradicts that A is finally derived from $\Gamma^{\Diamond}$.

For the right-left direction, suppose that there is a Δ such that (i) $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} A \vee Dab(\Delta)$, and (ii) $\Delta \cap U(\Gamma^{\Diamond}) = \emptyset$. In that case, some line j in an $\mathbf{AJ^r}$-proof from $\Gamma^{\Diamond}$ has A as its second element and Δ as its fifth (in view of RC). In view of (ii) any extension of the proof in which line j is marked can be further extended in such a way that j is unmarked. But then, A is finally derived at line j. Hence, $\Gamma^{\Diamond} \vdash_{\mathbf{AJ^r}} A$. ■

Theorem 9 *If* $\Gamma^{\Diamond} \vdash_{\mathbf{AJ^r}} A$, *then* $\Gamma^{\Diamond} \vDash_{\mathbf{AJ^r}} A$. *(Soundness)*

Proof. Suppose that $\Gamma^{\Diamond} \vdash_{\mathbf{AJ^r}} A$. In view of Theorem 8, there is a Δ such that $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} A \vee Dab(\Delta)$, and $\Delta \cap U(\Gamma^{\Diamond}) = \emptyset$. Hence, $\Gamma^{\Diamond} \vDash_{\mathbf{S5}} A \vee Dab(\Delta)$, in view of Theorem 4. As $\Delta \cap U(\Gamma^{\Diamond}) = \emptyset$, $Dab(\Delta)$ is false in all reliable models of $\Gamma^{\Diamond}$. But then, A is true in all reliable models of $\Gamma^{\Diamond}$, and $\Gamma^{\Diamond} \vDash_{\mathbf{AJ^r}} A$. ■

Theorem 10 *If $\Gamma^{\Diamond} \vDash_{\mathbf{AJ^r}} A$, then $\Gamma^{\Diamond} \vdash_{\mathbf{AJ^r}} A$. (Completeness)*

Proof. Suppose that $\Gamma^{\Diamond} \nvdash_{\mathbf{AJ^r}} A$. Let $B_1, B_2, \ldots$ be the sequence from the proof of Lemma 1, and define:

$$\Delta_0 = Cn_{\mathbf{S5}}(\Gamma^{\Diamond} \cup \{C \supset A\} \mid C \in \Omega - U(\Gamma^{\Diamond})\})$$

$$\Delta_{i+1} = Cn_{\mathbf{S5}}(\Delta_i \cup \{B_{i+1}\})$$

if $A \notin Cn_{\mathbf{S5}}(\Delta_i \cup \{B_{i+1}\})$, and

$$\Delta_{i+1} = \Delta_i$$

otherwise. Finally,

$$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \ldots$$

Each of the following is easily provable:

(i) $\Gamma^{\Diamond} \subseteq \Delta$ (by the definition of Δ).
(ii) $A \notin \Delta$. Suppose that $A \in \Delta$. By the definition of Δ, it follows that $A \in \Delta_0$. It is easily observed that the latter is impossible. Suppose that $A \in \Delta_0$. In that case, there is a $\Delta \subset \Omega - U(\Gamma^{\Diamond})$ such that $\Gamma^{\Diamond} \cup \{Dab(\Delta) \supset A\} \vdash_{\mathbf{S5}} A$. (In view of the fact that any proof is finite, and $A \supset B$, $C \supset B \vdash_{\mathbf{S5}} (A \vee C) \supset B$.) But then, by the Deduction Theorem, $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} (Dab(\Delta) \supset A) \supset A$, and hence, $\Gamma^{\Diamond} \vdash_{\mathbf{S5}} Dab(\Delta) \vee A$. However, as $\Delta \cap U(\Gamma^{\Diamond}) = \emptyset$, $\Gamma^{\Diamond} \vdash_{\mathbf{AJ^r}} A$ (by Theorem 8). This contradicts the main supposition.
(iii) For every $D \in \Omega - U(\Gamma^{\Diamond})$, $D \notin \Delta$. To see this, suppose that $D \in \Omega - U(\Gamma^{\Diamond})$. In that case, $D \supset A \in \Delta_0$. Hence, if $D \in \Delta$, then $A \in \Delta$. This contradicts (ii).

