UNIVERSITEIT GENT

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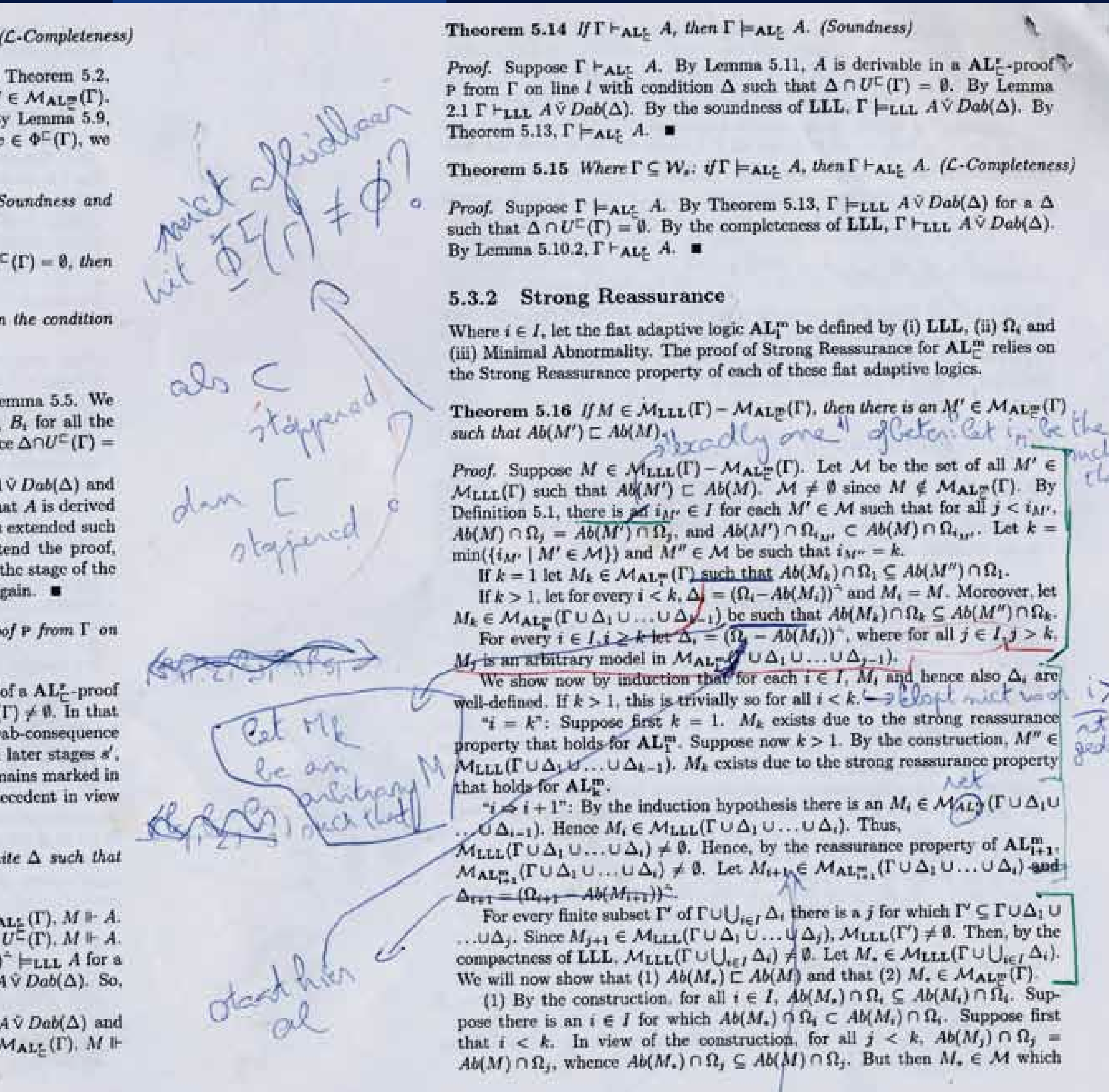
Generic Formats for

# Prioritized Adaptive Logics

with Applications in Deontic Logic, Abduction and Belief Revision.

Proefschrift voorgedragen tot het bekomen van de graad van Doctor in de Wijsbegeerte
Promotor: Prof. dr. Joke Meheus



**Theorem 5.14** *If $\Gamma \vdash_{\mathbf{AL}_{\sqsubset}} A$, then $\Gamma \models_{\mathbf{AL}_{\sqsubset}} A$. (Soundness)*

*Proof.* Suppose $\Gamma \vdash_{\mathbf{AL}_{\sqsubset}} A$. By Lemma 5.11, $A$ is derivable in a $\mathbf{AL}_{\sqsubset}$-proof $\mathsf{P}$ from $\Gamma$ on line $l$ with condition $\Delta$ such that $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. By Lemma 2.1 $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By the soundness of **LLL**, $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By Theorem 5.13, $\Gamma \models_{\mathbf{AL}_{\sqsubset}} A$. ■

**Theorem 5.15** *Where $\Gamma \subseteq \mathcal{W}_s$: if $\Gamma \models_{\mathbf{AL}_{\sqsubset}} A$, then $\Gamma \vdash_{\mathbf{AL}_{\sqsubset}} A$. ($\mathcal{L}$-Completeness)*

*Proof.* Suppose $\Gamma \models_{\mathbf{AL}_{\sqsubset}} A$. By Theorem 5.13, $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta$ such that $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. By the completeness of **LLL**, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By Lemma 5.10.2, $\Gamma \vdash_{\mathbf{AL}_{\sqsubset}} A$. ■

### 5.3.2 Strong Reassurance

Where $i \in I$, let the flat adaptive logic $\mathbf{AL}^{\mathrm{m}}_i$ be defined by (i) **LLL**, (ii) $\Omega_i$ and (iii) Minimal Abnormality. The proof of Strong Reassurance for $\mathbf{AL}^{\mathrm{m}}_{\sqsubset}$ relies on the Strong Reassurance property of each of these flat adaptive logics.

**Theorem 5.16** *If $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_{\sqsubset}}(\Gamma)$, then there is an $M' \in \mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_{\sqsubset}}(\Gamma)$ such that $Ab(M') \subset Ab(M)$.*

*Proof.* Suppose $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_{\sqsubset}}(\Gamma)$. Let $\mathcal{M}$ be the set of all $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $Ab(M') \subset Ab(M)$. $\mathcal{M} \neq \emptyset$ since $M \notin \mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_{\sqsubset}}(\Gamma)$. By Definition 5.1, there is ~~an~~ $i_{M'} \in I$ for each $M' \in \mathcal{M}$ such that for all $j < i_{M'}$, $Ab(M) \cap \Omega_j = Ab(M') \cap \Omega_j$, and $Ab(M') \cap \Omega_{i_{M'}} \subset Ab(M) \cap \Omega_{i_{M'}}$. Let $k = \min(\{i_{M'} \mid M' \in \mathcal{M}\})$ and $M'' \in \mathcal{M}$ be such that $i_{M''} = k$.

If $k = 1$ let $M_k \in \mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_1}(\Gamma)$ such that $Ab(M_k) \cap \Omega_1 \subseteq Ab(M'') \cap \Omega_1$.

If $k > 1$, let for every $i < k$, $\Delta_i = (\Omega_i - Ab(M_i))^{\check{\neg}}$ and $M_i = M$. Moreover, let $M_k \in \mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_k}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k-1})$ be such that $Ab(M_k) \cap \Omega_k \subseteq Ab(M'') \cap \Omega_k$.

For every $i \in I$, ~~$i \geq k$ let $\Delta_i$~~ $= (\Omega_i - Ab(M_i))^{\check{\neg}}$, where for all $j \in I$, $j > k$, ~~$M_j$ is an arbitrary model in $\mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_j}$~~$(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{j-1})$.

We show now by induction that for each $i \in I$, $M_i$ and hence also $\Delta_i$ are well-defined. If $k > 1$, this is trivially so for all $i < k$.

"$i = k$": Suppose first $k = 1$. $M_k$ exists due to the strong reassurance property that holds for $\mathbf{AL}^{\mathrm{m}}_1$. Suppose now $k > 1$. By the construction, $M'' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k-1})$. $M_k$ exists due to the strong reassurance property that holds for $\mathbf{AL}^{\mathrm{m}}_k$.

"$i \Rightarrow i+1$": By the induction hypothesis there is an $M_i \in \mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_i}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{i-1})$. Hence $M_i \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i)$. Thus, $\mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i) \neq \emptyset$. Hence, by the reassurance property of $\mathbf{AL}^{\mathrm{m}}_{i+1}$, $\mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_{i+1}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i) \neq \emptyset$. Let $M_{i+1} \in \mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_{i+1}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i)$ ~~and $\Delta_{i+1} = (\Omega_{i+1} - Ab(M_{i+1}))^{\check{\neg}}$~~.

For every finite subset $\Gamma'$ of $\Gamma \cup \bigcup_{i \in I} \Delta_i$ there is a $j$ for which $\Gamma' \subseteq \Gamma \cup \Delta_1 \cup \ldots \cup \Delta_j$. Since $M_{j+1} \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_j)$, $\mathcal{M}_{\mathbf{LLL}}(\Gamma') \neq \emptyset$. Then, by the compactness of **LLL**, $\mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \bigcup_{i \in I} \Delta_i) \neq \emptyset$. Let $M_\star \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \bigcup_{i \in I} \Delta_i)$. We will now show that (1) $Ab(M_\star) \subset Ab(M)$ and that (2) $M_\star \in \mathcal{M}_{\mathbf{AL}^{\mathrm{m}}_{\sqsubset}}(\Gamma)$.

(1) By the construction, for all $i \in I$, $Ab(M_\star) \cap \Omega_i \subseteq Ab(M_i) \cap \Omega_i$. Suppose there is an $i \in I$ for which $Ab(M_\star) \cap \Omega_i \subset Ab(M_i) \cap \Omega_i$. Suppose first that $i < k$. In view of the construction, for all $j < k$, $Ab(M_j) \cap \Omega_j = Ab(M) \cap \Omega_j$, whence $Ab(M_\star) \cap \Omega_j \subseteq Ab(M) \cap \Omega_j$. But then $M_\star \in \mathcal{M}$ which

niet afleidbaar uit $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$?

als $\sqsubset$ stappend

dan $\sqsubset$ stappend

Let $M_k$ be an arbitrary $M$ such that

staat hier al

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# Generic Formats for Prioritized Adaptive Logics

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# Contents

# Chapter 1

# Introduction

*I thank Joke Meheus and Dagmar Provijn for the valuable comments and intensive discussions that helped shape this introduction.*

This thesis is about *prioritized adaptive logics*, which as I will argue, are very good candidates to explicate forms of *prioritized defeasible reasoning*. First and foremost, it concerns generic formats in which such logics may be defined. Second, it is about concrete (prioritized) adaptive logics that explicate particular forms of human reasoning. This introductory chapter spells out the philosophical and logical motivations of the present study.

I will first explain what is meant by prioritized defeasible reasoning (Section 1.1). Then I will indicate what makes adaptive logics specific, in comparison to other approaches to defeasible reasoning in formal logic and AI (Section 1.2). I will explain the need for a thorough study of the metatheory of prioritized adaptive logics in Section 1.3. After that, I will spell out a number of restrictions of the current study (Section 1.4). I end this chapter with an overview of the other chapters' content (Section 1.5).

## 1.1 Prioritized defeasible reasoning

The term prioritized (defeasible) reasoning is used for all sorts of inferential processes. I do not wish to restrict its use here to only one very specific type of inferences, since it is my aim to argue that the research I will present is applicable in a very broad range of contexts. Rather, I will try to illustrate the variety and ubiquity of what I call prioritized defeasible reasoning forms in everyday life. While doing so, a number of typical features of these reasoning forms will be highlighted.

To get a first idea of what this thesis is about, let us consider some toy-examples. The first one is inspired by similar examples from [68] and will also be used in Chapter 7. The third is based on a true story – see [66].

*Example 1.* After having a car accident, Mary has to stay at the site of the accident to fill in her insurance papers. However, she also promised her mother to pick her up from the supermarket and take her home. She cannot do both at the same time, but gives priority to her legal duties. Hence she concludes that she should stay at the site of the accident and break the promise.

*Example 2.* Tim is walking with his son in the park, when they see two black birds on a branch of a tree. The son points at the two birds, and Tim tells his son that those two birds are crows. The day after, Tim's son sees another black bird in the garden of their house, and shouts: "look daddy, a crow!". However, when Tim looks, he sees that it is actually a magpie, and so he tells his son. At that point, his son readily changes his mind about the bird and believes it is a magpie.

*Example 3.* Benjamin notices that the overhead light in his office is broken, and asks the security warden of the building how he can get it repaired. The warden advises him to send an email to the Administrative Assistent of the building, and so he does. However, the morning after, Benjamin receives an angry email from the Administrative Assistent, telling him he should first make an official request to the security warden, who then (if he approves the request) makes over an official request to the Administrative Assistent. Since the Administrative Assistent has a higher authority than the security warden, Benjamin decides to follow the procedure she proposed.

To grasp the similarities between these and other examples, we may use the following informal definition of prioritized defeasible reasoning (henceforth PDR):

PDR is any kind of defeasible reasoning in which priorities co-determine which conclusions can, and which cannot, be drawn from a given body of evidence.

Let me briefly explain some of the notions used in this definition. Obviously, PDR refers to particular kinds of *reasoning*, i.e. inferential processes by means of which we draw conclusions from a given body of evidence. The reasoning forms are moreover *defeasible*. This means that in these processes, we may draw conclusions at a given moment in time, at a later moment retract some of them, and at a still later moment retrieve some of the retracted inferences – I will provide such an example in the next section.

Note that the definition does not delineate a subject of this reasoning: it can be about beliefs, about generalizations, about explanations, about obligations, etc. Hence, the term *evidence* refers to an equally broad range of things: experimental data, new information that contradicts our initial beliefs, propositions that express what is obligatory according to a given normative system, etc. Likewise, the intended conclusions of PDR can be of various sorts – they can be explanatory hypotheses, statements about what our actual obligations are (in view of the normative systems at hand), predictions about the outcome of the next experiment, etc.

What is indispensable however, are the *priorities*. Again, the precise meaning of this term depends on that which we reason about: it can refer to degrees of plausibility (of beliefs), of urgence or importance (of a duty), of explanatory power (of an explanation), of specificity (of a default rule), of the reliability of a specific type of inference, etc. Sometimes the priorities are seen as part of the evidence itself, as in Example 3, where Benjamin reasons about two advices which he got from sources with a different degree of authority. However, the priorities can also be internal to the reasoning process, e.g. when two different argumentation techniques receive a different weight in an general argumentation strategy. Finally, it may be the case that we reason on the basis of various kinds of information and methods, as e.g. in the case of Tim's son, who uses both inductive generalization and what his father tells him.

The definition of PDR stipulates that the priorities *co-determine* the conclusions which we can or cannot draw – obviously, the evidence itself also determines which conclusions can be drawn. The priorities are themselves not a result or the subject of the reasoning process. In this sense, PDR can (at least ideally) be distinguished from reasoning *about* priorities, and from reasoning processes through which we *obtain* priorities.

From this description and the above examples, it should be clear that many defeasible reasoning forms have a prioritized counterpart, or more precisely, that non-prioritized types of defeasible reasoning are often mere idealizations of PDRs. For instance, when we revise our beliefs in view of new information, it is hard to imagine that we consider each of these as equally plausible. Likewise, when we reason on the basis of conflicting obligations, we usually consider some of these as more important than others.

## 1.2 Defeasible Reasoning and Adaptive Logics

In formal logic and Artificial Intelligence, there is a vast body of research on processes that fall under the general header of PDR. Prioritized (non-monotonic) consequence relations have been studied e.g. in belief revision [36, 57, 114, 5], deontic logic [2, 39, 68], default logic [81, 42, 40] and circumscription [99, 66] – to name only the most prominent approaches. Just as many defeasible reasoning forms have a prioritized counterpart (cf supra), most frameworks for non-monotonic logics have been extended in order to account for forms of PDR.[1]

Notwithstanding the variety in approaches and applications, this field of research is bound by a number of restrictions. First and foremost, most of the existing models do not have a proof theory that explicates how agents draw conclusions on the basis of prioritized information, and *retract* inferences in view of new insights into this information – I will clarify this point below. Second, these models usually consider PDR as reasoning with (one type of) prioritized defeasible information. In this terminology, "information" can refer to beliefs, obligations, background assumptions, default rules, etc. Third, the priority order on the information is often assumed to be a well-founded, strict partial order.[2] Fourth and last, although some authors consider the possibility that this order

[1]A logic **L** is said to be non-monotonic if there are premise sets $\Gamma, \Gamma'$ such that $\Gamma \subseteq \Gamma'$, but the set of **L**-consequences of $\Gamma$ is not a subset of $\Gamma'$.

[2]This means that (i) the order is transitive, anti-symmetric and irreflexive and (ii) there are no infinite sequences of elements with ever higher priority.

is itself subject to revisions – see e.g. [42, 66, 123, 113, 84, 47, 129], the standard approaches start from a static order on the information.

In this thesis, PDR will be approached in a way that departs from the more common models, especially with respect to the first two of the above restrictions. That is, (i) it not only provides a static (semantic) characterization of prioritized non-monotonic consequence relations, but also equips them with a dynamic proof theory, and (ii) it models defeasible reasoning in terms of the abnormality of formulas with a specific logical form, and thereby allows us to compare and combine qualitatively distinct reasoning methods – this will be explained below. Both these features are intrinsic to the general framework I will be working with, viz. *adaptive logics*. To understand this, let us take a look at a slightly more elaborate example of PDR.

**Example 1.1** *When John has finished working, he passes by the house of his friend Peter on his way home. He notices that the lights of the house are on, which seems to indicate that Peter is at home. Hence John decides to pay his friend an unannounced visit.*

*However, when he comes near Peter's door, he also sees that Peter's car is nowhere in the street, even though there are quite a few empty places to park his car. "If Peter would be at home", so John thinks, "then how can I explain the fact that his car is not around?" At this point, John starts to doubt whether Peter is really at home, or whether he just went out and forgot to put the lights off.*

*However, after giving it some further thought, John figures out that there might be several explanations for the fact that Peter is at home, whereas his car is not around. For instance, he might have brought it to the garage, because something was wrong with it. Or he might have lend it to a relative or friend who urgently needed a car. Hence, John returns to his conclusion that Peter will be at home.*

*After John has rung the bell twice, Peter still does not show up. At that point, John gives up ringing, and concludes that Peter is after all not at home. As a result, he also drops all possible hypotheses about Peter's car – if Peter went out, then it is no wonder that his car is gone as well.*

*When John is back at home, he decides to call Peter on his mobile phone. After a brief conversation, it turns out that he was in the shower at the moment John rang the bell, and that Peter's brother – who does not have a car of his own – used his car to do some shopping.*

In the above example, John changes his mind quite a few times about whether or not Peter is at home. In other words, John's reasoning process is *dynamic*. This is inevitable, since the aim of this reasoning is to find an answer to the question "Is Peter at home?", on the basis of continuously growing, yet fairly weak evidence (the lights, the car, the door bell). Only later on, when Peter tells John that he was indeed at home, the conclusion can be called "deductive" – although it still relies on the assumption that Peter is telling the truth.

What is perhaps less obvious, is that two sorts of dynamics are at play. On the one hand, John sometimes changes his mind in the light of new evidence. For instance, when he notices that the car is not around, he withdraws his belief that Peter is at home. Likewise, when Peter does not answer the door bell, John concludes that Peter is not at home. This type of dynamics has been called the *external* dynamics of a reasoning process: additional information, or information that is external or new to the process, leads to the retraction of some previously drawn conclusions.