As in the proof of Lemma 1, let Δ be extended to $\Delta^{\star}$, and let an **S5**-model $\mathcal{M}$ be defined from $\Delta^{\star}$. In view of (i) and (ii), $\mathcal{M}$ verifies all members of $\Gamma^{\Diamond}$, and falsifies A. In view of (iii), $\mathcal{M}$ is an $\mathbf{AJ^r}$-model of $\Gamma^{\Diamond}$. Hence, $\Gamma^{\Diamond} \nvDash_{\mathbf{AJ^r}} A$. ■

Corollary 2 $\Gamma^{\Diamond} \vDash_{\mathbf{AJ^r}} A$ *iff* $\Gamma^{\Diamond} \vdash_{\mathbf{AJ^r}} A$.

8 The Discussive Adaptive Logic $\mathbf{D_2^r}$

The discussive adaptive logic $\mathbf{D_2^r}$ is defined from the modal adaptive logic $\mathbf{AJ^r}$ in the following way:

Definition 11 *Where $A \in \mathcal{W}$ and $\Gamma \subseteq \mathcal{W}$, $\Gamma \vdash_{\mathbf{D_2^r}} A$ iff $\Gamma^{\Diamond} \vdash_{\mathbf{AJ^r}} \Diamond A$.*

As is the case for other discussive logics, $\mathbf{D_2^r}$ does not allow for the derivation of contradictions (unless one of the premises is self-contradictory):

Theorem 11 *For any $A \in \mathcal{W}$ and for any $\Gamma \subseteq \mathcal{W}$ such that $\Gamma^{\Diamond}$ has* **S5***-models, $\Gamma \nvdash_{\mathbf{D_2^r}} A \wedge {\sim}A$.*

Proof. Suppose that the antecedent holds true, for some $\Gamma \subseteq \mathcal{W}$. It follows that $\Gamma^\Diamond$ has $\mathbf{AJ^r}$-models (in view of Corollary 1). But then, as $\Diamond(A \wedge {\sim}A)$ is false in any **S5**-model, there are $\mathbf{AJ^r}$-models of $\Gamma^\Diamond$ that falsify $\Diamond(A \wedge {\sim}A)$. Hence, in view of Definition 11, $\Gamma \nvdash_{\mathbf{D_2^r}} A \wedge {\sim}A$. ■

Moreover, its consequence set is not closed under (classical) conjunction:

Theorem 12 *For some* $\Gamma \subseteq \mathcal{W}$ *and some* $A, B \in \mathcal{W}$, $\Gamma \vdash_{\mathbf{D_2^r}} A$, $\Gamma \vdash_{\mathbf{D_2^r}} B$, *and* $\Gamma \nvdash_{\mathbf{D_2^r}} A \wedge B$.

Proof. Where $\Gamma = \{p,\ {\sim}p\}$, $\Gamma \vdash_{\mathbf{D_2^r}} p$ and $\Gamma \vdash_{\mathbf{D_2^r}} {\sim}p$, but $\Gamma \nvdash_{\mathbf{D_2^r}} p \wedge {\sim}p$ (in view of Theorem 11). ■

Still, $\mathbf{D_2^r}$ enables one to conjoin a statement that behaves consistently to any other statement:

Theorem 13 *For any* $\Gamma \subseteq \mathcal{W}$ *and any* $A, B \in \mathcal{W}$, *if* $\Gamma \vdash_{\mathbf{D_2^r}} A$ *and* $\Gamma \vdash_{\mathbf{D_2^r}} B$, *then* $\Gamma \vdash_{\mathbf{D_2^r}} A \wedge B$, *if* $\omega(A) \cap U(\Gamma^\Diamond) = \emptyset$ *or* $\omega(B) \cap U(\Gamma^\Diamond) = \emptyset$.

Proof. Suppose that the antecedent holds true and that $\omega(A) \cap U(\Gamma^\Diamond) = \emptyset$. It follows that $\Gamma^\Diamond \vdash_{\mathbf{AJ^r}} \Diamond A$ and $\Gamma^\Diamond \vdash_{\mathbf{AJ^r}} \Diamond B$. But then, as $\omega(A) \cap U(\Gamma^\Diamond) = \emptyset$, $\Gamma^\Diamond \vdash_{\mathbf{AJ^r}} \Box A$ (in view of the proof for Theorem 6 and the fact that $\vdash_{\mathbf{Triv}} \Diamond A \supset \Box A$) and hence, $\Gamma^\Diamond \vdash_{\mathbf{AJ^r}} \Diamond(A \wedge B)$ and $\Gamma \vdash_{\mathbf{D_2^r}} A \wedge B$.

Analogously for the case where $\omega(B) \cap U(\Gamma^\Diamond) = \emptyset$. ■

Thus, as was mentioned already in the introduction, $\mathbf{D_2^r}$ validates Adjunction 'as much as possible'. An important consequence of this property is that $\mathbf{D_2^r}$ is *non-monotonic*:

Theorem 14 *For some* $\Gamma \cup \Gamma' \subseteq \mathcal{W}$ *and some* $A \in \mathcal{W}$, $\Gamma \vdash_{\mathbf{D_2^r}} A$ *and* $\Gamma \cup \Gamma' \nvdash_{\mathbf{D_2^r}} A$. *(non-monotonicity)*

Proof. Let $\Gamma = \{p,\ q\}$ and $\Gamma' = \{{\sim}p \vee {\sim}q\}$. As $U(\Gamma^\Diamond) = \emptyset$, all $\mathbf{AJ^r}$-models of $\Gamma^\Diamond$ falsify $\Diamond p \wedge \Diamond{\sim}p$ as well as $\Diamond q \wedge \Diamond{\sim}q$, and hence, verify each of the following: $\Box p$, $\Box q$, $\Box(p \wedge q)$, $\Diamond(p \wedge q)$. But then, in view of Definition 11, $\Gamma \vdash_{\mathbf{D_2^r}} p \wedge q$.

Let $\mathcal{M}$ be an **S5**-model that verifies $\Diamond p$, $\Diamond q$ and $\Box({\sim}p \vee {\sim}q)$, but that verifies no other formula of the form $\exists(\Diamond A \wedge \Diamond{\sim}A)$ than $\Diamond p \wedge \Diamond{\sim}p$ and $\Diamond q \wedge \Diamond{\sim}q$. It is easily observed that $\mathcal{M}$ is an **S5**-model of $\Gamma^\Diamond \cup \Gamma'^\Diamond$, that it is reliable and that it falsifies $\Diamond(p \wedge q)$. Hence, in view of Definition 11, $\Gamma \cup \Gamma' \nvdash_{\mathbf{D_2^r}} p \wedge q$. ■

Another important consequence is that, for consistent sets of premises, the consequence set of $\mathbf{D_2^r}$ equals that of **CL**:

Theorem 15 *If* Γ *is consistent, then* $Cn_{\mathbf{D_2^r}}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$.

Proof. If Γ is consistent, $U(\Gamma^\Diamond) = \emptyset$. Hence, the $\mathbf{AJ^r}$-models of $\Gamma^\Diamond$ are its **Triv**-models (which are analogues of **CL**-models). But then, $\Gamma \vdash_{\mathbf{DL^r}} A$ iff $\Gamma^\Diamond \vdash_{\mathbf{AJ^r}} \Diamond A$ iff $\Gamma \vdash_{\mathbf{CL}} A$. ■

In view of these properties, the logic $\mathbf{D_2^r}$ is an example of an inconsistency-adaptive logic: it localizes the specific inconsistencies that follow from a set of premises, and adapts its rules of inferences to these. There are, however, some important differences between $\mathbf{D_2^r}$ and other inconsistency-adaptive logics, for instance, **ACLuN1** and **ACLuN2** from [3], $\mathbf{LP}^m$ from [24] and **ANA** from [21]. A first one is that $\mathbf{D_2^r}$ does not allow for the derivation of contradictions. Another difference concerns the rules of **CL** that are turned into conditional rules. In nearly all inconsistency-adaptive logics, the distinction between conditional and unconditional rules is related to properties of the negation. For instance, in **ACLuN1** and **ACLuN2** all positive rules of **CL** (*Modus Ponens*, Adjunction, ...) are unconditionally valid and all negative rules (Double Negation, Disjunctive Syllogism, *Modus Tollens*, ...) are conditionally valid.[9] $\mathbf{D_2^r}$ proceeds along a completely different line. In $\mathbf{D_2^r}$, the distinction between unconditional and conditional rules coincides with that between single-premise rules and multiple-premise rules. Whereas the former are unconditionally valid in $\mathbf{D_2^r}$, the latter are only conditionally valid.