However, there is also an *internal* dynamics in the above example.[3] This is most appearant in the third paragraph of the story. There, John changes his mind, solely on the basis of his own further reasoning. In this particular case, the reasoning leads to a choice between two conflicting hypotheses: "Peter is at home", and "Peter is not at home". The choice is made on the basis of the fact that Peter's being absent is only one of several equally likely hypotheses that explain why his car is not around.

In more complex cases, the internal dynamics can also be of a different sort. That is, it may be the case that a reasoner draws a number of different conclusions, and that only after a while, i.e. after he made several additional inferences, he notes that these conclusions are mutually inconsistent. At that point (and not before), he will retract some of those conclusions in order to reinstall consistency.

Formally, the external dynamics of a reasoning method corresponds to the non-monotonic character of a logic. As noted above, there are several competing models for non-monotonic consequence relations, e.g. default logic, belief revision, and circumscription. These frameworks allow one to explicate how the addition of new evidence forces a rational subject to change its mind about a given matter. However, to explicate the internal dynamics, we need a more fine-grained model, i.e. one that not only defines an "output" (consequence set) for each given "input" (premise set), but also describes how a reasoner stepwise obtains the former from the latter. In other words, we need a *proof theory*, just like monotonic systems such as classical logic have one. However, to make this proof theory capable of dealing with the dynamics of defeasible reasoning, it will also have to differ from the classical proof theory in some respects. Adaptive logics have such a proof theory, as will be shown in the next chapter.

A second distinctive feature of adaptive logics, is that they model defeasible reasoning in terms of the abnormality of formulas with a specific logical form. To understand the difference with other approaches, consider the following propositions:

(i) "If the light is on, then Peter is at home."
(ii) "If he does not answer the door bell, then Peter is not at home."

We may treat (i) and (ii) as default assumptions held by John, where (ii) has priority over (i). Roughly speaking, this means that we use (i) and (ii) as rules of thumb, and derive their consequent from their antecedent, unless this leads to problems. If is turns out that applying both rules is problematic, but applying only one of them is not, we only apply rule (ii). This treatment explains some of the dynamics in the above example, and seems to be fairly intuitive.

[3]The distinction between internal and external is very common in the research on adaptive logics – see e.g. [21]. Pollock refers to the same distinction, when he speaks of the synchronic versus diachronic defeasibility of non-monotonic reasoning [122].

However, what about Peter's car? In this case, John's reasoning is about possible explanations for a given fact. Although it might in principle be formalized in terms of default assumptions, we can also frame it in a more direct way, i.e. in terms of an abductive inference.[4] Such an inference is of the following form:

If $A$ is the case, then $B$ naturally follows.
$B$ is the case.
Hence, $A$ is the case.

The overall aim of an abduction is to find an explanation for a given fact. However, in John's case, it plays a slightly different role: the fact that he cannot use a very obvious hypothesis to explanain his previously drawn conclusion, causes him to withdraw this conclusion. The reason is that he considers the fact that a given observation cannot be explained, as *abnormal.* Only after John has come up with a number of other explanations, his mind is relieved and he again assumes that Peter is at home.

What counts as an abnormal fact? This depends on the context in which we reason and on the reasoning method we are implementing. Sometimes, we consider it as abnormal that we hold a certain belief, but that this belief is contradicted by new (supposedly reliable) information. On other occasions, we consider it as abnormal that a certain phenomenon cannot be explained by means of reference to a more general rule, or that we cannot generalize a specific feature of an object to the class this object belongs to.

The working hypothesis of the adaptive logic program is that for every defeasible reasoning method, we can specify a corresponding notion of "abnormality" in terms of one or several logical forms. Formulas that have this form, are called "abnormalities", and are "as much as possible" considered to be false by the adaptive logic that is defined from them.[5] What matters to us here, is that different forms of defeasible reasoning come with different types of abnormalities. Hence adaptive logics allow us to distinguish these defeasible reasoning forms, to treat them differently, yet still to combine them into one single system. Examples of different notions of abnormality that correspond to different forms of defeasible reasoning are given in Table 1.1.

In other approaches to defeasible reasoning such as e.g. circumscription, there is also the idea that certain states of the world are more abnormal than others, and that we should only consider those interpretations of our premises that correspond to minimally abnormal states of the world. However, what is missing in those approaches, is the fact that abnormality is explicated by means of a logical *form*, and that various notions of abnormality can be used to model defeasible reasoning.

In the context of prioritized defeasible reasoning, this has an important consequence. Just as different pieces of information can receive a different priority, also different sorts of abnormalities can receive a different weight. This way,

[4] I am not arguing here that we can always clearly distinguish between default assumptions and abductive hypotheses. For instance, the antecedents of propositions (i) and (ii) can also be interpreted as abductive explanations for their respective consequents.

[5] Actually, as we will see in Chapter 2, every adaptive logic is defined by a triple, of which the set of abnormalities is the second member.

| *formal framework* | *reasoning form* | *abnormality* |
|---|---|---|
| Epistemic Logic | Reasoning with defeasible background assumptions | "$p$" is a background assumption, but "$p$" is false |
| Deontic Logic | Reasoning with prima facie obligations | "$q$" is a prima facie obligation, but not an actual one |
| Predicate Logic | Inductive Generalization | there is an object that has property "$R$", but not every object has property "$R$" |
| Predicate Logic | Singular Fact Abduction | "$\forall x(Px \supset Qx)$" and "$Qa$" are true, but "$Pa$" is false |

Table 1.1: Some examples of defeasible reasoning forms, and examples of corresponding abnormalities.

adaptive logics allow us to model cases in which qualitatively distinct reasoning methods are combined in a prioritized way. A simple example of such a case is the reasoning of Tim's son in Example 2 above. As noted before, Tim's son uses both inductive inferences, and inferences on the basis of what his father tells him. If we model this example in terms of an adaptive logic, we can distinguish between these (defeasible) types of inferences, and attach different priority degrees to them. Again, how this can be done precisely, will be explained in subsequent chapters.

## 1.3 Prioritized Adaptive Logics: A Gap

The term adaptive logics (henceforth ALs) refers to a very broad class of formal systems developed in the field of philosophical logic. The first adaptive logics were designed by Diderik Batens, to capture non-monotonic reasoning with inconsistent premises – see e.g. [10, 9, 12, 30]. These are nowadays called *inconsistency-adaptive logics* – an example of them will be given in Chapter 2. It soon became clear that the general idea behind inconsistency-ALs (cf. infra) could be applied to several other types of inference – most often types which were considered to be beyond the scope of formal logic, in view of their non-monotonicity and dynamic flavor.

In recent years, scholars have developed ALs for numerous forms of human reasoning: inductive generalization [29, 17, 13, 24, 101], abduction [111, 106, 107, 108, 58], reasoning on the basis of conflicting norms [109, 138, 140, 35], factual detachment in a deontic context [139], abstract argumentation [144, 142, 143], reasoning with vague premises [156, 162, 154, 152, 153], analogical reasoning [104], revision of conceptual knowledge in view of anomalies [147], etc.[6] In addition, many consequence relations from the literature have been reformulated as ALs, see e.g. [31, 34, 136, 164].

The consequence relations of ALs are often non-monotonic and usually lack a positive test.[7] However, this is not a matter of principle. Rather, what is crucial is that ALs model both the internal and external dynamics of defeasible reasoning processes, by means of a dynamic proof theory (cf supra).

Roughly speaking, one may discern two complementary aspects of the adaptive logic program. First of all, it has aimed to characterize particular reasoning methods in terms of concrete adaptive logics, with special attention to their dynamic character. This research went hand in hand with the philosophical analysis of those reasoning methods, and detailed case-studies from the history of science – see e.g. [102, 103, 58, 125, 148].

Second, more theoretic work focused on the overall structure of adaptive logics, which eventually led to the formulation of a so-called *standard format* [16, 11, 21, 33]. This not only simplified the metatheory of the particular ALs, but also paved the road to new applications, and to the incorporation of other consequence relations from the literature. As a result, the standard format provides a unifying framework for the study of a great variety of human reasoning forms. Such unification has several advantages:

(i) It allows us to compare various systems from a structural point of view, and with respect to their application.
(ii) It also allows us to combine different logics in a very straightforward way – some ways to combine ALs in standard format will be discussed in subsequent chapters.
(iii) Once we have developed a single logic for a specific application, we can easily vary some of its components while staying within the same general framework. Hence we can adapt the logic to more specific circumstances, and distinguish between several variants of a reasoning method, or several ways to obtain a specific goal. An illustration of this is provided in Chapter 9.

Moreover, the focus on the general structure of ALs has led to a very rich body of concepts that allows us to discuss various aspects of the defeasibility of human reasoning.

Every adaptive logic in standard format is characterized by a triple: a lower limit logic (henceforth LLL), a set of abnormalities (usually denoted by $\Omega$) and a strategy. The LLL is a monotonic logic, the rules of which are unconditionally valid in the AL. The AL strengthens its LLL by considering a certain set of formulas (the elements of $\Omega$) as abnormal, and by interpreting premises "as

[6] I refer to [33] for a longer list of references to applications of adaptive logics. Unpublished papers in the bibliography (and many others) are available from the internet address `http://logica.UGent.be/centrum/writings/`.

[7] There is a positive test for a given property and a class of objects, iff there is a mechanical procedure that leads to the answer YES whenever a particular object has the property. If the property does not hold, the procedure may lead to the answer NO or may continue ad infinitum. Note that decidability implies that there is a positive test.

normally as possible". For all ALs in standard format, it is required that the abnormalities are specified in terms of a logical form. The precise interpretation of the phrase "as normally as possible" depends on the strategy of the AL – again, all this will be specified in Chapter 2. As we will see there, every AL in standard format is sound and complete with respect to a static selection semantics in the vein of Shoham [133], and has a number of nice properties such as Idempotence, Cautious Monotonicity, and Cumulative Transitivity.[8]

However, so far, the standard format is only able to deal with non-prioritized, or "flat" defeasible reasoning forms. To model forms of PDR, it seems that one has to trespass the safe boundaries of the standard format. In the remainder, I will therefore often use the term "flat adaptive logics" to refer to adaptive logics in standard format.

This does not imply that PDR has been neglected by adaptive logicians – quite to the contrary. Various ALs were developed to capture reasoning with prioritized belief bases (see e.g. [32, 164, 163]). There are also examples in the literature of prioritized logics for inductive generalization [24] and prioritized inconsistency-adaptive logics [22, 19]. Examples of prioritized ALs that combine qualitatively distinct reasoning methods can be found in [105, 91, 139, 144]. Finally, in [17, 29, 13, 24], some logics are presented to formalize the interaction of induction and background assumptions. For the time being, I will use the term *prioritized adaptive logics* (PALs) in a loose way to denote these systems. That is, PALs are logics that bear a significant number of similarities with ALs in standard format, but are developed to capture forms of prioritized defeasible reasoning. What these similarities exactly are, will become clear in subsequent chapters.

While ALs in standard format are well-studied, the existing PALs have been comparatively neglected from the meta-theoretic point of view.[9] In the literature, many ALs that deal with prioritized reasoning are defined in terms of a sequential superposition of flat ALs – how this is done exactly, will be explained in Chapter 3. Other systems, e.g. those from [164, 29] were defined in a more direct way, i.e. in terms of a proof theory and semantics that have a prioritized flavor. However, the metatheory of these logics has remained largely unknown, and more recently it turned out that they have several disadvantages – more on this in Chapter 3. Moreover, as will be shown in Chapters 4 and 5, there are several other ways to characterize prioritized reasoning by adaptive logic tools, which score at least as good as superpositions of ALs in various respects.

This explains the need to investigate these formats, and to try to achieve the same level of insight into their metatheory, as that obtained for the standard format. Such research is motivated by the same reasons as the research on the standard format: if successful, it warrants the well-behavedness of PALs; it provides a unifying framework for PALs (with all the advantages mentioned above); it allows us to focus on the properties that are typical for prioritized defeasible reasoning in general, and to make various distinctions on a more abstract level.

There are two additional reasons why we should take the metatheory of PALs

[8]The precise meaning of these terms is explained in Chapter 2.

[9]Notable exceptions are [25] and [137] – both are of very recent vintage and will be often referred to in the remainder of this thesis.

serious. First of all, as will be explained in the next section, the current PALs are restricted in various respects. In order to remove one or more of these restrictions, we should first get to know better the internal structure, power and limitations of the current systems. I will argue at the end of Chapter 6 that, on the basis of the results presented here, we can think of various ways to further extend the scope of adaptive logics. Second, as I will argue especially in Chapter 10, the existing formats of PALs are not only interesting for the development of adaptive logics; some of the ideas that underlie them might also be used in other formalisms.

## 1.4 The Subject of this Thesis

As indicated at the start of this chapter, I will be mostly concerned with the metatheory of PALs in this thesis (Part I). That is, I will compare these logics to adaptive logics in standard format, focusing on their proof theory, their semantic chacterization(s) and a list of meta-theoretic properties that are known to hold for all flat ALs. This will be done by means of a study of three generic formats of such logics.

After that, I will also consider the application of PALs in three contexts: prioritized normative reasoning, abduction, and the revision of a prioritized set of beliefs (Part II). These applications substantiate the claim that the various formats of PALs from Part I are applicable in a broad range of contexts. A more detailed overview of the other chapters is provided below. However, let me first point at some restrictions of the research presented in the current thesis.

Some features of (prioritized) ALs will remain largely outside the scope of the current study, e.g. the alternative dynamic semantics for ALs in terms of blocks [7, 8], the proof-procedures for final derivability [18, 160], the game-theoretic interpretations of dynamic proofs [23, 158], and issues concerning the computational complexity of ALs [157, 26] – the list is by far not complete. The research on ALs has been very successful over the past few decades, and it is impossible to even explain all these results in one thesis, let alone to see whether they can be extended to the prioritized case.

More generally, some aspects of prioritized defeasible reasoning will not be studied at length in this thesis. First of all, the existing PALs can only handle forms of PDR in which the priorities are ordered by a well-founded, modular order. In the context of prioritized information (obligations, beliefs, etc), this means that we can divide the information into distinct layers, each associated with a priority degree. In the context of combining various (qualitatively distinct) methods in a prioritized way, this means that each method receives a place of its own in a fixed hierarchy.[10]

Needless to say, this restriction implies that various applications call for other logics, and possibly also extensions of the existing formats. For instance, we may have three obligations $p, q, r$, where we only know for sure that $p$ is more important than $q$, but we have no information on the relation between $p$ and $r$, or between $q$ and $r$. The formats I will consider in this thesis seem unable to

[10] For a general definition of the concept of a modular order, see e.g. [88].

adequately represent such situations, and to deal with them in an appropriate way.

A second restriction, which PALs share with most of the existing approaches to PDR, concerns the dynamics of priorities. As argued e.g. in [123], not only the conclusions we can or should draw, but also the priorities themselves may change as a reasoning process proceeds. For instance, we may start off with a "flat" belief base, and only when it turns out that some beliefs are problematic in view of the new information, attach different degrees of reliability to them. Although such dynamics are particularly interesting from the philosophical point of view, there has been no successful way to deal with them in terms of adaptive logics, so far.

I will return to these restrictions in Chapter 6, where I indicate how one might tackle some of them on the basis of the present results.

## 1.5 Overview of the Other Chapters

**Part I.** Chapter 2 gives a detailed account of the semantics, proof theory and metatheory of the standard format. This chapter contains no new material, but is indispensible for several reasons: it introduces concepts that are used throughout the rest of the thesis (including the notion of an adaptive proof and the definition of a selection semantics), and numerous important metatheorems are stated here for further reference.

In the three subsequent chapters, I will discuss three generic formats of prioritized ALs: sequential superpositions (Chapter 3), hierarchic ALs (Chapter 4) and lexicographic ALs (Chapter 5). I will present semantic characterizations and proof theories for each of these, and consider a fixed list of metatheoretic properties, comparing the results with those of the standard format and the other formats of prioritized ALs. As the reader will note, Chapters 2-5 do not have a concluding section. A summary of the results of these chapters is presented at the end of Chapter 6.

In Chapter 6, I will compare the various formats of prioritized adaptive logics in terms of their logical strength, and the relation between their respective semantic characterizations. This chapter ends with an extensive conclusion and outlook to further research, on the basis of the technical results from Chapters 3-6.

**Part II.** In Chapter 7, the prioritized deontic logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ is presented. This logic allows us to derive a set of actual and "unconflicted" obligations from a (possibly conflicting) ordered set of prima facie obligations. The $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-consequence relation is compared to several other systems from the literature by means of concrete examples.

Chapter 8 discusses two logics for a specific kind of abduction, called *Abduction of Generalizations*. The logics allow us to derive generalizations that explain facts such as "all chocolate tastes sweet" or "all iron tools in this garage are corroded". Special attention is paid to a reconstruction of this pattern in terms of a (prioritized) combination of singular fact abduction and inductive generalization.

Chapter 9 is not about prioritized defeasible reasoning, but is used as a stepping stone to Chapter 10. In Chapter 9, a specific notion of relevance in the theory of belief change is explained, and it is shown how various adaptive logics allow us to model the internal and external dynamics of relevant belief revision.

Finally, Chapter 10 discusses the notion of relevance in a prioritized context, and explains how we can generalize the logics from Chapter 9 in order to deal with prioritized belief bases in an accurate and relevance-sensitive way. It also shows how the idea of superposing adaptive logics can be translated to the context of belief revision.

**Shortcuts for the Selective Reader.** I am aware that this thesis is fairly lengthy, and that many chapters, especially those in Part I, might be rather hard to digest for the reader who is not acquinted with the formal framework of adaptive logics. I have opted for a compact presentation of this framework, mainly for reasons of space and since my own research is concerned with extensions and applications of it.

For systematic reasons, the three generic formats of prioritized ALs, together with their proof theory, semantics and metatheory are spelled out in the first part of this thesis. In Part II, it will be assumed that the reader is acquinted with the definitions and results from Part I. However, readers who are more interested in specific formats or applications of (prioritized) adaptive logics, may want to restrict themselves to certain chapters. The following selections of chapters are more or less self-contained, and can be read independently from the other chapters:

- Chapters 2-6
- Chapters 2, 5, 7
- Chapters 2, 3, 8
- Chapters 2, 9, Appendix C
- Chapters 2, 3, 9, 10

# Part I

# Generic Formats For Prioritized Adaptive Logics

# Chapter 2

# Flat Adaptive Logics

*I am indebted to Peter Verdée, Mathieu Beirlaen and Joke Meheus for their valuable comments on previous versions of this chapter.*

In this chapter, the standard format of adaptive logics (henceforth ALs) and its most salient properties are explained.[1] This chapter contains no new material, but merely summarizes and illustrates results from the literature and highlights some specific facts that are relevant for the rest of this thesis. I refer to the available literature for more details, examples and meta-theoretic proofs.[2]

The outline of this chapter is as follows. I will first present the basic ingredients of every adaptive logic, and explain the general idea behind each of these (Section 2.1). Next, I will show how we can use them to obtain a selection semantics (Section 2.2), and a dynamic proof theory (Section 2.3). Both semantics and syntax of the standard format are illustrated by means of two simple examples in Section 2.4. In Section 2.5, various properties of the consequence relations of flat adaptive logics are spelled out.

After this general introduction to the standard format, I will consider a few specific topics that are of particular interest for the study of prioriotized adaptive logics. First, some observations will be made about the relation between the two strategies of flat adaptive logics in standard format (Section 2.6). Second, I will discuss the role of the so-called "checked connectives" in adaptive proofs, and establish some properties concerning a specific class of premise sets that have such connectives in them (Section 2.7). The chapter ends with a couple of generic lemmas that will shorten some proofs in subsequent chapters (Section 2.8).

**Some Conventions** Before we continue, let me introduce some general conventions that will be used in the remainder of this thesis. All formulas will be

[1] Recall that I use the name flat ALs to denote ALs in standard format, in contradistinction to the prioritized ALs introduced in subsequent chapters.

[2] Especially [15] and [21] can serve as introductions to the standard format. In [25], the reader can find numerous examples and an extensive discussion of all the important results in the field. Another highly recommendable piece of work is the recent [137], in which a wide variety of applications of the standard format is studied.

assumed to be finite strings in a recursively enumerable language. I will use $A, B, C, \ldots$ as metavariables for formulas and $\Gamma, \Delta, \Theta, \ldots$ as metavariables for sets of formulas.[3] Where $\mathbb{N} = \{1, 2, \ldots\}$, I will use $i, j, k, \ldots$ as metavariables for members of $\mathbb{N}$, and $I, J, K, \ldots$ as metavariables for initial subsequences of $\mathbb{N}$.

In accordance with the traditional style of the Ghent group, I will use $\mathcal{L}, \mathcal{L}', \ldots$ to refer to specific languages, and $\mathcal{W}, \mathcal{W}', \ldots$ to refer to the associated sets of formulas. In this thesis, it is assumed that languages and sets of formulas are infinite. Table A.1 in Appendix A gives an overview of all the logics and languages used in this thesis.

I will use $\Gamma \vdash_{\mathbf{L}} A$ to denote that $A$ is $\mathbf{L}$-derivable from $\Gamma$. Let $Cn_{\mathbf{L}}(\Gamma) =_{\mathsf{df}} \{A \mid \Gamma \vdash_{\mathbf{L}} A\}$. Logic $\mathbf{L}$ are conceived as functions that map every premise set $\Gamma$ to a consequence set $Cn_{\mathbf{L}}(\Gamma)$. I write $\Gamma \vdash_{\mathbf{L}} \Delta$ whenever $\Gamma \vdash_{\mathbf{L}} A$ for every $A \in \Delta$, and $\Gamma \dashv\vdash_{\mathbf{L}} \Delta$ as an abbreviation for ($\Gamma \vdash_{\mathbf{L}} \Delta$ and $\Delta \vdash_{\mathbf{L}} \Gamma$).

Where $M$ is an $\mathbf{L}$-model, $M \Vdash A$ denotes that $A$ is valid in $M$. $M$ is an $\mathbf{L}$-model of $\Gamma$ iff it is an $\mathbf{L}$-model and $M \Vdash A$ for all $A \in \Gamma$. The set of $\mathbf{L}$-models of $\Gamma$ is denoted by $\mathcal{M}_{\mathbf{L}}(\Gamma)$. $A$ is a semantic $\mathbf{L}$-consequence of $\Gamma$, $\Gamma \models_{\mathbf{L}} A$ iff $A$ is verified by all $\mathbf{L}$-models of $\Gamma$.

Let $\mathcal{L}_{\mathbf{L}}$ be the language associated with the logic $\mathbf{L}$. Then $\mathbf{L}$ is called a *Tarski-logic* iff the following holds for every $\Gamma \subseteq \mathcal{W}_{\mathbf{L}}$:

*Reflexivity*: $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$

*Transitivity*: if $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$, then $Cn_{\mathbf{L}}(\Gamma') \subseteq Cn_{\mathbf{L}}(\Gamma)$

*Monotonicity*: $Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$

It is common knowledge that every Tarski-logic $\mathbf{L}$ has the *Fixed Point* property, or alternatively, that Tarski-logics are *idempotent*: $Cn_{\mathbf{L}}(Cn_{\mathbf{L}}(\Gamma)) = Cn_{\mathbf{L}}(\Gamma)$. $\mathbf{L}$ is said to be *compact* whenever the following holds for every $\Gamma \subseteq \mathcal{W}_{\mathbf{L}}$: if $\Gamma \vdash_{\mathbf{L}} A$, then there are $B_1, \ldots, B_n \in \Gamma$, such that $\{B_1, \ldots, B_n\} \vdash_{\mathbf{L}} A$. A premise set $\Gamma \subseteq \mathcal{W}_{\mathbf{L}}$ is $\mathbf{L}$-*trivial* iff $Cn_{\mathbf{L}}(\Gamma) = \mathcal{W}_{\mathbf{L}}$.

Let $\mathcal{L}$ be the language associated with both $\mathbf{L}$ and $\mathbf{L}'$. $\mathbf{L}$ is *at least as strong* as $\mathbf{L}'$ iff for every $\Gamma \subseteq \mathcal{W}$, $Cn_{\mathbf{L}'}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma)$. $\mathbf{L}$ is *stronger than* $\mathbf{L}'$ ($\mathbf{L}'$ is *weaker than* $\mathbf{L}$) iff $\mathbf{L}$ is *at least as strong* as $\mathbf{L}'$ and for some $\Gamma \subseteq \mathcal{W}$, $Cn_{\mathbf{L}'}(\Gamma) \subset Cn_{\mathbf{L}}(\Gamma)$.

As we will see below, (propositional) classical logic (henceforth $\mathbf{CL}$) takes a special place in the AL framework. In this thesis, I will use $\mathcal{L}_c$ to refer to the language of propositional classical logic, which is obtained by adding the regular connectives $\bot, \neg, \wedge, \vee, \supset, \equiv$ to the set of sentential letters, $\mathcal{S} =_{\mathsf{df}} \{p, q, r, \ldots\}$.[4] The associated set of formulas will be denoted by $\mathcal{W}_c$. The set of *literals* of $\mathcal{W}_c$ is denoted by $\mathcal{W}_c^l =_{\mathsf{df}} \mathcal{S} \cup \{\neg A \mid A \in \mathcal{S}\}$. A full axiomatization of $\mathbf{CL}$ can be found in Appendix B.

## 2.1 General Characteristics of AL

As mentioned in Chapter 1, every logic $\mathbf{AL}$ is characterized by a triple:

[3] For some very specific sets of formulas, I will violate these conventions in order to stay in line with the vocabulary that is common in the adaptive logic program. Such violations will always be made explicit.

[4] As usual, $\top =_{\mathsf{df}} \bot \supset \bot$.

1. A *lower limit logic* **LLL**: a compact Tarski-logic that has a proof theory and a characteristic semantics, and that includes classical logic
2. A *set of abnormalities* $\Omega$: a set of formulas, characterized by a (possibly restricted) logical form; or a union of such sets[5]
3. An *adaptive strategy*: Reliability or Minimal Abnormality

The strategy is indicated by a superscript: $\mathbf{AL^r}$ for ALs that have Reliability as their strategy, $\mathbf{AL^m}$ for those that have Minimal Abnormality as strategy. Many definitions and theorems are applicable to both classes of logics – in that case, the generic name **AL** is used. I will also sometimes write $\mathbf{AL^x}$, where $\mathbf{x}$ can be either $\mathbf{r}$ or $\mathbf{m}$. The role of the strategy of an AL will become clear in the next two sections, where the semantics and proof theory of $\mathbf{AL^r}$, resp. $\mathbf{AL^m}$ are defined.

Some examples might help the reader to grasp the intuitive meaning of each of the first two elements of a flat adaptive logic. For instance, inconsistency-adaptive logics usually have a paraconsistent lower limit logic, i.e. a logic that does not trivialize (all) formulas of the form $A \wedge {\sim}A$ (I use $\sim$ for the paraconsistent negation, not to be confused with the classical negation $\neg$). The set of abnormalities of a prototypical inconsistency-AL contains all formulas of the form $A \wedge {\sim}A$. An example of such a system is the logic $\mathbf{CLuN^m}$, the propositional fragment of which is presented at length in Section 2.4. $\mathbf{CLuN^m}$ interprets premises "as consistently as possible". To see what this amounts to, consider the following facts:

- $\{p, {\sim}p\} \nvdash_{\mathbf{CLuN^m}} q$
- $\{p, {\sim}p, {\sim}p \vee q\} \nvdash_{\mathbf{CLuN^m}} q$
- $\{p, {\sim}p \vee q\} \vdash_{\mathbf{CLuN^m}} q$
- $\{p, {\sim}p, q, {\sim}q \vee r\} \vdash_{\mathbf{CLuN^m}} r$

In view of the first fact, $\mathbf{CLuN^m}$ does not trivialize inconsistent premise sets. In view of the other three, it only validates disjunctive syllogism in a "case-sensitive" way.

Another simple example is the logic $\mathbf{IL^r}$ of inductive generalization from [29] – we will encounter a variant of this system in Chapter 8. Roughly speaking, the lower limit logic of $\mathbf{IL^r}$ is a classical predicate logic, and its set of abnormalities contains all formulas of the form $\exists x Ax \wedge \exists x \neg Ax$. That is, from the viewpoint of a logic of induction, it counts as an abnormality that one object has a given property $A$, whereas another object does not have this property. This means that $\mathbf{IL^r}$ interprets premises "as uniformly as possible", or in other words, that it validates the rule $\exists x Ax \supset \forall x Ax$ "as much as possible". Again, this can be illustrated by some simple examples:

- $\{Pa\} \vdash_{\mathbf{IL^r}} \forall x Px$
- $\{Pa, \neg Pb\} \nvdash_{\mathbf{IL^r}} \forall x Px$
- $\{Pa, Qa, \neg Pb, Qb\} \vdash_{\mathbf{IL^r}} \forall x Qx$
- $\{Pa, Qa, \neg Pb, \neg Qb\} \vdash_{\mathbf{IL^r}} \forall x(Px \supset Qx), \forall x(Qx \supset Px)$

[5]Since it is assumed that $\mathcal{L}$ is infinite, $\Omega$ will also typically be an infinite set.

In this case, the adaptive logic allows us to derive universal generalizations, unless this leads to contradictions. How this is possible, will be explained below.

Let us now consider each of the three elements of every AL in standard format from a more abstract point of view, starting with the lower limit logic **LLL**. This logic is obtained from a Tarski-logic $\mathbf{LLL_s}$, which is often a well-known system from the literature – examples are **CL** (for ALs of induction and abduction), the paraconsistent logic **CLuN** (for inconsistency-adaptive logics) or a non-aggregative deontic logic such as Goble's logic **P** (for deontic adaptive logics). In this thesis, Kripke's modal logic **K** will also often serve as lower limit logic of adaptive systems. Henceforth, let $\mathcal{L}_s$ be the language of $\mathbf{LLL_s}$, and let $\mathcal{W}_s$ be the set of closed formulas of $\mathcal{L}_s$.

As mentioned in Chapter 1, the logic **AL** enriches its lower limit logic, by considering certain formulas as abnormal, and by avoiding abnormalities "as much as possible". Under one interpretation of this phrase, abnormalities are considered to be false by the adaptive logic, unless they are part of a minimal (classical) disjunction of abnormalities that is $\mathbf{LLL_s}$-derivable from the premises – this interpretation corresponds to the Reliability Strategy (see below). Another interpretation is slightly more complex, but also makes use of classical disjunctions that are $\mathbf{LLL_s}$-derivable from the premise set.

However, such a construction requires that $\mathbf{LLL_s}$ ranges over classical disjunctions. Also, to express the falsehood of an abnormality, we need a classical negation. Since the standard format has the aim to encompass a very broad range of systems, in which the connectives of $\mathbf{LLL_s}$ can behave non-classically in several ways, it is convenient to add a layer of classical connectives to the lower limit logic, which can be used in the metatheory and generic definitions of ALs. As will be explained in Section 2.7, this small amendment is also indispensable to obtain a specific kind of completeness[6] for all adaptive logics in standard format (even if the connectives of $\mathbf{LLL_s}$ behave classically).

The additional connectives are marked by a check: $\check{\neg}, \check{\vee}, \check{\wedge}, \check{\supset}, \check{\equiv}$, and are usually referred to as the *checked connectives*.[7] The language $\check{\mathcal{L}}_s$ is obtained by extending $\mathcal{L}_s$ with the checked connectives, where it is assumed that these symbols are not in $\mathcal{L}_s$. $\check{\mathcal{W}}_s$ is obtained by superimposing the checked connectives on $\mathcal{W}_s$, i.e. $\check{\mathcal{W}}_s$ is the smallest set such that:

(i) $\mathcal{W}_s \subset \check{\mathcal{W}}_s$
(ii) where $A, B \in \check{\mathcal{W}}_s$: $\check{\neg} A, A \check{\vee} B, A \check{\wedge} B, A \check{\supset} B, A \check{\equiv} B \in \check{\mathcal{W}}_s$

Unless specified differently, I will henceforth use $\Gamma$ as a metavariable for subsets of $\check{\mathcal{W}}_s$.

To model inferences on the basis of $\check{\mathcal{L}}_s$, $\mathbf{LLL_s}$ is upgraded to $\mathbf{LLL} : \wp(\check{\mathcal{W}}_s) \rightarrow \wp(\check{\mathcal{W}}_s)$. As will become clear, **LLL** is a conservative extension of $\mathbf{LLL_s}$, i.e. for every $\Gamma \subseteq \mathcal{W}_s$, $Cn_{\mathbf{LLL_s}}(\Gamma) \cap \mathcal{W}_s = Cn_{\mathbf{LLL}}(\Gamma) \cap \mathcal{W}_s$. How **LLL** is obtained precisely, requires some explanation.

[6]To be precise: for all $\Gamma \subseteq \mathcal{W}_s$, $\Gamma \vdash_{\mathbf{AL}} A$ iff $\Gamma \models_{\mathbf{AL}} A$. However, for some $\Gamma \subseteq \check{\mathcal{W}}_s$ and $A \in \check{\mathcal{W}}_s$, $\Gamma \nvdash_{\mathbf{AL}} A$ whereas $\Gamma \models_{\mathbf{AL}} A$. So the completeness of ALs is restricted to premise sets that contain no checked connectives. Why this is so, is explained in Section 2.7.

[7]Obviously, the connectives $\check{\wedge}$ and $\check{\supset}$ can be defined from $\check{\vee}$ and $\check{\neg}$. It is not necessary to add the connective $\check{\bot}$ to the language of $\mathbf{LLL_s}$.

For the **LLL**-semantics, we define a model validity relation $\Vdash$ that extends the validity relation $\Vdash_s$ of $\mathbf{LLL_s}$, as follows. Let $M$ be an $\mathbf{LLL_s}$-model. Define (1) for all $A \in \mathcal{W}_s$: $M \Vdash A$ iff $M \Vdash_s A$, (2) for all $A \in \check{\mathcal{W}}_s$: $M \nVdash A$ iff $M \Vdash \check{\neg} A$, (3) for all $A, B \in \check{\mathcal{W}}_s$: ($M \Vdash A$ or $M \Vdash B$) iff $M \Vdash A \check{\vee} B$, and likewise for the other checked connectives. Henceforth, we say that $M$ is an **LLL**-model of $\Gamma \subseteq \check{\mathcal{W}}_s$, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ iff $M$ is an $\mathbf{LLL_s}$-model and $M \Vdash A$ for every $A \in \Gamma$. We write $\Gamma \models_{\mathbf{LLL}} A$ iff for all **LLL**-models $M$ of $\Gamma$: $M \Vdash A$.

In the triple characterization of **AL**, a sound and complete axiomatization for **LLL** is presupposed. Where $\mathbf{LLL_s}$ is supraclassical, one can obtain the axiomatization for **LLL** by a generic procedure – it suffices to make each checked classical connective equivalent to its classical counterpart in $\mathcal{L}_s$. This will be illustrated in Section 2.4.2. However, if $\mathbf{LLL_s}$ has weak or very non-standard connectives, it becomes a lot tougher to find a generic procedure that gives a sound and complete axiomatization for **LLL**. Nevertheless, for concrete cases, the adaptive logician's job of devising a syntax for **LLL** will usually be fairly easy, as will be illustrated in Section 2.4.1.

Every logic **AL** is a function $\wp(\check{\mathcal{W}}_s) \to \wp(\check{\mathcal{W}}_s)$. Since **AL** was intended to explicate defeasible reasoning processes on the basis of premises in $\mathcal{L}_s$, premises of **AL** are often assumed to be subsets of $\mathcal{W}_s$. One possible interpretation of the relation between **AL**, $\mathcal{L}_s$ and $\check{\mathcal{L}}_s$ is that **AL** provides an explication of a reasoning based on formulas in $\mathcal{L}_s$, but that for this explication, it uses formulas in $\check{\mathcal{L}}_s$ — this will become clear when we present the **AL**-proof theory.

The set of abnormalities $\Omega \subseteq \check{\mathcal{W}}_s$ represents those formulas that **AL** assumes to be false "as much as possible", in view of the premises. As we saw before, the phrase "as much as possible" can have various interpretations – every such interpretation corresponds to an adaptive strategy.[8]

Every flat AL also has an *upper limit logic* **ULL** : $\wp(\check{\mathcal{W}}_s) \to \wp(\check{\mathcal{W}}_s)$, which is obtained by considering all abnormalities to be false. In the remainder of this thesis, let $\Theta^{\check{\neg}} =_{\mathsf{df}} \{\check{\neg} A \mid A \in \Theta\}$ for any $\Theta \subseteq \check{\mathcal{W}}_s$. Syntactically, **ULL** is defined as follows: $\Gamma \vdash_{\mathbf{ULL}} A$ iff $\Gamma \cup \Omega^{\check{\neg}} \vdash_{\mathbf{LLL}} A$. Semantically, we speak of *normal models* as those **LLL**-models $M$ for which $M \Vdash \check{\neg} A$ for every $A \in \Omega$. $\Gamma$ is a *normal premise set* iff it has normal models; alternatively, iff $\Gamma \cup \Omega^{\check{\neg}}$ is **LLL**-satisfiable. Finally, $\Gamma \models_{\mathbf{ULL}} A$ iff for every normal model $M$ of $\Gamma$, $M \Vdash A$. Note that in view of these definitions, the upper limit logic trivializes every premise set $\Gamma$ that entails a (classical) disjuntion of abnormalities, since for all such $\Gamma$, every **LLL**-model of $\Gamma$ verifies at least one abnormality.

## 2.2 The AL-Semantics

As stipulated at the start of this chapter, the semantic consequence relations of every logic **L** in this thesis is defined as a function of the set of all **L**-models of a premise set. Hence, it suffices to specify what the **AL**-models of a given premise set $\Gamma$ are, in order to obtain the relation $\models_{\mathbf{AL}}$.

[8]Other strategies than Reliability and Minimal Abnormality are e.g. Counting, Normal Selections and the Flip-Flop-Strategy. These are strictly speaking not part of the standard format, but can be obtained from it under a translation – see [25, Chapter 6].

As mentioned in the introduction, ALs have a semantics similar to Shoham's preferential semantics [133]: from the set of all **LLL**-models, **AL** selects a subset. In most interesting cases, the set inclusion is proper. Also, whenever $\Gamma$ has normal models, then **AL** will select only these models.

Before we can explain the exact selection procedure, we first need a few extra definitions. Where $\Delta$ is a finite subset of $\Omega$, $Dab(\Delta) =_{\mathsf{df}} \check{\bigvee}\Delta$ is called a *Dab-formula*. Where $\Delta = \{A\}$, $Dab(\Delta)$ denotes $A$; where $\Delta = \emptyset$, $Dab(\Delta)$ denotes the empty string. Where $\Delta \neq \emptyset$, $Dab(\Delta)$ is a *minimal Dab-consequence of* $\Gamma$ iff $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ for which $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta')$.

Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-consequences of $\Gamma$, let $\Sigma(\Gamma) = \{\Delta_1, \Delta_2, \ldots\}$. We say that $U(\Gamma) =_{\mathsf{df}} \bigcup\Sigma(\Gamma)$ is the set of *unreliable* formulas with respect to $\Gamma$. In view of the preceding definitions and the fact that **LLL** is idempotent, we can infer:

**Fact 2.1** $\Sigma(Cn_{\mathbf{LLL}}(\Gamma)) = \Sigma(\Gamma)$, *whence also* $U(Cn_{\mathbf{LLL}}(\Gamma)) = U(\Gamma)$ *and* $\Phi(Cn_{\mathbf{LLL}}(\Gamma)) = \Phi(\Gamma)$.