It is also interesting to compare $\mathbf{D_2^r}$ to consequence relations that proceed in terms of maximal consistent subsets. These consequence relations were first presented in [26], and are today quite popular in handling inconsistent databases (see, for instance, [12] and [13]).[10]

The idea behind these consequence relations is that conjoining is allowed up to maximal consistency. The consequence relations differ from each other with respect to the restrictions they impose on "to follow from". For instance, the so-called *strong consequences* of (a possibly inconsistent) Γ are those that follow by **CL** from *every* maximal consistent subset of Γ; the *weak consequences* are those that follow by **CL** from *some* maximal consistent subset of Γ.

Thus, where $\Gamma = \{p, {\sim}p, r, s\}$, $r \wedge s$ is a strong consequence of Γ, and the latter as well as $p \wedge r$, $p \wedge s$, ${\sim}p \wedge r$, and ${\sim}p \wedge s$ are weak consequences. For consistent sets of premises, both consequence relations deliver the same consequence set as **CL**. Still, the consequence relations are non-adjunctive: $p \wedge {\sim}p$ is neither a strong nor a weak consequence relation of Γ.

It may seem that the consequence relation of $\mathbf{D_2^r}$ simply coincides with the weak consequence relation. There are, however, two important differences. The first is that the weak consequence relation is *monotonic*. Thus, $p \wedge q$ is a weak consequence of all possible extensions of $\Gamma = \{p, {\sim}p, q\}$. As we have seen, this is not the case for $\mathbf{D_2^r}$: although $p \wedge q$ follows by $\mathbf{D_2^r}$ from Γ, it does not follow from $\Gamma \cup \{{\sim}q\}$. The second difference concerns the sensitivity with respect to the formulation of the premises. The weak consequence relation (like other consequence relations based on the idea of maximal consistent subsets) only considers maximal consistent subsets of the *premises*. As a result, this consequence relation is extremely sensitive to the formulation of the premises. For instance, r is a weak consequence of $\Gamma = \{p \wedge q,\ p \supset r,\ {\sim}q\}$, but not of $\Gamma' = \{p \wedge q,\ (p \supset r) \wedge {\sim}q\}$. This is not the case for the consequence relation of $\mathbf{D_2^r}$. The latter is invariant under all classical transformations of the premises that do not lead to formulas that are self-contradictory. Thus, in the previous example r is not only a $\mathbf{D_2^r}$-consequence of $\{p \wedge q,\ p \supset r,\ {\sim}q\}$, but also of $\{p \wedge q,\ (p \supset r) \wedge {\sim}q\}$ and $\{p \wedge {\sim}q,\ (p \supset r) \wedge q\}$.

[9] **ANA** is an exception. Its only conditional rules are constructive rules that are weakening (for instance, $A/A \vee B$, and $A/B \supset A$) or paradoxical (for instance, $A/A \vee (B \wedge {\sim}B)$).

[10] Interesting unifications of these consequence relations in terms of adaptive logics can be found in [6] and [11].

9 In Conclusion

$\mathbf{D_2^r}$ is the first discussive logic that is adaptive. As compared to other discussive logics, it has the enormous advantage that it conditionally validates *all* multiple-premise rules of **CL**, without the introduction of discussive connectives. $\mathbf{D_2^r}$ is also the first inconsistency-adaptive logic that is based on a non-adjunctive paraconsistent logic.

$\mathbf{D_2^r}$ deserves further study. An important open problem concerns the design of a *direct* proof theory for $\mathbf{D_2^r}$—one that does not refer to **S5**, but proceeds entirely in the language of **CL**. Another open problem concerns the comparison of $\mathbf{D_2^r}$ with other inconsistency-adaptive systems,[11] for instance with respect to applications in the history and philosophy of science.

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