Where $M$ is an **LLL**-model, let $Ab(M) =_{\mathsf{df}} \{B \in \Omega \mid M \Vdash B\}$. We call $Ab(M)$ the *abnormal part* of $M$. Every logic **AL** selects a subset of models $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$, in view of their abnormal part. The precise criterion for a model to be selected depends on the strategy:

**Definition 2.1** $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}}(\Gamma)$ *iff* ($M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ *and* $Ab(M) \subseteq U(\Gamma)$).

**Definition 2.2** $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}}(\Gamma)$ *iff* ($M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ *and there is no* $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ *such that* $Ab(M') \subset Ab(M)$).

$\mathcal{M}_{\mathbf{AL}^{\mathbf{r}}}(\Gamma)$ is called the set of *reliable* models, $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}}(\Gamma)$ the set of $\subset$-*minimally abnormal* models, or more briefly, *minimally abnormal* models.[9]

Although the above definition of $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}}(\Gamma)$ is more direct, we can also define the semantics of Minimal Abnormality in terms of the minimal *Dab*-consequences of $\Gamma$. This requires some notational preparation. Let $\Psi = \{\Delta_i \subseteq \Omega \mid i \in I\}$ for a given $I \subseteq \mathbb{N}$. We say that $\varphi \subseteq \Omega$ is a *choice set* of $\Psi$ iff for every $i \in I$, $\varphi \cap \Delta_i \neq \emptyset$. For the border case where $\Psi = \emptyset$, this means that every set $\varphi \subseteq \Omega$ is a choice set of $\Psi$, including the empty set.

$\varphi$ is a $\subset$-*minimal* choice set of $\Psi$ iff there is no choice set $\psi$ of $\Psi$ such that $\psi \subset \varphi$. In the context of the standard format, we speak of "minimal choice sets" to refer to "$\subset$-minimal choice sets". The following is proven in [25, Chapter 5]:

**Fact 2.2** *If every* $\Delta \in \Psi$ *is finite, then* $\Psi$ *has minimal choice sets.* [25, Fact 5.2.1]

$\Phi(\Gamma)$ is the set of minimal choice sets of $\Sigma(\Gamma)$. Note that when $\Sigma(\Gamma) = \emptyset$, $\Phi(\Gamma) = \{\emptyset\}$. It is easily provable that $U(\Gamma) = \bigcup\Phi(\Gamma)$. Also, remark that since all the members of $\Sigma(\Gamma)$ are finite, $\Phi(\Gamma) \neq \emptyset$ for every $\Gamma \subseteq \check{\mathcal{W}}_s$ by Fact 2.2. The following theorems are immediate consequences of Lemma 4 from [21]:

[9] Readers that are familiar with the standard format might wonder why the reference to the subset-relation is made explicit – this will become clear in Chapter 5, where I will also speak of "$\sqsubset$-minimal" abnormal models.

**Theorem 2.1** *$M \in \mathcal{M}_{\mathbf{AL^m}}(\Gamma)$ iff ($M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $Ab(M) \in \Phi(\Gamma)$).*

**Theorem 2.2** *If $\Gamma$ has **LLL**-models, then $\Phi(\Gamma) = \{Ab(M) \mid M \in \mathcal{M}_{\mathbf{AL^m}}(\Gamma)\}$.*

Note that in view of Theorem 2.1 and the preceding paragraph, $Ab(M) \in \Phi(\Gamma)$ implies that $Ab(M) \subseteq U(\Gamma)$. Also, $U(\Gamma) = \{A \in \Omega \mid M \Vdash A \text{ for an } M \in \mathcal{M}_{\mathbf{AL^m}}(\Gamma)\}$. It follows immediately that every minimally abnormal model is a reliable model:

**Theorem 2.3** $\mathcal{M}_{\mathbf{AL^m}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL^r}}(\Gamma)$

As can be observed from the above definitions, the **AL**-semantics is fairly straightforward: we specify a selection criterion in terms of abnormal parts of models, and we select all the **LLL**-models of $\Gamma$ that obey this criterion. The real challenge is to define a *proof theory*, i.e. a procedure in terms of inference rules and an associated notion of derivability, that matches this semantic consequence relation. This proof theory will be presented below. To see how it relates to the semantic consequence relation, the following two theorems are of particular interest:

**Theorem 2.4** *$\Gamma \models_{\mathbf{AL^r}} A$ iff there is a $\Delta \subset \Omega$ such that $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$*

**Theorem 2.5** *$\Gamma \models_{\mathbf{AL^m}} A$ iff for every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\varphi \cap \Delta = \emptyset$ and $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$*

In the next section, we will see criteria for the membership of $Cn_{\mathbf{AL^r}}(\Gamma)$, resp. $Cn_{\mathbf{AL^m}}(\Gamma)$, which parallel the above two theorems – see Theorems 2.6 and 2.7.

## 2.3 Proof Theory of AL

The adaptive proof theory mirrors the dynamic character of defeasible reasoning forms, as discussed in Chapter 1: not only can new lines be added to any proof (as is the case in any proof theory), but also, some derivations can be canceled or retracted in view of other derivations in the proof. Every **AL**-proof consists of lines that have four elements: a line number $i$, a formula $A$, a justification (consisting of a series of line numbers and a derivation rule) and a condition $\Delta \subseteq \Omega$. Where $\Gamma$ is the set of premises, the inference rules are given by:

PREM If $A \in \Gamma$:
$$\begin{array}{ll} \vdots & \vdots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:
$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Theta)$:
$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

The rule PREM states that a premise may be introduced at any line of a proof on the empty condition. The unconditional rule RU states that, if $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$ and $A_1, \ldots, A_n$ occur in the proof on the respective conditions $\Delta_1, \ldots, \Delta_n$, then we may add $B$ on the condition $\Delta_1 \cup \ldots \cup \Delta_n$. RU also allows us to introduce any **LLL**-theorem $B$ on the condition $\emptyset$.

The strength of an adaptive logic comes with the conditional rule RC, which works analogously to RU, but allows us to "push" abnormalities from the formula to the condition. Put differently, if we can derive the formula $A$ in disjunction with one or more abnormalities, then RC states that we may derive $A$, relying on the (defeasible) assumption that those abnormalities are false.

A *stage* of a proof can be seen as a sequence of lines, obtained by the application of the above rules. $s'$ is an *extension* of a stage $s$ iff every line in $s$ occurs in $s'$. A proof is a sequence of stages $(s, s', s'', \ldots)$. Adding lines to a proof by applying the inference rules brings the proof to a next stage. New lines can be added anywhere in the proof, as long as the inference rules are applied correctly and each line is either obtained by the introduction of a premise or **LLL**-theorem, or from lines that precede it.[10]

In view of the inference rules, the condition of any line $l$ is necessarily finite, and the following lemma holds:

**Lemma 2.1** *There is an* **AL***-proof from* $\Gamma$ *that contains a line at which* $A$ *is derived on a condition* $\Delta \subset \Omega$ *iff* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$*. [21, Lemma 1]*

Lemma 2.1 allows us to introduce a derived rule RD, which can be represented as follows:

RD If $A \vdash_{\mathbf{LLL}} \check{\neg} B$:
$$\begin{array}{ll} A & \Delta \\ B & \Theta \\ \hline Dab(\Delta) \check{\vee} Dab(\Theta) & \emptyset \end{array}$$

[10] It is necessary to allow for inserting lines between existing lines in a proof, in order to be able to extend a stage that consists of infinitely many lines.

That is, if $A$ is derived on the condition $\Delta$ and $B$ on the condition $\Theta$, then by the left-right direction of Lemma 2.1, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Gamma \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Theta)$. But then, if $A \vdash_{\mathbf{LLL}} \check{\neg} B$, it follows that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta) \check{\vee} Dab(\Theta)$. Hence by the right-left direction of Lemma 2.1, we may derive $Dab(\Delta) \check{\vee} Dab(\Theta)$ on the empty condition in an **AL**-proof from $\Gamma$.

A distinctive feature of the adaptive proof theory is the marking definition – see below. At every stage of a proof, this definition determines for each line in the proof whether it is marked or not. If a line that has as its second element $A$ is marked at stage $s$, this indicates that according to our best insights at this stage, $A$ cannot be considered derivable. If the line is unmarked at stage $s$, we say that $A$ is derived at stage $s$ of the proof. To prepare for the marking definitions, we need some more conventions.

Where $\Delta$ is a finite and non-empty subset of $\Omega$, $Dab(\Delta)$ is a Dab-formula at stage $s$ of a proof iff it is the second element of a line at stage $s$ with an empty condition. $Dab(\Delta)$ is a *minimal* Dab-formula at stage $s$ iff there is no other Dab-formula $Dab(\Delta')$ at stage $s$ for which $\Delta' \subset \Delta$. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-formulas at stage $s$ of a proof, let $\Sigma_s(\Gamma) = \{\Delta_1, \Delta_2, \ldots\}$. Let $U_s(\Gamma) =_{\mathsf{df}} \bigcup \Sigma_s(\Gamma)$ and let $\Phi_s(\Gamma)$ be defined as the set of minimal choice sets of $\Sigma_s(\Gamma)$. By Fact 2.2, $\Phi_s(\Gamma) \neq \emptyset$ at every stage $s$ of a proof from $\Gamma$.

**Definition 2.3** $\mathbf{AL^r}$*-Marking: a line $l$ is marked at stage $s$ iff, where $\Delta$ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

**Definition 2.4** $\mathbf{AL^m}$*-Marking: a line $l$ with formula $A$ is marked at stage $s$ iff, where its condition is $\Delta$: (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (ii) for a $\varphi \in \Phi_s(\Gamma)$, there is no line at which $A$ is derived on a condition $\Theta$ for which $\Theta \cap \varphi = \emptyset$.*

Put differently: where the strategy is Minimal Abnormality, a line with formula $A$ is *un*marked at stage $s$ iff its condition has an empty intersection with at least one $\varphi \in \Phi_s(\Gamma)$, and for every $\psi \in \Phi_s(\Gamma)$, there is a line at which $A$ is derived on a condition $\Delta$ such that $\Delta \cap \psi = \emptyset$. The difference between these two marking definitions will be illustrated by means of two simple examples in the next section.

As a line may be marked at stage $s$, unmarked at a later stage $s'$ and marked again at a still later stage $s''$, we also need to define a stable notion of derivability:

**Definition 2.5** *$A$ is* finally derived *from $\Gamma$ on line $l$ of a finite stage $s$ of an* **AL**-*proof iff (i) $A$ is the second element of line $l$, (ii) line $l$ is unmarked at stage $s$, and (iii) every extension of the proof at stage $s$, in which line $l$ is marked may be further extended in such a way that line $l$ is unmarked again.*

**Definition 2.6** $\Gamma \vdash_{\mathbf{AL^x}} A$ *iff $A$ is finally derived on a line of an* $\mathbf{AL^x}$*-proof from $\Gamma$.*

Note that, on the one hand, finiteness of the stage is a prerequisite for final derivability – I return to this point in Section 2.7. On the other hand, extensions may be infinite, and for the Minimal Abnormality Strategy, infinite extensions are necessary in order to warrant soundness of the proof theory.[11]

There is an interesting game-theoretic interpretation of adaptive proofs, in terms of a proponent-opponent dialogue. If $\Gamma \vdash_{\mathbf{AL}} A$, then the proponent only needs a finite proof to argue in favor of $A$. Moreover, whatever the opponent's counterargument may be, the proponent can always reply in such a way that this counterargument is defeated. If $\Gamma \nvdash_{\mathbf{AL}} A$, then either the proponent cannot produce a finite proof for $A$, or if he can, there is always a counterargument the opponent can yield, that defeats every further argumentation pro $A$.

The following two theorems were already announced in Section 2.2. They link the dynamic proof theory nicely to the static semantics for Reliability, resp. Minimal Abnormality, and are therefore crucial for the proofs of soundness and completeness:

**Theorem 2.6** *Each of the following holds:* [21, Th. 6]

1. *If $\Gamma \vdash_{\mathbf{AL^r}} A$, then there is a $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$*
2. *Where $\Gamma \subseteq \mathcal{W}_s$: if there is a $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$, then $\Gamma \vdash_{\mathbf{AL^r}} A$*

**Theorem 2.7** *Each of the following holds:* [21, Th. 8]

1. *If $\Gamma \vdash_{\mathbf{AL^m}} A$, then for every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\varphi \cap \Delta = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$*
2. *Where $\Gamma \subseteq \mathcal{W}_s$: if for every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\varphi \cap \Delta = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, then $\Gamma \vdash_{\mathbf{AL^m}} A$*

Relying on Theorems 2.4 and 2.6, and the soundness and completeness of **LLL**, we can immediately infer that $\mathbf{AL^r}$ is sound and complete. Likewise, by Theorems 2.5 and 2.7 and the soundness and completeness of **LLL**, $\mathbf{AL^m}$ is sound and complete. However, a small warning is in place: note that since the right-left directions of Theorems 2.6 and 2.7 only hold for $\Gamma \subseteq \mathcal{W}_s$, we cannot obtain a general completeness theorem for ALs – this will be explained in Section 2.7.

## 2.4 Two Examples

Before we look at the generic metatheory of ALs, let us consider two exemplary systems, one for reasoning with inconsistent premises, the other for reasoning with defeasible background assumptions. These systems will help the reader to get a better grip on the abstract definitions from the preceding sections, and will also play a role in the remainder of this thesis.

### 2.4.1 The Logics $\mathbf{CLuN^m}$ and $\mathbf{CLuN^r}$

The first example I will use is the inconsistency-adaptive logic $\mathbf{CLuN^m}$, which was already mentioned in Section 2.1. For reasons of simplicity, I will only consider the propositional fragment of this system.

[11] See [25, Section 4.9.2] for more details and illustrations of this fact.

**CLuN$^{\mathbf{m}}$** is an adaptive logic based on the monotonic paraconsistent system **CLuN**, which stands for "**C**lassical **L**ogic with gl**u**ts for the **N**egation". Where $\sim$ is the paraconsistent negation, we obtain **CLuN** by adding the axiom $A \vee \sim A$ to all axioms that characterize the positive fragment of **CL** (see Appendix B), and closing the resulting set under modus ponens. It can easily be verified that **CLuN** invalidates disjunctive syllogism: $\{A, \sim A \vee B\} \nvdash_{\mathbf{CLuN}} B$. **CLuN+**, the lower limit logic of **CLuN$^{\mathbf{m}}$**, is obtained by enriching **CLuN** with the checked connectives, as explained in Section 2.1.

Let me illustrate the way a logic **LLL** is obtained from a logic **LLL$_{\mathbf{s}}$**, for the case of **CLuN$^{\mathbf{m}}$**. The regular connectives of **CLuN** are $\bot, \sim, \vee, \wedge, \supset, \equiv$. Except for the paraconsistent negation $\sim$, all these connectives behave classically. Together with $\mathcal{S}$ (the set of sentential letters), this gives us the language $\mathcal{L}_{\sim}$ and the associated set of formulas $\mathcal{W}_{\sim}$.

To obtain $\check{\mathcal{L}}_{\sim}$, we add the checked connectives, i.e. $\check{\neg}, \check{\vee}, \check{\wedge}, \check{\supset}, \check{\equiv}$, which gives us the language of **CLuN+**. The set $\check{\mathcal{W}}_{\sim}$ is obtained by superimposing the checked connectives, as described on page 18.

It remains to be specified which axioms we need to add to **CLuN**, in order to obtain the logic **CLuN+**. First of all, every logic **LLL** contains the **CL**-axioms for the checked connectives:

**A$\check{\supset}$1** $A \check{\supset} (B \check{\supset} A)$<br>
**A$\check{\supset}$2** $((A \check{\supset} B) \check{\supset} A) \check{\supset} A)$<br>
**A$\check{\supset}$3** $(A \check{\supset} (B \check{\supset} C)) \check{\supset} ((A \check{\supset} B) \check{\supset} (A \check{\supset} C))$<br>
**A$\check{\wedge}$1** $(A \check{\wedge} B) \check{\supset} A$<br>
**A$\check{\wedge}$2** $(A \check{\wedge} B) \check{\supset} B$<br>
**A$\check{\wedge}$3** $A \check{\supset} (B \check{\supset} (A \check{\wedge} B))$<br>
**A$\check{\vee}$1** $A \check{\supset} (A \check{\vee} B)$<br>
**A$\check{\vee}$2** $B \check{\supset} (A \check{\vee} B)$<br>
**A$\check{\vee}$3** $(A \check{\supset} C) \check{\supset} ((B \check{\supset} C) \check{\supset} ((A \check{\vee} B) \check{\supset} C))$<br>
**A$\check{\equiv}$1** $(A \check{\equiv} B) \check{\supset} (A \check{\supset} B)$<br>
**A$\check{\equiv}$2** $(A \check{\equiv} B) \check{\supset} (B \check{\supset} A)$<br>
**A$\check{\equiv}$3** $(A \check{\supset} B) \check{\supset} ((B \check{\supset} A) \check{\supset} (A \check{\equiv} B))$<br>
**A$\check{\neg}$1** $A \check{\supset} (\check{\neg} A \check{\supset} B)$<br>
**A$\check{\neg}$2** $(A \check{\supset} \check{\neg} A) \check{\supset} \check{\neg} A$

Moreover, for every specific logic **LLL**, we need to add a number of axioms that link the checked connectives to the regular connectives of that logic. In the current case, these axioms are:

**A$\check{\neg}\sim$** $\check{\neg} A \check{\supset} \sim A$<br>
**A$\check{\vee}\vee$** $(A \check{\vee} B) \check{\equiv} (A \vee B)$<br>
**A$\check{\supset}\supset$** $(A \check{\supset} B) \check{\equiv} (A \supset B)$<br>
**A$\check{\wedge}\wedge$** $(A \check{\wedge} B) \check{\equiv} (A \wedge B)$<br>
**A$\check{\equiv}\equiv$** $(A \check{\equiv} B) \check{\equiv} (A \equiv B)$

Note that all checked connectives are equivalent to their unchecked counterparts, except $\check{\neg}$, which is stronger than $\sim$. We close the whole of these axioms under the rule MP+: from $A$ and $A \check{\supset} B$, infer $B$.

A **CLuN**-model $M$ is fully determined by an assignment function $v$, which assigns a truth value to schematic letters *and formulas of the form* $\sim A$. The valuation function $v_M$ determined by the model $M$ is obtained by the usual clauses, replacing the clause for the negation with the following:

C$\sim$ $v_M({\sim}A) = 1$ iff $v_M(A) = 0$ or $v({\sim}A) = 1$

This means that whenever $A$ is false, ${\sim}A$ has to be true in a model, but it can be the case that a model verifies both $A$ and ${\sim}A$, viz. when $v({\sim}A) = 1$. The semantics of **CLuN+** is obtained from the **CLuN**-semantics in the way explained in Section 2.1, e.g. by letting $M \Vdash \check{\neg} A$ iff $M \nVdash A$, $M \Vdash A \check{\supset} B$ iff either $M \nVdash A$ or $M \Vdash B$, etc.

Having introduced the lower limit logic **CLuN+**, we can now turn to the adaptive logic based on it. The set of abnormalities of $\mathbf{CLuN^m}$ is $\Omega_{\mathbf{CLuN}} =_{\mathsf{df}} \{A \wedge {\sim}A \mid A \in \mathcal{W}_{\sim}\}$. Hence contradictions (with respect to any formula $A \in \mathcal{W}_{\sim}$) are avoided as much as possible. As a result, $\mathbf{CLuN^m}$ is much richer than **CLuN**, without trivializing inconsistent premises sets $\Gamma \subseteq \mathcal{W}_{\sim}$.

Consider the premise set $\Gamma_1 = \{p, {\sim}p \vee q, {\sim}q, {\sim}p \vee r, q \vee r\}$. Note that the following Dab-formula is **CLuN+**-derivable from $\Gamma_1$, which implies that we are dealing with an inconsistent premise set:

$$(p \wedge {\sim}p) \check{\vee} (q \wedge {\sim}q) \tag{2.1}$$

On the semantic level, every **CLuN+**-model of $\Gamma_1$ verifies either $p \wedge {\sim}p$ or $q \wedge {\sim}q$, or both. For every minimally abnormal **CLuN+**-model $M$ of $\Gamma_1$, either $Ab(M) = \{p \wedge {\sim}p\}$ or $Ab(M) = \{q \wedge {\sim}q\}$. Suppose that for some such model $M$, $Ab(M) = \{p \wedge {\sim}p\}$. In view of the premise set, $M \Vdash {\sim}q$ and $M \Vdash q \vee r$. Since also $M \nVdash q \wedge {\sim}q$, $M \nVdash q$ and $M \Vdash r$. I leave it to the reader to see that also the second class of minimally abnormal models verify $r$. As a result, $r$ is a semantic $\mathbf{CLuN^m}$-consequence of $\Gamma_1$.

Consider the following $\mathbf{CLuN^m}$-proof from $\Gamma_1$:

| | | | |
|---|---|---|---|
| 1 | $p$ | PREM | $\emptyset$ |
| 2 | ${\sim}p \vee q$ | PREM | $\emptyset$ |
| 3 | ${\sim}q$ | PREM | $\emptyset$ |
| 4 | ${\sim}p \vee r$ | PREM | $\emptyset$ |
| 5 | $q \vee r$ | PREM | $\emptyset$ |

Note that the fourth element is $\emptyset$, indicating that premises are introduced on the empty condition. We may now derive $r$ from lines 1 and 4:

| | | | |
|---|---|---|---|
| 6 | $(p \wedge {\sim}p) \check{\vee} r$ | 1,4;RU | $\emptyset$ |
| 7 | $r$ | 6;RC | $\{p \wedge {\sim}p\}$ |

In the remainder of this thesis, I will denote the stage consisting of lines $1 - -n$ by stage $n$ in concrete examples. At stage 7 of the proof, $r$ is derived. However, we can continue the proof as follows, showing that the condition on line 7 is problematic:

| | | | |
|---|---|---|---|
| 6 | $(p \wedge {\sim}p) \vee r$ | 1,4;RU | $\emptyset$ |
| 7 | $r$ | 6;RC | $\{p \wedge {\sim}p\}\checkmark^{8}$ |
| 8 | $(p \wedge {\sim}p) \check{\vee} (q \wedge {\sim}q)$ | 1,2,3;RU | $\emptyset$ |

Where $i \in \mathbb{N}$, I will henceforth use $\checkmark^i$ to denote the marking of a line at stage $i$. At stage 8, line 7 is marked. Recall that in order to find out which lines are marked at stage $s$, we had to look at the set $\Phi_s(\Gamma_1)$. Since $\Sigma_8(\Gamma_1) = \{\{p \wedge \sim p, q \wedge \sim q\}\}$, the minimal choice sets at stage 8 are $\varphi_1 = \{p \wedge \sim p\}$ and $\varphi_2 = \{q \wedge \sim q\}$.

Clearly, the condition of line 7 has an empty intersection with $\varphi_2$. But $r$, the formula on line 7, has not been derived on a condition that has an empty intersection with $\varphi_1$. Hence the marking definition for Minimal Abnormality stipulates that line 7 is marked.

So how can line 7 become unmarked again? This is done by showing that $r$ can be derived in the proof on a yet different condition:

| | | | |
|---|---|---|---|
| 6 | $(p \wedge \sim p) \check{\vee} r$ | 1,4;RU | $\emptyset$ |
| 7 | $r$ | 6;RC | $\{p \wedge \sim p\}$ |
| 8 | $(p \wedge \sim p) \check{\vee} (q \wedge \sim q)$ | 1,2,3;RU | $\emptyset$ |
| 9 | $r$ | 3,5;RC | $\{q \wedge \sim q\}$ |

Note that throughout the stages $8 - 9$, the set of minimal choice sets remains the same, which means that lines 7 and 9 are unmarked at stage 9.

At stage 9 of the above proof, $r$ is finally derived from $\Gamma_1$. To see why, note that the only minimal Dab-consequence of $\Gamma_1$ is derived on line 8. Hence at every later stage $s$ of this proof, $\Phi_s(\Gamma_1) = \Phi_9(\Gamma_1)$. It follows that there is no extension of the proof in which lines 7 and 9 are marked. By Definitions 2.5 and 2.6, $\Gamma_1 \vdash_{\mathbf{CLuN^m}} r$.[12]

The difference with the Reliability Strategy can also be clarified by the above example: in $\mathbf{CLuN^r}$, $r$ is not finally derivable from $\Gamma_1$. The reason is that from stage 8 on, the set of unreliable formulas is $\{p \wedge \sim p, q \wedge \sim q\}$. In view of Definition 2.3, both lines 7 and 9 are marked if Reliability is the strategy. This is in agreement with the $\mathbf{CLuN^r}$-semantics: there is a $M \in \mathcal{M}_{\mathbf{CLuN^r}}(\Gamma) - \mathcal{M}_{\mathbf{CLuN^m}}(\Gamma)$ for which $Ab(M) = \{p \wedge \sim p, q \wedge \sim q\}$ and $M \not\Vdash r$.

### 2.4.2 The Logics $\mathbf{K_1^r}$ and $\mathbf{K_1^m}$

The logics $\mathbf{K_1^r}$ and $\mathbf{K_1^m}$ are very basic systems, which makes them very suitable candidates to explain and study the behavior of adaptive logics from a metatheoretic point of view.[13] They are based on Kripke's minimal modal logic, which I will call $\mathbf{K_s}$ for reasons that will become clear below. Let the standard modal language (with the classical negation $\neg$ and the modal operator $\Box$) be denoted by $\mathcal{L}_m$, and let $\mathcal{W}_m$ be the associated set of well-formed closed formulas.[14]

[12] In cases where there are infinitely many minimal Dab-consequences of the premise set $\Gamma$, it is possible that we never arrive at a finite stage $s$ such that $\Gamma \vdash_{\mathbf{CLuN^m}} A$ if and only if $A$ is derived on an unmarked line at stage $s$, and remains unmarked in every further extension. It is for these cases that the definition of final derivability refers to (possibly infinite) extensions of a proof.

[13] Some readers might wonder why there is a subscript 1 in the name of these two logics. The reason will become clear in the next two chapters, where we discuss a whole range of logics $\mathbf{K_1^x}, \mathbf{K_2^x}, \ldots$ and some ways to combine these systems.

[14] For the present purposes, we need not restrict the language in any way. However, as will become clear, the intended application of $\mathbf{K_1^r}$ and $\mathbf{K_1^m}$ are modeled by premise sets that consist of formulas that have either the form $A$ or $\Diamond A$, where $A$ is a non-modal formula.

$\mathbf{K_s}$ is axiomatized by the propositional fragment of **CL** together with the following axioms:

K $\quad \Box(A \supset B) \supset (\Box A \supset \Box B)$
RN $\quad$ if $\vdash A$ then $\vdash \Box A$

As usually, $\Diamond A =_{\mathsf{df}} \neg\Box\neg A$.

$\mathbf{K_s}$-models are defined as pointed-Kripke frames with the standard valuations. More precisely, a $\mathbf{K_s}$-model $M$ is a quadruple $\langle W, R, v, w_0\rangle$, where $W$ is a set of possible worlds, $R$ an accessibility relation on $W$, $v : \mathcal{S} \times W \to \{0,1\}$ an assignment function and $w_0 \in W$ the actual world. The valuation $v_M : \mathcal{W}_m \to \{0,1\}$ defined by the model $M$ is characterized by:

C1 $\quad$ where $A \in \mathcal{S}$, $v_M(A, w) = v(A, w)$
C2 $\quad v_M(\neg A, w) = 1$ iff $v_M(A, w) = 0$
C3 $\quad v_M(A \vee B, w) = 1$ iff $v_M(A, w) = 1$ or $v_M(B, w) = 1$
C4 $\quad v_M(A \wedge B, w) = 1$ iff $v_M(A, w) = 1$ and $v_M(B, w) = 1$
C5 $\quad v_M(A \supset B, w) = 1$ iff $v_M(A, w) = 0$ or $v_M(B, w) = 1$
C6 $\quad v_M(\Box A, w) = 1$ iff, $v_M(A, w') = 1$ for all $w'$ such that $Rww'$

Where $M$ is a $\mathbf{K_s}$-model, $M \Vdash A$ iff $v_M(A, w_0) = 1$. We say that $M$ is a $\mathbf{K_s}$-model of $\Gamma$ iff $M \Vdash A$ for every $A \in \Gamma$. $\Gamma \models_{\mathbf{K_s}} A$ iff all $\mathbf{K_s}$-models of $\Gamma$ verify $A$.

As for **CLuN**, we have to upgrade $\mathbf{K_s}$, in order obtain the **LLL** of our adaptive logics. In line with the preceding, I use $\check{\mathcal{L}}_m$ and $\check{\mathcal{W}}_m$ to denote the extension of $\mathcal{L}_m$, resp. $\mathcal{W}_m$ with the checked connectives. An axiomatization for the enriched logic, which I will call **K**, is obtained as follows: add to $\mathbf{K_s}$ all the axioms from Section 2.4.1, but replace $\mathbf{A}\check{\neg}\sim$ by

$\mathbf{A}\check{\neg}\neg \qquad \check{\neg}A \check{\equiv} \neg A$

and close the whole under MP+. In the remainder, I will not refer to $\mathbf{K_s}$ again, but I will often refer to (adaptive logics based on) **K**, which, as should be clear by now, is a conservative extension of $\mathbf{K_s}$.

The adaptive logic $\mathbf{K^r_1}$ is defined by the following triple:[15]

1. the lower limit logic **K**
2. the set of abnormalities $\Omega^{\mathbf{K}}_1 =_{\mathsf{df}} \{\Diamond A \wedge \neg A \mid A \in \mathcal{W}^l_c\}$
3. the Reliability Strategy

$\mathbf{K^r_1}$ is intended to explicate reasoning from plausible knowledge, also referred to as *background assumptions*. It is but a variation on an existing theme in the adaptive logics program.[16] That a formula $A$ is plausible, is expressed by $\Diamond A$. We thus translate the original set of background assumptions $\Gamma$ into $\Gamma^\Diamond = \{\Diamond A \mid A \in \Gamma\}$. We may also reason from facts together with plausible knowledge. In this case, a premise set contains formulas of the form $\Diamond A$ and $A$, where $A$ is a non-modal formula. Where $A$ is a literal, the adaptive logic enables one to derive $A$ from $\Diamond A$, just in case $\Diamond A \wedge \neg A$ does not occur in a minimal Dab-consequence

[15] Recall that $\mathcal{W}^l_c$ is the set of literals, i.e. sentential letters and their negation.
[16] A similar logic based on Feys' modal logic **T** is presented in [32].

of $\Gamma$. This makes sense in view of the fact that our plausible knowledge can be contradicted by other plausible knowledge, or by a given set of facts.

Some readers might wonder why not just any formula of the form $\Diamond A \wedge \neg A$ counts as an abnormality. The reason is that, without the restriction to literals, $\mathbf{K_1^r}$ would be a so-called *flip-flop logic*. This is an adaptive logic which is equivalent to its lower limit logic whenever the premises are abnormal (i.e. whenever they entail at least one Dab-formula). Consider e.g. the premise set $\Gamma_{\mathsf{ff}} = \{\Diamond p, \Diamond q, \neg p\}$. Intuitively, we expect there to be only a problem with the plausible knowledge $p$, since this is contradicted by the fact $\neg p$. However, the following is a minimal disjunction that can be $\mathbf{K}$-derived from $\Gamma_{\mathsf{ff}}$:

$$(\Diamond q \wedge \neg q) \check{\vee} (\Diamond(p \vee \neg q) \wedge \neg(p \vee \neg q))$$

If this would be a disjunction of abnormalities, then we would not be able to derive $q$ from $\Gamma_{\mathsf{ff}}$. This problem is overcome by restricting the abnormalities in such a way that only literals can behave abnormally.[17]

Let us now take a look at the $\mathbf{K_1^r}$-proof theory. Consider the following premise set: $\Gamma = \{\Diamond p, \Diamond q, \Diamond r, \neg p \vee \neg q\}$. We start a $\mathbf{K_1^r}$-proof from $\Gamma$ by writing down the premises:

| | | | |
|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\emptyset$ |
| 2 | $\Diamond q$ | PREM | $\emptyset$ |
| 3 | $\Diamond r$ | PREM | $\emptyset$ |
| 4 | $\neg p \vee \neg q$ | PREM | $\emptyset$ |

Note that the fourth element is $\emptyset$, indicating that premises are derived on the empty condition. We may now derive $p$ from line 1, using RU and RC:

| | | | |
|---|---|---|---|
| 5 | $p \check{\vee} \neg p$ | RU | $\emptyset$ |
| 6 | $p \check{\vee} (\Diamond p \wedge \neg p)$ | 1,5; RU | $\emptyset$ |
| 7 | $p$ | 6; RC | $\{\Diamond p \wedge \neg p\}$ $\checkmark^8$ |

For the time being, ignore the $\checkmark^8$ at line 7. At stage 7 of the proof, there are no unreliable formulas: $U_7(\Gamma) = \emptyset$, and $p$ is derived on an unmarked line. However, we may immediately add the following line:

| | | | |
|---|---|---|---|
| 8 | $(\Diamond p \wedge \neg p) \check{\vee} (\Diamond q \wedge \neg q)$ | 1,2,4; RU | $\emptyset$ |

This means that $U_8(\Gamma) = \{\Diamond p \wedge \neg p, \Diamond q \wedge \neg q\}$. As a consequence line 7 is marked at stage 8 of the proof, which is indicated by the $\checkmark^8$. Moreover, since $\Gamma \nvdash_{\mathbf{K}} \Diamond q \wedge \neg q$, the line will be marked in every extension of the proof. Nevertheless, we may still apply RC to derive $r$ on an unmarked line:

| | | | |
|---|---|---|---|
| 9 | $r \check{\vee} \neg r$ | RU | $\emptyset$ |
| 10 | $r \check{\vee} (\Diamond r \wedge \neg r)$ | 3,9; RU | $\emptyset$ |
| 11 | $r$ | 10; RC | $\{\Diamond r \wedge \neg r\}$ |

[17] This restriction has no negative impact on the (conditional) derivability of formulas, since whenever $\Gamma \vdash_{\mathbf{K}} \Diamond A$, then $\Gamma \vdash_{\mathbf{K}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega_1^{\mathbf{K}}$. This follows immediately from the fact that every $A \in \mathcal{W}_c$ has a conjunctive normal form $\bigwedge_{i \in I} \bigvee_{j \in J_i} B_j$, so that $\Diamond A \vdash_{\mathbf{K}} A \check{\vee} \check{\bigvee}\{\Diamond B_j \wedge \neg B_j \mid j \in J_i, i \in I\}$.

Line 10 is unmarked in every extension of the proof, since the only minimal Dab-consequence of $\Gamma$ is the formula on line 7, hence no new Dab-formula can render $\Diamond r \wedge \neg r$ unreliable. By Definitions 2.5 and 2.6, we can infer that $\Gamma \vdash_{\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}} r$.

The difference in strength between Reliability and Minimal Abnormality can also be clarified by the above example: $\Gamma \vdash_{\mathbf{K}^{\mathbf{m}}_{\mathbf{1}}} p \vee q$, while $\Gamma \nvdash_{\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}} p \vee q$. The **K**-models of $\Gamma$ that verify $\Diamond p \wedge \neg p$ and $\Diamond q \wedge \neg q$ are not minimally abnormal, since there are models that verify only one of both abnormalities.

This implies that there must be a $\mathbf{K}^{\mathbf{m}}_{\mathbf{1}}$-proof in which $p \vee q$ is finally derived. Since the choice of the strategy only affects the marking in a proof, we may simply continue the preceding proof to achieve this goal:

| | | | |
|---|---|---|---|
| 12 | $p \vee q$ | 7; RU | $\{\Diamond p \wedge \neg p\}$ |
| 13 | $q \check{\vee} (\Diamond q \wedge \neg q)$ | 2; RU | $\emptyset$ |
| 14 | $q$ | 13; RC | $\{\Diamond q \wedge \neg q)\}\checkmark^{14}$ |
| 15 | $p \vee q$ | 14; RU | $\{\Diamond q \wedge \neg q)\}$ |

Note that throughout the stages 12-15, $\Phi_s(\Gamma)$ remains the same, that is, $\Phi_s(\Gamma) = \Phi(\Gamma) = \{\{\Diamond p \wedge \neg p\}, \{\Diamond q \wedge \neg q\}\}$. Let us call the choice sets $\varphi_1$ and $\varphi_2$ respectively. From stage 12 to stage 14, line 12 is marked. That is, as long as $p \vee q$ is not derived on a line with condition $\Delta$ such that $\Delta \cap \varphi_1 = \emptyset$, the marking definition stipulates that line 11 is marked. However, at stage 15, line 12 is unmarked, because at that stage of the proof we know that $p \vee q$ is true both when $\Diamond p \wedge \neg p$ is false, and when $\Diamond q \wedge \neg q$ is false. Line 14 will however be marked in every extension of the proof, hence $q$ is treated in exactly the same way as $p$.

## 2.5 Metatheory of the Standard Format

In this section, I mention some of the most significant meta-theoretic properties of the standard format. I merely recapitulate these to illustrate the merits of the standard format, and in preparation for the new results that will be discussed from the next chapter on. Where necessary, I will briefly explain the importance and meaning of certain properties for the logics under consideration.

**The Restriction to $\mathcal{L}_s$.** In Section 2.7, an example is presented of a $\Gamma \subseteq \check{\mathcal{W}}_s$ and $A \in \mathcal{W}_s$ for which $\Gamma \nvdash_{\mathbf{AL}} A$, whereas $\Gamma \models_{\mathbf{AL}} A$. Hence completeness does not hold for **AL** in general. Nevertheless, for all $\Gamma \subseteq \mathcal{W}_s$, completeness is provable – I will explain both the general incompleteness and the restricted completeness of ALs in Section 2.7. As I will show there, some other properties such as e.g. Fixed Point also have to be restricted to $\Gamma \subseteq \mathcal{W}_s$.

This should not be seen as a fundamental problem for ALs, since as I explained in Section 2.1, they were developed to explicate a reasoning process on the basis of premise sets $\Gamma \subseteq \mathcal{W}_s$. It is relevant though, especially for the proofs of other metatheoretic properties such as Fixed Point and Cumulative Transitivity – see below. Hence I will state the meta-theory about **AL** for any $\Gamma \subseteq \check{\mathcal{W}}_s$, whenever possible. Recall that, unless specified differently, I use $\Gamma$ to refer to an arbitrary subset of $\check{\mathcal{W}}_s$.

**Some well-known properties.** A number of well-known properties are inherent to ALs in standard format, among which the following are the most salient ones (I mention the original theorems in the literature between square brackets):

**Theorem 2.8** *For every* $\Gamma \subseteq \mathcal{W}_s$*:* $\Gamma \vdash_{\mathbf{AL}} A$ *iff* $\Gamma \models_{\mathbf{AL}} A$*. (Soundness and Completeness)* [21, Corr. 2, Th. 9]

**Theorem 2.9** $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma)$*. (Reflexivity)* [21, Th. 11.2]

**Theorem 2.10** *For every* $\Gamma \subseteq \mathcal{W}_s$*:* $Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$*. (Fixed Point)* [21, Th. 11.6, Th. 11.7]

Furthermore, for $\mathbf{AL^m}$, the Deduction Theorem holds:

**Theorem 2.11** *If* $\Gamma \cup \{A\} \vdash_{\mathbf{AL^m}} B$*, then* $\Gamma \vdash_{\mathbf{AL^m}} A \check{\supset} B$*.* [21, Th. 14]

The Deduction Theorem does not hold for $\mathbf{AL^r}$ – see [21, Theorem 13.3] for a simple counterexample.

**Reassurance and Strong Reassurance.** In Section 2.2, it was explained that every **AL** selects a subset of the **LLL**-models of $\Gamma$. Now suppose an **LLL**-model $M$ of $\Gamma$ is not selected. In that case, it seems desirable to have as a property of the logic that there is an **LLL**-model $M'$ of $\Gamma$ that *is* selected, and that is less abnormal than $M$. This property is called *Strong Reassurance* in the literature.

**Theorem 2.12** *If* $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}}(\Gamma)$*, then there is an* $M' \in \mathcal{M}_{\mathbf{AL}}(\Gamma)$ *such that* $Ab(M') \subset Ab(M)$*.* [21, Th. 4&5] *(Strong Reassurance)*

Note that the abnormal part-relation and $\subset$ impose a partial order on the set of all **LLL**-models: $M \prec M'$ iff $Ab(M) \subset Ab(M')$. Strong Reassurance boils down to the claim that, for every $\Gamma$, the order $\prec$ is smooth with respect to the set of **LLL**-models of $\Gamma$.[18] It also entails that whenever $\Gamma$ has **LLL**-models, $\Gamma$ has **AL**-models. The syntactic counterpart of this property states that unless $\Gamma$ is **LLL**-trivial, **AL** will not trivialize $\Gamma$.[19] So we have:

**Theorem 2.13** *If* $\Gamma$ *has* **LLL***-models, then it has* **AL***-models. (Semantic Reassurance)*

**Theorem 2.14** *If* $\Gamma$ *is not* **LLL***-trivial, then* $\Gamma$ *is not* **AL***-trivial. (Syntactic Reassurance)*

---

[18] A partial order $\prec$ on a set $X$ is smooth with respect to a set $Y \subseteq X$ iff for all $a \in Y$ either $a$ is $\prec$-minimal in $Y$, or there is a $\prec$-minimal element $b \in Y$ for which $b \prec a$.

[19] According to the conventions introduced at the start of this chapter, $\Gamma$ is **LLL**-trivial iff $Cn_{\mathbf{LLL}}(\Gamma) = \check{\mathcal{W}}_s$.

**Cumulative Indifference.** Suppose we have established for some $\Gamma, A$, that $\Gamma \vdash_{\mathbf{AL}} A$. In that case, it seems desirable that the **AL**-closure of $\Gamma \cup \{A\}$ is not different from the **AL**-closure of $\Gamma$ itself. That is, adding $A$ as a premise to $\Gamma$ should not lead to a different consequence set. This is warranted by the Cumulative Indifference principle:

**Theorem 2.15** *For every $\Gamma \subseteq \mathcal{W}_s$: if $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$, then $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$. (Cumulative Indifference)* [21, Th. 11.10]

The Fixed Point property, i.e. that $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma))$, is derivable from Theorem 2.15 together with the reflexivity of **AL** (see Section 2.8 where this is proven generically for any logic **L**). Cumulative Indifference is often divided in two parts: Cautious Monotonicity ($Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$) and Cumulative Transitivity ($Cn_{\mathbf{AL}}(\Gamma) \supseteq Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$). The cautious monotonicity of ALs can be proven for the more general case where $\Gamma \subseteq \check{\mathcal{W}}_s$, whereas their cumulative transitivity only holds for $\Gamma \subseteq \mathcal{W}_s$.

**Relations Between Logics.** The following theorem summarizes the difference in strength between the different logics **LLL**, $\mathbf{AL^r}$, $\mathbf{AL^m}$ and **ULL**:

**Theorem 2.16** $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL^r}}(\Gamma) \subseteq Cn_{\mathbf{AL^m}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$. [21, Th. 11.1]

Obviously, in all interesting cases, **AL** is stronger than **LLL**. Also, $\mathbf{AL^r}$ is often weaker than $\mathbf{AL^m}$, as the examples in Section 2.4 illustrated. A related property is that if a premise set $\Gamma$ is normal, then **AL** is equipowerful to **ULL**:[20]

**Theorem 2.17** *If $\Gamma$ is normal, then $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}})$.* [21, Th. 12.1]

Hence if **AL** can avoid abnormalities altogether, it will do so. Nevertheless, if the premise set is not normal, it will still in most cases render more consequences than **LLL**, without yielding triviality as **ULL** would. In other words, **AL** oscillates between **LLL** and **ULL**, adapting itself to the premises. Finally, **AL** is both closed and invariant under **LLL**, as the following theorems state:

**Theorem 2.18** *Where $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$.* [21, Th. 11.8] *(**LLL**-Closure)*

**Theorem 2.19** *Where $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{AL}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$.* [21, Th. 15.2] *(**LLL**-Invariance)*

**Equivalent Premise Sets.** In [33], it is argued that ALs have certain advantages over numerous other formal approaches to defeasible reasoning methods. The most important argument there is one concerning transparency: there are various criteria to decide when two premise sets are **AL**-equivalent. For a lengthy discussion, I refer to the original paper; here I simply mention the three criteria for equivalence (the original Theorems from [33] are given between square brackets).

[20]See page 19 for the definition of a normal premise set.

**Theorem 2.20** *Where* $\Gamma, \Gamma' \subseteq \mathcal{W}_s$, $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma')$ *if one of the following holds:*

*(C1)* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ *and* $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma')$ [Th. 6]
*(C2)* *Where* **L** *is a Tarski-logic weaker than or identical to* **AL**: $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$ [Th. 7]
*(C3)* *Where* **L** *is a Tarski-logic and for every* $\Theta \subseteq \mathcal{W}_s$, $Cn_{\mathbf{AL}}(\Theta) = Cn_{\mathbf{L}}(Cn_{\mathbf{AL}}(\Theta))$: $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$ [Th. 7]

The second criterion can be strengthened with the aid of the following theorem:

**Theorem 2.21** *Every monotonic logic that is weaker than or identical to* **AL** *is weaker than or identical to* **LLL**. *(Maximality of* **LLL***)* [Th. 10]

In view of Theorem 2.20, we thus obtain:

**Theorem 2.22** *Where* **L** *is a monotonic logic weaker than or identical to* **LLL**: *if* $\Gamma$ *and* $\Gamma'$ *are* **L**-*equivalent, then they are* **AL**-*equivalent. (C2')*

Hence there are several shortcuts to decide whether $\Gamma$ and $\Gamma'$ are **AL**-equivalent, depending on the many different Tarski-logics weaker than **LLL**. In Chapter 5, we will see that Theorems 2.20-2.22 can easily be generalized to the format of lexicographic ALs, presented in that chapter.

## 2.6 Reliability versus Minimal Abnormality

In this section, I will adress some differences between Reliability and Minimal Abnormality. As before, I will focus on what is important for the remainder of this thesis.

First of all, as might be clear in view of the definitions for both strategies, Reliability is clearly grounded in proof theoretic intuitions (whence the crucial role of $U_s(\Gamma)$ and its abstract counterpart $U(\Gamma)$), whereas Minimal Abnormality is most easily understood from a semantic perspective. This may help us to understand some of the other differences between both strategies which I will now discuss.

Another salient point is that Reliability is computationally less complex than Minimal Abnormality, as has been shown in [79, 157]. This relates to a fact mentioned in Section 2.3, i.e. that for the Reliability Strategy, it suffices to speak of finite extensions of a proof in the definition of final derivability. Since every finite proof (hence also every finite extension of a finite proof) has a Gödel number, and in view of the definition of final derivability, one can derive that $\mathbf{AL^r}$ is $\Sigma^0_3$-complex.[21] To the contrary, for logics $\mathbf{AL^m}$, it is possible to construct a $\Gamma$ such that $Cn_{\mathbf{AL^m}}(\Gamma)$ is $\Pi^1_1$-complex – see [157] where this is done for $\mathbf{CLuN^m}$.

[21] Contrary to what the authors of [79] thought, we cannot do without infinite extensions in the definition of final derivability for $\mathbf{AL^m}$. If only finite extensions are taken into account, the proof theory is not sound with respect to the semantics – a counterexample can be found in [25, Chapter 4, Section 4.9.2].

Although computational complexity is not a central topic in the current thesis, I will briefly return to this point in Chapter 4.

Third, Minimal Abnormality is often stronger than Reliability in a non-trivial way. Although this point was already spelled out before (see Theorem 2.16) and illustrated in Section 2.4, it is worthwhile stressing it another time. If an AL is intended to approximate a given certain standard of normality – as embodied by an upper limit logic – as much as possible, than it seems that Minimal Abnormality is simply the best candidate to do this job.

From these three facts, we can conclude that we had and have very good reasons to pay equal attention to adaptive logics that use the Reliability Strategy, as to those that use the Minimal Abnormality Strategy. More specifically, when devising logics that deal with prioritized defeasible reasoning forms, as will be done in the next three chapters, the Minimal Abnormality-variants of these systems should receive just as much attention as the Reliability-variants.

A last remark may perhaps seem more superficial than the previous ones, but is of crucial importance for the next two chapters. For the Reliability Strategy, it is possible to characterize the set of selected models in terms of a set of formulas and **LLL**, as the following lemma states:

**Lemma 2.2** *$M \in \mathcal{M}_{\mathbf{AL^r}}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^r}}(\Gamma))$.* [137, Lemma 2.3.3]

For Minimal Abnormality, this is not possible: there are cases in which no (possibly infinite) set of formulas suffices to characterize the set of $\mathbf{AL^m}$-models. Hence it is not in general true that $\mathcal{M}_{\mathbf{AL^m}}(\Gamma) = \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^m}}(\Gamma))$.

Let me illustrate this by means of a concrete example, using the logic $\mathbf{K_1^m}$ from Section 2.4.2. In the remainder of this section, let $!A$ abbreviate $\Diamond A \wedge \neg A$. Let $\Gamma = \{\Diamond p_i \mid i \in \mathbb{N}\} \cup \{\neg p_i \vee \neg p_j \mid i, j \in \mathbb{N}, i \neq j\}$ – variations on this premise set will be used throughout the first part of this thesis.

Note that by Definition 2.2, every $\mathbf{K_1^m}$-model of $\Gamma$ is a $\mathbf{K}$-model. Also, in view of the soundness of $\mathbf{K_1^m}$, every $\mathbf{K_1^m}$-model of $\Gamma$ verifies every member of $Cn_{\mathbf{K_1^m}}(\Gamma)$. It follows that $\mathcal{M}_{\mathbf{K_1^m}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{K}}(Cn_{\mathbf{K_1^m}}(\Gamma))$. In general, $\mathcal{M}_{\mathbf{AL^m}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^m}}(\Gamma))$. However, the opposite direction of the set inclusion is more problematic. In this particular case, we can show that there is an $M \in \mathcal{M}_{\mathbf{K}}(Cn_{\mathbf{K_1^m}}(\Gamma)) - \mathcal{M}_{\mathbf{K_1^m}}(\Gamma)$.

Consider an arbitrary $\mathbf{K}$-model $M$ of $\Gamma$, such that $M \Vdash p_1$. It follows that $M \Vdash \neg p_i$ for every $i \in \mathbb{N} - \{1\}$, whence also $M \Vdash !p_i$ for all $i \in \mathbb{N} - \{1\}$. By the same reasoning, every $\mathbf{K}$-model $M'$ that verifies $p_2$, will verify all abnormalities $!p_i$ with $i \in \mathbb{N} - \{2\}$. So, any model must verify all abnormalities $!p_i$ but one. On the other hand, it is possible to falsify all abnormalities $A \in \Omega_1^{\mathbf{K}} - \{!p_i \mid i \in \mathbb{N}\}$. As a result, the set of minimal abnormal models of $\Gamma$ is $\mathcal{M}_{\mathbf{K_1^m}}(\Gamma) = \{M \in \mathcal{M}_{\mathbf{K}}(\Gamma) \mid Ab(M) = \{!p_i \mid i \in \mathbb{N} - k\}$ for a $k \in \mathbb{N}\}$.

Now assume that (†) $Cn_{\mathbf{K_1^m}}(\Gamma) \cup \{!p_i \mid i \in \mathbb{N}\}$ has no $\mathbf{K}$-models. By the compactness of $\mathbf{K}$, it follows that for an $n \in \mathbb{N}$, $Cn_{\mathbf{K_1^m}}(\Gamma) \cup \{!p_i \mid i \leq n\}$ has no $\mathbf{K}$-models. By **CL**-properties, $Cn_{\mathbf{K_1^m}}(\Gamma) \vdash_{\mathbf{K}} \check{\neg} \bigwedge\{!p_i \mid i \leq n\}$. Since $Cn_{\mathbf{K_1^m}}(\Gamma)$ is closed under $\mathbf{K}$ (see Theorem 2.18), we obtain that (‡) $\check{\neg} \bigwedge\{!p_i \mid i \leq n\} \in Cn_{\mathbf{K_1^m}}(\Gamma)$.

However, let $m$ be an arbitary natural number such that $m > n$. Note that there is an $M \in \mathcal{M}_{\mathbf{K_1^m}}(\Gamma)$ such that $Ab(M) = \{!p_i \mid i \in \mathbb{N} - \{m\}\}$. It follows that $M \not\Vdash \check{\neg} \bigwedge\{!p_i \mid i \leq n\}$. But then, by the soundness of $\mathbf{K_1^m}$, $\check{\neg} \bigwedge\{!p_i \mid i \leq n\} \notin Cn_{\mathbf{K_1^m}}(\Gamma)$, which contradicts (‡).

Hence (†) fails: there is an $M \in \mathcal{M}_{\mathbf{K}}(Cn_{\mathbf{K_1^m}}(\Gamma) \cup \{!p_i \mid i \in \mathbb{N}\})$. In view of the preceding, $M \notin \mathcal{M}_{\mathbf{K_1^m}}(\Gamma)$, whence $\mathcal{M}_{\mathbf{K_1^m}}(\Gamma) \neq \mathcal{M}_{\mathbf{K}}(Cn_{\mathbf{K_1^m}}(\Gamma))$.

For flat adaptive logics, this poses no genuine problem, and in no way does it harm the soundness and completeness result for Minimal Abnormality. However, it does show us that the selection of Minimal Abnormal models sometimes goes beyond what can be expressed by means of a set of formulas, i.e. finite strings built up from a denumerable language $\mathcal{L}_s$. If we would allow for infinite disjunctions in the language, we would be able to express the information "embodied in" $\mathcal{M}_{\mathbf{K_1^m}}(\Gamma)$ by $\bigvee_{i \in \mathbb{N}} p_i$. But since the consequence relation of $\mathbf{AL^m}$ is defined as a function that maps sets of finite formulas to sets of finite formulas, it cannot carry over such information.

This fact will be important in the next two chapters, where, on the one hand, the syntactic consequence relations of several flat ALs will be combined in a specific way, and, on the other hand, minimal abnormal selections in terms of several sets of abnormalities are combined. For the Reliability-variants of these systems, this will pose no genuine problem, and soundness and completeness are provable with the aid of Lemma 2.2. For the Minimal Abnormality-variants, only a restricted form of soundness and completeness is available, in view of the following Lemma:[22]

**Lemma 2.3** *If $\Phi(\Gamma)$ is finite and $\Gamma \subseteq \mathcal{W}_s$, then $M \in \mathcal{M}_\Gamma^{\mathbf{AL^x}}$ iff ($M \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ and $M \models Cn_{\mathbf{AL^x}}(\Gamma)$).* [137, Lemma 3.2.5]

## 2.7 About the Checked Connectives

### 2.7.1 The Problem

In this section, I present the example promised in Section 2.5, which shows that adaptive logics are not always complete with respect to their semantics, unless we restrict premise sets to formulas in $\mathcal{W}_s$. Eventually, the example also shows why several other metatheoretic properties are restricted to $\Gamma \subseteq \mathcal{W}_s$. This observation can almost immediately be carried over to the prioritized adaptive logics studied in the next chapter. Hence, the reader should not be surprised to find that in the remainder of this thesis, numerous theorems are preceded by this restriction.[23]

As in the previous section, I will make use of the (very simple) logics $\mathbf{K_1^r}$ and $\mathbf{K_1^m}$. As before, let $!A$ abbreviate $\Diamond A \wedge \neg A$. The problematic premise set we will consider is $\Gamma = \Delta_1 \cup \Delta_2 \cup \Delta_3$, where

$$\Delta_1 = \{(!q_1 \check{\vee} !q_2) \supset (p \check{\vee} !q_1)\}$$
$$\Delta_2 = \{!q_1 \check{\vee} !q_i \mid i \in \mathbb{N} - \{1\}\}$$
$$\Delta_3 = \{(!q_1 \check{\vee} !q_{i+1}) \supset !q_i \mid i \in \mathbb{N} - \{1, 2\}\}$$

[22] In Chapter 6, we will see that Lemma 2.3 can be generalized to a broader class of premise sets.

[23] To the best of my knowledge, the example I present is the first one that illustrates this fact for both the Minimal Abnormality Strategy and the Reliability Strategy. An example for the Minimal Abnormality Strategy can be found in [25, Section 4.9.3] and [137, Section 2.8].

First of all, note that $!q_1 \notin U(\Gamma)$. That is, for every formula $!q_1 \check{\vee} !q_i \in Cn_{\mathbf{K}}(\Gamma)$ with $i \neq 1$, we can **K**-derive $!q_i$ in view of $\Delta_2$ and $\Delta_3$. By Definition 2.1, this means that (1) every $\mathbf{K_1^r}$-model of $\Gamma$ falsifies the abnormality $!q_1$. Also, in view of $\Delta_1$ and $\Delta_2$, $\Gamma \vdash_{\mathbf{K}} p \check{\vee} !q_1$, whence by the soundness of **K**, also (2) $\Gamma \models_{\mathbf{K}} p \check{\vee} !q_1$. By (1) and (2), every $\mathbf{K_1^r}$-model of $\Gamma$ verifies $p$, whence $\Gamma \models_{\mathbf{K_1^r}} p$. Since the minimal abnormal models are a subset of the reliable models, also $\Gamma \models_{\mathbf{K_1^m}} p$.

Let me first explain why completeness fails for $\mathbf{K_1^r}$. Suppose we want to derive $p$ in a $\mathbf{K_1^r}$-proof from $\Gamma$. At first sight, we may try to derive it on the condition $!q_1$, as follows:

| | | | |
|---|---|---|---|
| 1 | $(!q_1 \check{\vee} !q_2) \supset (p \check{\vee} !q_1)$ | PREM | $\emptyset$ |
| 2 | $!q_1 \check{\vee} !q_2$ | PREM | $\emptyset$ |
| 3 | $p \check{\vee} !q_1$ | 1,2;RU | $\emptyset$ |
| 4 | $p$ | 3;RC | $\{!q_1\}\checkmark^4$ |

However, line 4 is marked in view of the Dab-formula on line 2, as indicated by the $\checkmark^4$-sign. Let us now try to unmark this line, by showing that the Dab-formula on line 2 is in fact not a minimal Dab-consequence of $\Gamma$:

| | | | |
|---|---|---|---|
| 1 | $(!q_1 \check{\vee} !q_2) \supset (p \check{\vee} !q_1)$ | PREM | $\emptyset$ |
| 2 | $!q_1 \check{\vee} !q_2$ | PREM | $\emptyset$ |
| 3 | $p \check{\vee} !q_1$ | 1,2;RU | $\emptyset$ |
| 4 | $p$ | 3;RC | $\{!q_1\}\checkmark^7$ |
| 5 | $(!q_1 \check{\vee} !q_3) \supset !q_2$ | PREM | $\emptyset$ |
| 6 | $!q_1 \check{\vee} !q_3$ | PREM | $\emptyset$ |
| 7 | $!q_2$ | 5,6;RU | $\emptyset$ |

Note that at stage 7, the Dab-formula on line 2 is no longer minimal in view of the Dab-formula on line 7. However, to derive the latter, we had to introduce a new Dab-formula that contains $!q_1$, and that is minimal at stage 7. Hence again, line 4 is marked. More generally, there is no finite proof in which we can derive $p$ on an unmarked line.

Only if we could write down an infinite proof, in which each $!q_i$ $(i > 1)$ is derived, then $p$ would be derived on an unmarked line. However, since the definition of final derivability stipulates that $p$ has to be derived on a line at a finite stage of a proof, such an infinite proof is simply a no-go. As a result, $\Gamma \nvdash_{\mathbf{K_1^r}} p$. By the same reasoning, $\Gamma \nvdash_{\mathbf{K_1^m}} p$.

It is worthwhile to stress that the real problem lies with the (philosophically justified) requirement that we derive $p$ on an unmarked line in a finite proof.[24] In view of the preceding, it is fairly easy to see that every extension of the above proof in which line 2 is marked, can be further extended (by infinitely many lines) such that line 2 is unmarked. However, since line 2 is marked at stage 7, the first condition of the definition of final derivability is not fulfilled.

[24]The philosophical justification for this requirement is pretty obvious, but let me just spell it out for the skeptic reader. When reasoning in a specific context, we only have finitary means to establish a certain result. We may jump to a meta-level, and at this meta-level speak of infinite proofs, but even at this level we only have finitary means to do so. Hence if we want proofs and provability to relate to reality somehow, finiteness of a proof is a prerequisite.

Upon further inspection, this example also shows that the theorems of **LLL**-Invariance, **LLL**-Closure, Fixed Point and Cautious Monotonicity fail for $\Gamma \subseteq \check{\mathcal{W}}_s$ – see Section 2.5 for the exact formulation of these properties. Let me explain in a nutshell why this is so for Reliability – the reasoning is completely analogous for Minimal Abnormality.

> **LLL**-*Invariance.* Note that $p \check{\vee} !q_1 \in Cn_{\mathbf{K}}(\Gamma)$. Hence, if $Cn_{\mathbf{K}}(\Gamma)$ is our premise set, we may simply introduce $p \check{\vee} !q_1$ as a premise and next derive $p$ on the condition $\{!q_1\}$ on the second line of our proof. In this very short proof, $p$ is derived on an unmarked line.
>
> **LLL**-*Closure.* Note that $\check{\neg} !q_1 \in Cn_{\mathbf{K_i^r}}(\Gamma)$. That is, we can derive $\check{\neg} !q_1 \vee !q_1$ on the empty condition by the rule RU, and on the second line of our proof, (finally) derive $\check{\neg} !q_1$. Also, since $p \check{\vee} !q_1 \in Cn_{\mathbf{K}}(\Gamma)$, it follows by Theorem 2.16 that $p \check{\vee} !q_1 \in Cn_{\mathbf{K_i^r}}(\Gamma)$. But then $p \in Cn_{\mathbf{K}}(Cn_{\mathbf{K_i^r}}(\Gamma)) - Cn_{\mathbf{K_i^r}}(\Gamma)$.
>
> *Fixed Point.* Immediate in view of the preceding paragraph, and the fact that by Theorem 2.16, $Cn_{\mathbf{K}}(Cn_{\mathbf{K_i^r}}(\Gamma)) \subseteq Cn_{\mathbf{K_i^r}}(Cn_{\mathbf{K_i^r}}(\Gamma))$.
>
> *Cautious Monotonicity.* Immediate in view of the preceding paragraph, and the fact that $Cn_{\mathbf{K_i^r}}(\Gamma) \subseteq Cn_{\mathbf{K_i^r}}(\Gamma)$.

### 2.7.2 How To Avoid It

At this point, the reader might become suspicious about the metatheoretic results presented in Section 2.5: how is it possible that one can avoid the above problem whenever $\Gamma \subseteq \mathcal{W}_s$? The reason is rather straightforward. Recall that in Section 2.1, Dab-formulas were defined as *checked* disjunctions of abnormalities. Such disjunctions however do not occur in any $\Gamma \subseteq \mathcal{W}_s$. Hence, in order to introduce them on a line in a proof from such $\Gamma \subseteq \mathcal{W}_s$, one has to *derive* them from the original premises. This allows the proponent of $A$ to introduce all the premises she needs to derive $A$, without being forced to introduce any Dab-formulas.[25]

Let me illustrate this with a variant of the above example. Let $\Gamma' = \Delta'_1 \cup \Delta'_2 \cup \Delta'_3$, where

$$\Delta'_1 = \{(!q_1 \vee !q_2) \supset (p \vee !q_1)\}$$
$$\Delta'_2 = \{!q_1 \vee !q_i \mid i \in \mathbb{N} - \{1\}\}$$
$$\Delta'_3 = \{(!q_1 \vee !q_{i+1}) \supset !q_i \mid i \in \mathbb{N} - \{1, 2\}\}$$

Note that $\Gamma' \subseteq \mathcal{W}_m$, or in other words, $\Gamma'$ contains no checked connectives. We may now apply exactly the same reasoning as before to derive $p$ on line 4 of a proof from $\Gamma'$:

| | | | |
|---|---|---|---|
| 1 | $(!q_1 \vee !q_2) \supset (p \vee !q_1)$ | PREM | $\emptyset$ |
| 2 | $!q_1 \vee !q_2$ | PREM | $\emptyset$ |
| 3 | $p \check{\vee} !q_1$ | 1,2;RU | $\emptyset$ |
| 4 | $p$ | 3;RC | $\{!q_1\}$ |

[25] More precisely: the proponent will only be forced to introduce formulas $Dab(\Delta)$ where $\Delta$ is a singleton, i.e. abnormalities in themselves. Such abnormalities can however not be a problem for the proponent – see the proof of Lemma 4.9.1 in [25, Chapter 4], or the proof for Lemma 5.5 in Chapter 5 of this thesis.

Note that this time, line 4 is not marked. The reason is that the formula on line 2 is not a Dab-formula – even though it is a (classical) disjunction of abnormalities, it is not a checked disjunction of abnormalities. Intuitively, this distinction can be justified as follows: in order for the formula on line 2 to have an impact on the marking at stage 4, the reasoner has to realize that it "is", or perhaps more precise, that it *implies* a disjunction of *abnormalities* – more intuitively still, that it indicates that there may be problems with some of our previous inferences. However, at stage 4, all the reasoner has done is to use the formula $!q_1 \vee !q_2$, as if it were just one "block", and applied modus ponens to this block and the implicative formula on line 1.[26]

Hence by differentiating between (classical) disjunctions of abnormalities on the one hand and Dab-formulas, or checked classical disjunctions of abnormalities on the other, it becomes possible to overcome the problem sketched in the previous section. Since no $\Gamma \subseteq \mathcal{W}_s$ contains checked formulas, completeness and all the other problematic properties are regained for these premise sets.

### 2.7.3 A Specific Kind of Completeness

As I will now show, there is a specific class of premise sets $\Gamma \subseteq \check{\mathcal{W}}_s$ for which **AL** is also complete, i.e. those premise sets $\Gamma$ that are closed under **LLL**. At first sight, this result may seem to have little applications. However, it provides the basis for several crucial steps in the proofs of lemmas and theorems in subsequent chapters.

**Lemma 2.4** *For every finite $\Delta \subset \Omega$, each of the following holds:*

1. $\Gamma \models_{\mathbf{LLL}} Dab(\Delta)$ *iff* $\Gamma \models_{\mathbf{AL}} Dab(\Delta)$.[27]
2. $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ *iff* $\Gamma \vdash_{\mathbf{AL}} Dab(\Delta)$. *(Dab-conservativity)*

*Proof. Ad 1.* ($\Rightarrow$) Immediate in view of the fact that every **AL**-model of $\Gamma$ is an **LLL**-model of $\Gamma$ — see Definitions 2.1 and 2.2. ($\Leftarrow$) Suppose $\Gamma \models_{\mathbf{AL}} Dab(\Delta)$. Let $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. If $M \in \mathcal{M}_{\mathbf{AL}}(\Gamma)$, it follows by the supposition that $M \Vdash Dab(\Delta)$. If $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}}(\Gamma)$, then by Theorem 2.12, there is an $M' \in \mathcal{M}_{\mathbf{AL}}(\Gamma)$ such that $Ab(M') \subset Ab(M)$. In view of the supposition, $M' \Vdash Dab(\Delta)$, whence $M' \Vdash A$ for an $A \in \Delta$. It follows immediately that also $M \Vdash A$, whence $M \Vdash Dab(\Delta)$.

*Ad 2.* ($\Rightarrow$) Immediate in view of item 1, the soundness of **LLL** and Theorem 2.16. ($\Leftarrow$) Immediate in view of item 1, the soundness of **AL** and the completeness of **LLL**. ■

**Lemma 2.5** *Each of the following holds:*

1. *If $\Gamma \models_{\mathbf{AL^r}} A$ then there is a $\Delta \subseteq \Omega - U(\Gamma)$ such that $\Gamma \vdash_{\mathbf{LLL}} A$.*
2. *If $\Gamma \models_{\mathbf{AL^m}} A$, then for every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subseteq \Omega - \varphi$ such that $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.*

[26] My terminology refers to the translation of regular proofs into so-called "block proofs", which allows us to capture the amount of information embodied by a proof at a certain stage. See [25, Section 4.10] for an introduction to this research area.

[27] Item 2 of this lemma is a semantic variant of the one for Theorem 10 in [21], but generalized to every $\Gamma \subseteq \check{\mathcal{W}}_s$.

*Proof. Ad 1.* Suppose $\Gamma \models_{\mathbf{AL^r}} A$. Assume that for no $\Delta \subseteq \Omega - U(\Gamma)$, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By the compactness of **LLL**, there is an $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $M \Vdash\!\!\!\!/\, A$ and $M \Vdash\!\!\!\!/\, Dab(\Delta)$ for every $\Delta \subseteq \Omega - U(\Gamma)$. It follows that $Ab(M) \subseteq U(\Gamma)$, whence $M \in \mathcal{M}_{\mathbf{AL^r}}(\Gamma)$. But then $\Gamma \not\models_{\mathbf{AL^r}} A$ — a contradiction.

*Ad 2.* Suppose $\Gamma \models_{\mathbf{AL^m}} A$. Assume that there is a $\varphi \in \Phi(\Gamma)$ for which there is no $\Delta \subseteq \Omega - \varphi$ such that $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. This implies that $\Gamma$ is **LLL**-satisfiable. By the compactness of **LLL** there is an $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ for which $M \Vdash\!\!\!\!/\, A$ and $M \Vdash\!\!\!\!/\, Dab(\Delta)$ for all $\Delta \subseteq \Omega - \varphi$. Hence, $M \Vdash\!\!\!\!/\, B$ for all $B \in \Omega - \varphi$. Hence, $Ab(M) \subseteq \varphi$. By Theorem 2.1 there is an $M' \in \mathcal{M}_{\mathbf{AL^m}}(\Gamma)$ for which $Ab(M') = \varphi$. Since $M'$ is minimally abnormal, $Ab(M) = Ab(M')$. Hence, by Theorem 2.1, $M \in \mathcal{M}_{\mathbf{AL^m}}(\Gamma)$. This is a contradiction since $M \Vdash\!\!\!\!/\, A$ and $\Gamma \models_{\mathbf{AL^m}} A$. ■

**Theorem 2.23** *Where $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$: if $\Gamma \models_{\mathbf{AL^x}} A$, then $\Gamma \vdash_{\mathbf{AL^x}} A$.*

*Proof.* I prove the theorem for $\mathbf{x} = \mathbf{m}$ – the Reliability case is analogous but much simpler, and therefore safely left to the reader.

Suppose that (†) $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$ and (‡) $\Gamma \models_{\mathbf{AL^m}} A$. By Fact 2.2, there is a $\varphi \in \Phi(\Gamma)$. By (‡) and Lemma 2.5.2, there is a $\Delta \subseteq \Omega - \varphi$ such that $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By the completeness of **LLL**, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, whence by (†), $A \check{\vee} Dab(\Delta) \in \Gamma$.

We start an $\mathbf{AL^m}$-proof from $\Gamma$ as follows:

(a) introduce the premise $A \check{\vee} Dab(\Delta)$ on line 1, and
(b) derive $A$ on line 2, using the rule RC, on the condition $\Delta$.

Let $s_2$ be the stage consisting of these two lines.

Suppose line 2 is marked at stage $s_2$. This implies that $A \check{\vee} Dab(\Delta)$ is a Dab-formula, whence also $A$ is a Dab-formula. But then, by (‡) and Lemma 2.4, $\Gamma \models_{\mathbf{LLL}} A$. By the completeness of **LLL**, $\Gamma \vdash_{\mathbf{LLL}} A$ whence by (†), $A \in \Gamma$. By the reflexivity of $\mathbf{AL^m}$, $A \in Cn_{\mathbf{AL^m}}(\Gamma)$.

Suppose line 2 is not marked at stage $s_2$. Suppose moreover that, in an extension of the proof, line 2 is marked. In view of the preceding, we may further extend the extended proof, such that (c) every minimal Dab-consequence of $\Gamma$ is derived in it, and (d) for every $\varphi' \in \Phi(\Gamma)$, $A$ is derived on a condition $\Delta'$ that has an empty intersection with $\varphi'$. Let $s$ be the stage of the second extension. In view of (c), $\Phi_s(\Gamma) = \Phi(\Gamma)$. Hence in view of (d), line 2 is unmarked at stage $s$. But then, by Definition 2.5, $A$ is finally derived at line 2, whence by Definition 2.6, $A \in Cn_{\mathbf{AL^m}}(\Gamma)$. ■

By the (unrestricted) soundness of flat adaptive logics, we can infer:

**Corollary 2.1** *Where $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$: $\Gamma \vdash_{\mathbf{AL^x}} A$ iff $\Gamma \models_{\mathbf{AL^x}} A$.*

From the preceding, we can also derive that for the same specific group of premise sets, **AL** is closed under its lower limit logic:

**Lemma 2.6** *If $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$, then $Cn_{\mathbf{AL^x}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL^x}}(\Gamma))$.*

*Proof.* That $Cn_{\mathbf{AL^x}}(\Gamma) \subseteq Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL^x}}(\Gamma))$ is immediate in view of the reflexivity of **LLL**. For the other direction, suppose that (†) $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$ and $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL^x}}(\Gamma))$. By the soundness of **LLL**, (‡) $A$ is true in every $M \in \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^x}}(\Gamma))$. By Definition 2.1, resp. 2.2 and the soundness of $\mathbf{AL^x}$, every $\mathbf{AL^x}$-model of $\Gamma$ is an **LLL**-model of $Cn_{\mathbf{AL^x}}(\Gamma)$. Hence by (‡), $\Gamma \models_{\mathbf{AL^x}} A$. By (†) and Theorem 2.23, $\Gamma \vdash_{\mathbf{AL^x}} A$. ■

## 2.8 Two Lemmas

In this section, I present two lemmas that will be useful in the remainder of this thesis, in that they show how certain meta-theoretic properties mentioned in Section 2.5 are easily derivable in the presence of others. The lemmas are stated generically for any logic **L** – in subsequent chapters, they will be used to prove certain properties of prioritized ALs. The proofs are mere generalizations of proofs in [21] and [33] respectively.

Recall the property of Cumulative Transitivity: **L** is cumulatively transitive iff, for every $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$: $Cn_{\mathbf{L}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{L}}(\Gamma)$. As I will now show, whenever **L** is cumulatively transitive and reflexive, then it is also idempotent:

**Lemma 2.7** *If* **L** *is reflexive and cumulatively transitive, then* **L** *is idempotent.*

*Proof.* ($Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(Cn_{\mathbf{L}}(\Gamma))$) Immediate in view of the reflexivity of **L**.

($Cn_{\mathbf{L}}(Cn_{\mathbf{L}}(\Gamma)) \subseteq Cn_{\mathbf{L}}(\Gamma)$) By the reflexivity of **L**, $Cn_{\mathbf{L}}(\Gamma) = \Gamma \cup Cn_{\mathbf{L}}(\Gamma)$. But then $Cn_{\mathbf{L}}(Cn_{\mathbf{L}}(\Gamma)) = Cn_{\mathbf{L}}(\Gamma \cup Cn_{\mathbf{L}}(\Gamma))$, whence by the cumulative transitivity of **L**, $Cn_{\mathbf{L}}(Cn_{\mathbf{L}}(\Gamma)) \subseteq Cn_{\mathbf{L}}(\Gamma)$. ■

Recall criterion (C1) from Section 2.5: $\Gamma$ and $\Gamma'$ are **L**-equivalent iff $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma')$ and $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$. Also, recall the property of Cumulative Indifference: **L** is cumulatively indifferent iff, where $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$, $Cn_{\mathbf{L}}(\Gamma \cup \Gamma') = Cn_{\mathbf{L}}(\Gamma)$.

I will now prove that cumulative indifference implies that (C1) holds:

**Lemma 2.8** *If* **L** *is cumulatively indifferent, then* **L** *obeys criterion (C1) for equivalence.*

*Proof.* Suppose (1) $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma')$ and (2) $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$. By cumulative indifference and (1), $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$. By cumulative indifference and (2), $Cn_{\mathbf{L}}(\Gamma') = Cn_{\mathbf{L}}(\Gamma' \cup \Gamma)$. Hence $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$. ■

# Chapter 3

# Superpositions of Adaptive Logics

*I am indebted to Christian Straßer for many valuable comments on previous versions of this chapter.*

In this Chapter, I present the first of three formats of prioritized adaptive logics: sequential superpositions of (flat) adaptive logics. This is by far the oldest and most often applied format of the three – see [11] where the basic idea of superposing ALs was presented, and see [24, 22, 19, 105, 139, 144, 13] for some applications of this idea.

I will first present a generic definition for systems in this format, illustrate it by means of some logics that deal with prioritized belief bases, and discuss some properties that hold for all logics in this format (Section 3.1). In Section 3.2, I will establish a couple of lemmas that are used throughout the next four chapters. I will proceed with a discussion of the semantics of superpositions of ALs, and present two previously unpublished proposals for their proof theory (Sections 3.3-3.5). Finally, I will give a short overview of some additional metatheoretic results for these systems (Section 3.6).

As indicated in Chapter 1, much of what follows is based on joint work with other logicians in the Ghent group; first and foremost Diderik Batens and Christian Straßer. Diderik Batens was the first to formulate the core ideas behind this format, and made many important observations about it during the preparation of his [25, Chapter 6]. Christian Straßer devoted a chapter to sequential superpositions in his [137], numerous results of which are used in the current thesis. He also contributed in a more concrete way to this chapter, in two respects. First of all, he thoroughly checkread earlier drafts of it, providing suggestions for improvements in various instances. Second, he is co-author of the paper [141], which provided the basis for Section 3.5. Needless to say, all remaining flaws and unclarities are mine.

# 3.1 General Characteristics of SAL

## 3.1.1 The Syntactic Consequence Relation

Let $\mathbf{AL_1}$, $\mathbf{AL_2}$, $\ldots$, $\mathbf{AL_n}$ be flat ALs in standard format. Then we may define the consequence set, obtained by the sequential superposition of these flat ALs as follows:

$$Cn_{\mathbf{SAL}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{AL_n}}(\ldots(Cn_{\mathbf{AL_2}}(Cn_{\mathbf{AL_1}}(\Gamma))\ldots),$$

where the right ... denotes a sequence of right brackets. So $Cn_{\mathbf{SAL}}(\Gamma)$ is the result of applying first $\mathbf{AL_1}$ to $\Gamma$, next applying $\mathbf{AL_2}$ to $Cn_{\mathbf{AL_1}}(\Gamma)$, etc., and finally applying $\mathbf{AL_n}$ to $Cn_{\mathbf{AL_{n-1}}}(\ldots(Cn_{\mathbf{AL_2}}(Cn_{\mathbf{AL_1}}(\Gamma))\ldots)$. For the superposition of infinitely many flat ALs, we need a slightly more technical definition – this will be spelled out below.

Let $\mathbf{LLL_1}, \mathbf{LLL_2}, \ldots$ be the respective lower limit logics of $\mathbf{AL_1}, \mathbf{AL_2}, \ldots$, $\Omega_1, \Omega_2, \ldots$ their sets of abnormalities, and $\mathbf{x_1}, \mathbf{x_2}, \ldots$ their strategies. Before we continue, I introduce a restriction. In this thesis, I will only consider sequential superpositions in which $\mathbf{LLL_1} = \mathbf{LLL_2} = \ldots$. This restriction is motivated by historical and pragmatic reasons. All superpositions of ALs from the literature – all those I am aware of – obey this restriction. Also, it seems much harder to deal with sequential superpositions of ALs that have a different lower limit logic, especially if one wants to obtain a unified proof theory (where the rules of the lower limit logic can be applied at any point in a proof) and semantics (where $\mathbf{SAL}$-models are obtained by a selection on a given set of models $\mathcal{M}_{\mathbf{L}}(\Gamma)$, for a monotonic logic $\mathbf{L}$).

In the remainder, I use the metavariable $\mathbf{SAL}$ for all superpositions that obey this restriction, and $\mathbf{LLL}$ to refer to the lower limit logic of all logics $\mathbf{AL_i}$. Whenever $\mathbf{x_1} = \mathbf{x_2} = \ldots$, I use the metavariable $\mathbf{SAL^x}$, where $\mathbf{x}$ refers to the strategy shared by all the flat ALs of the combined logic.[1]

Recall that I use $I$ as a metavariable for an initial subsequence of $\mathbb{N} = \{1, 2, \ldots\}$. To include the infinite case, it will also be useful to introduce a metavariable for the supremum of $I$. Where $I = \{1, \ldots, n\}$, let $\vec{I} =_{\mathsf{df}} n$; where $I = \mathbb{N}$, let $\vec{I} =_{\mathsf{df}} \infty$. The syntactic consequence relation of $\mathbf{SAL}$, obtained by the superposition of the sequence of flat adaptive logics $\langle \mathbf{AL_i} \rangle_{i \in I}$, is defined as follows:[2]

**Definition 3.1** *Let* $\mathbf{SAL_0} =_{\mathsf{df}} \mathbf{LLL}$.
*For every $i \in I$, let* $Cn_{\mathbf{SAL_i}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{AL_i}}(\ldots(Cn_{\mathbf{AL_2}}(Cn_{\mathbf{AL_1}}(\Gamma)))\ldots)$.
$Cn_{\mathbf{SAL}}(\Gamma) =_{\mathsf{df}} \limsup_{i \to \vec{I}} Cn_{\mathbf{SAL_i}}(\Gamma) = \bigcup_{i \in I} Cn_{\mathbf{SAL_i}}(\Gamma)$.[3]

Since the distinctive feature of ALs is their dynamic proof theory, some readers might complain that $\mathbf{SAL}$ is not a genuine adaptive logic: defined as such,

[1] Most superpositions of ALs in the literature are of the format $\mathbf{SAL^x}$. One notable exception – the only one I am aware of – is discussed in [138]. However, since the metatheory of logics $\mathbf{SAL}$ is not much more complicated then that of $\mathbf{SAL^x}$, I decided to broaden the scope to all logics $\mathbf{SAL}$ in this thesis.

[2] The precise formulation of this definition is due to Christian Straßer.

[3] Note that the sequence $\langle Cn_{\mathbf{SAL_i}}(\Gamma)\rangle_{i \in I}$ converges to its limes superior due to the fact that the sequence is (by definition) monotonic – see Fact 3.1.2 below.

$Cn_{\mathbf{SAL}}(\Gamma)$ is not a function of the proof theory. Indeed, combining flat adaptive logics in the above way seems a rather abstract undertaking and has little to do with the core business of adaptive logicians, i.e. to explicate defeasible reasoning forms by means of dynamic proof theories. However, a semantics and proof theory has been proposed for several logics in **SAL**-format – see e.g. [144, 11, 138]. In many cases, these proposals are adequate with respect to the above definition; in other cases, problems arise, as we will shortly see. One of the motivations behind the current chapter is to provide a generic proof theory for a very large group of superpositions of ALs.

In other words, the above abstract definition is not the real solution to the problem of how to capture prioritized defeasible reasoning in the AL framework – it should rather be seen as a goal or working hypothesis for the adaptive logician, as the consequence relation she or he may try to characterize by proof theoretic and semantic means.

One crucial remark should be made, before I continue. In this and the next chapter, prioritized ALs are studied, whose syntactic consequence relation is defined in terms of the consequence relations of flat ALs. On the one hand, where $\mathbf{L}$ is a logic obtained by such a combination, I will always use $A \in Cn_{\mathbf{L}}(\Gamma)$ to denote that $A$ is a member of the consequence set, obtained by combining a number of flat ALs in a specific way. I will also define a proof theory, which yields a derivability relation $\vdash_{\mathbf{L}}$ that is adequate with respect to $Cn_{\mathbf{L}}(\Gamma)$. On the other hand, a semantic consequence relation $\models_{\mathbf{L}}$ will be provided, which is not always equivalent to the corresponding syntactic consequence relation – this will be explained below. Nevertheless, I will use the same name $\mathbf{L}$, because in many cases, Soundness and Completeness can be guaranteed (in the same sense as for flat ALs, see Section 2.5 in Chapter 2). So the reader should be warned that e.g. the relations $\vdash_{\mathbf{SAL}}$ and $\models_{\mathbf{SAL}}$ do not always coincide, and that we cannot always transfer properties from the syntax to the semantics.

Note that every logic $\mathbf{SAL^x}$ can be characterized by a triple $\langle \mathbf{LLL}, \langle\Omega_i\rangle_{i\in I}, \mathbf{x}\rangle$, where $\mathbf{LLL}$ is the lower limit logic shared by all the flat ALs in the combination, each $\Omega_i$ is the set of abnormalities of the logic $\mathbf{AL_i}$, and $\mathbf{x}$ is the strategy these flat ALs have in common.

**More Notational Conventions.** Let me introduce some notational conventions for the following chapters. Let $I \subseteq \mathbb{N}$ be given. For every $i \in I$, let $\Omega_i \subseteq \check{\mathcal{W}}_s$ be a set of abnormalities, i.e. a set of formulas characterized by one or several logical forms. For all $i \in I$, let $\Omega_{(i)} =_{\mathsf{df}} \Omega_1 \cup \ldots \cup \Omega_i$ and let $\Omega_{[i]} =_{\mathsf{df}} \Omega_i - \Omega_{(i-1)}$, where it is is stipulated that $\Omega_{[1]} =_{\mathsf{df}} \Omega_1$. Let $\Omega =_{\mathsf{df}} \bigcup_{i\in I}\Omega_i$.

Unless specified differently, I will use the term Dab-formulas to refer to any "checked" disjunction of the members of a finite $\Delta \subset \Omega$. Likewise, I use $\Phi(\Gamma)$ to denote the set of minimal choice sets of $\Sigma(\Gamma) =_{\mathsf{df}} \{\Delta \subset \Omega \mid \Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ and for no $\Delta' \subset \Delta : \Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)\}$. As before, let $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$. I will use $\mathbf{AL^x}$ to refer to the flat adaptive logic that is characterized by the triple $\langle \mathbf{LLL}, \bigcup_{i\in I}\Omega_i, \mathbf{x}\rangle$. Finally, I use $Ab(M)$ to denote the set $\{A \in \Omega \mid M \Vdash A\} = \{A \in \Omega_i \mid M \Vdash A, i \in I\}$.

It will be useful to introduce metavariables for the Dab-consequences of $\Gamma$ associated with each priority level $i \in I$, and associated sets of unreliable formulas and minimal choice sets. Where $i \in I$, $Dab(\Delta)$ is a minimal $\text{Dab}_i$-consequence of $\Gamma$ iff $\Delta \subset \Omega_i$, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ for which $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta')$. Where $i \in I$ and $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal $\text{Dab}_i$-consequences of $\Gamma$, let $\Sigma^i(\Gamma) =_{\mathsf{df}} \{\Delta_1, \Delta_2, \ldots\}$, $U^i(\Gamma) =_{\mathsf{df}} \bigcup \Sigma^i(\Gamma)$ and define $\Phi^i(\Gamma)$ as the set of minimal choice sets of $\Sigma^i(\Gamma)$.

In the remainder, I will sometimes consider disjunctions of abnormalities *up to level* $i \in I$. $Dab(\Delta)$ is a minimal $\text{Dab}_{(i)}$-consequence of $\Gamma$ iff $\Delta \subset \Omega_{(i)}$, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ for which $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta')$. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal $\text{Dab}_{(i)}$-consequences of $\Gamma$, let $\Sigma^{(i)}(\Gamma) =_{\mathsf{df}} \{\Delta_1, \Delta_2, \ldots\}$, $U^{(i)}(\Gamma) = \bigcup \Sigma^{(i)}(\Gamma)$ and define $\Phi^{(i)}(\Gamma)$ as the set of minimal choice sets of $\Sigma^{(i)}(\Gamma)$.

Where the logic **SAL** is associated with a sequence $\langle \Omega_i \rangle_{i \in I}$ and a lower limit logic **LLL**, let the logics $\mathbf{AL}^{\mathbf{x}}_{\mathbf{(i)}}$ be characterized by the triple $\langle \mathbf{LLL}, \Omega_{(i)}, \mathbf{x} \rangle$. I will use the metavariables $\mathbf{SAL}^{\mathbf{x}}_{\mathbf{(I)}}$ for logics obtained by the superposition of logics $\langle \mathbf{AL}^{\mathbf{x}}_{\mathbf{(i)}} \rangle_{i \in I}$, and $\mathbf{SAL}^{\mathbf{x}}_{\mathbf{(i)}}$ for the logic obtained by the superposition of logics $\langle \mathbf{AL}^{\mathbf{x}}_{\mathbf{(j)}} \rangle_{j \leq i}$.

### 3.1.2 Examples of Logics

In Section 2.4, we saw an example of the flat logics $\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}$ and $\mathbf{K}^{\mathbf{m}}_{\mathbf{1}}$, which capture reasoning with (possibly inconsistent or false) plausible knowledge. In these systems, a formula $A$ is either considered as a fact, or as a plausible statement, the truth of which is presumed *ceteris normalibus*.

However, in real life, we often distinguish between different degrees of plausibility. In this case, we may start from what is called a *prioritized* belief base, i.e. a sequences of sets of beliefs, each associated with a distinct priority level. Formally, a prioritized belief base is a set of the form $\Psi = \langle \Theta_1, \Theta_2, \ldots \rangle$, where each $\Theta_i$ is a set of formulas, and the index of the sets denotes their plausibility degree: $\Theta_1$ is the set of most plausible beliefs, $\Theta_2$ of the second most plausible beliefs, and so on.

Several ALs have been developed to explicate reasoning with prioritized belief bases – see [32], [164] and [163]. These logics typically use a certain logical operator or a sequence of such operators to express that a belief has a certain degree of plausibility. I will discuss only one such system here, in order to illustrate how superpositions of ALs deal with prioritized defeasible reasoning.

As before, I restrict the logic to the propositional level, where $\mathcal{W}^l_s$ denotes the set of literals. To express the plausibility degree of a piece of information, sequences of diamonds are used: $\Diamond\Diamond \ldots \Diamond A$. The longer the sequence, the less plausible the information. A sequence of $i$ diamonds will be abbreviated by $\Diamond^i$. Starting from a prioritized belief base $\Psi = \langle \Theta_1, \Theta_2, \ldots \rangle$, we translate this into the premise set $\Psi^{\Diamond} = \bigcup_{i \in \mathbb{N}} \{\Diamond^i A \mid A \in \Theta_i\}$. In the examples below, we will also consider some propositions as facts – these are represented by non-modal formulas.

Where $A \in \mathcal{W}_c$, let $!^i A$ abbreviate $\Diamond^i A \wedge \neg A$. Let $\Omega^{\mathbf{K}}_i =_{\mathsf{df}} \{\Diamond^i A \wedge \neg A \mid A \in \mathcal{W}^l_c\}$ and $\Omega^{\mathbf{K}}_{(i)} =_{\mathsf{df}} \Omega^{\mathbf{K}}_1 \cup \ldots \cup \Omega^{\mathbf{K}}_i$. Where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$, we first define the logics $\mathbf{K}^{\mathbf{x}}_{\mathbf{i}}$ as flat ALs in standard format, characterized by the triple $\langle \mathbf{K}, \Omega^{\mathbf{K}}_i, \mathbf{x} \rangle$. Likewise, the flat ALs $\mathbf{K}^{\mathbf{x}}_{(\mathbf{i})}$ are characterized by the triple $\langle \mathbf{K}, \Omega^{\mathbf{K}}_{(i)}, \mathbf{x}\rangle$. If we superpose some of these flat ALs, we obtain the the following systems:

$\mathbf{SK}^{\mathbf{r}} =_{\mathsf{df}} \langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_{i}\rangle_{i\in\mathbb{N}}, \mathbf{r}\rangle$
$\mathbf{SK}^{\mathbf{m}} =_{\mathsf{df}} \langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_{i}\rangle_{i\in\mathbb{N}}, \mathbf{m}\rangle$
$\mathbf{SK}^{\mathbf{r}}_{(\mathbb{N})} =_{\mathsf{df}} \langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_{(i)}\rangle_{i\in\mathbb{N}}, \mathbf{r}\rangle$
$\mathbf{SK}^{\mathbf{m}}_{(\mathbb{N})} =_{\mathsf{df}} \langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_{(i)}\rangle_{i\in\mathbb{N}}, \mathbf{m}\rangle$

Note that these logics can deal with cases in which infinitely many priority levels are used. However, in many cases, I will confine myself to the much simpler logics that use only two priority levels, i.e.:[4]

$\mathbf{SK2}^{\mathbf{r}} =_{\mathsf{df}} \langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_{i}\rangle_{i\leq 2}, \mathbf{r}\rangle$
$\mathbf{SK2}^{\mathbf{m}} =_{\mathsf{df}} \langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_{i}\rangle_{i\leq 2}, \mathbf{m}\rangle$
$\mathbf{SK2}^{\mathbf{r}}_{(\mathbf{2})} =_{\mathsf{df}} \langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_{(i)}\rangle_{i\leq 2}, \mathbf{r}\rangle$
$\mathbf{SK2}^{\mathbf{m}}_{(\mathbf{2})} =_{\mathsf{df}} \langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_{(i)}\rangle_{i\leq 2}, \mathbf{m}\rangle$

Let us consider some examples of premise sets, to illustrate how these logics deal with prioritized beliefs.

**Example 3.1** *Let* $\Gamma_{\mathsf{e1}} = \{\Diamond p, \Diamond\Diamond q, \neg p \vee \neg q\}$. *Note that* $\Gamma$ *has no minimal* $Dab_1$*-consequences* $Dab(\Delta)$. *As a result, we can finally* $\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}$*-derive* $p$ *from* $\Gamma$, *on the condition* $\{\Diamond p \wedge \neg p\}$. *But then also* $\neg q$ *is a* $\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}$*-consequence of* $\Gamma$. *Hence* $\Diamond\Diamond q \wedge \neg q$ *is a minimal* $Dab_2$*-consequence of* $Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma)$. *This implies that we cannot derive* $q$ *in a* $\mathbf{K}^{\mathbf{r}}_{\mathbf{2}}$*-proof from* $Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma)$. *To summarize,* $p, \neg q \in Cn_{\mathbf{SK2}^{\mathbf{r}}}(\Gamma)$. *By the same reasoning,* $p, \neg q \in Cn_{\mathbf{SK2}^{\mathbf{m}}}(\Gamma)$.

**Example 3.2** *Let* $\Gamma_{\mathsf{e2}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg p \vee \neg r\}$. *Note that the following Dab-formulas are* $\mathbf{K}$*-derivable from* $\Gamma_{\mathsf{e2}}$*:*

*(1)* $!^1 p \,\check{\vee}\, !^1 q$
*(2)* $!^1 p \,\check{\vee}\, !^2 r$

*In view of (1), neither* $p$ *nor* $q$ *are* $\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}$*-derivable from* $\Gamma_{\mathsf{e2}}$. *This also means that we cannot derive* $!^2 r$ *from* $\Gamma_{\mathsf{e2}}$ *by the first logic of the superposition. As a result, there is no* $Dab_2$*-formula* $Dab(\Delta)$, *such that* $Dab(\Delta) \in Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma_{\mathsf{e2}})$. *But then the abnormality* $!^2 r$ *is considered reliable by the logic* $\mathbf{K}^{\mathbf{r}}_{\mathbf{2}}$, *in view of* $Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma_{\mathsf{e2}})$. *Hence we may finally* $\mathbf{K}^{\mathbf{r}}_{\mathbf{2}}$*-derive* $r$ *and* $\neg p$ *from* $Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma_{\mathsf{e2}})$, *on the condition* $\{!^2 r\}$. *By the same reasoning,* $r, \neg p \in Cn_{\mathbf{SK2}^{\mathbf{m}}}(\Gamma_{\mathsf{e2}})$.

**Example 3.3** *Let* $\Gamma_{\mathsf{e3}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg p \vee \neg r, \neg q \vee \neg r\}$. *Note that in addition to (1) and (2), we may also* $\mathbf{K}$*-derive the Dab-formula* $!^1 q \vee !^2 r$ *from* $\Gamma_{\mathsf{e3}}$. *This means that now also* $\neg q$ *can be* $\mathbf{SK2}^{\mathbf{r}}$*-derived from* $\Gamma_{\mathsf{e3}}$*:* $r, \neg p, \neg q \in Cn_{\mathbf{SK2}^{\mathbf{r}}}(\Gamma_{\mathsf{e3}})$. *On the other hand, we can* $\mathbf{K}^{\mathbf{m}}_{\mathbf{1}}$*-derive* $!^2 r$ *from* $\Gamma_{\mathsf{e3}}$, *on the conditions* $\{!^1 p\}$ *and* $\{!^1 q\}$. *As a result,* $\neg r \in Cn_{\mathbf{SK2}^{\mathbf{m}}}(\Gamma_{\mathsf{e3}})$.

[4] It is noteworthy that most if not all the metatheoretic difficulties concerning prioritized ALs already pop up in this simplified case. This is most apparent in view of Appendix C, where a number of negative claims about prioritized ALs in various formats are established.

In view of the third example, $\mathbf{SK2^r}$ and $\mathbf{SK2^m}$ are incomparable. Roughly speaking, this can be explained as follows. If we superpose logics, then the first logic may allow us to derive certain Dab-formulas $Dab(\Delta)$, where $\Delta \subset \Omega_2$, which in turn block the derivation of formulas by the second logic. If the first logic has Minimal Abnormality as its strategy, then this results in *more* such Dab-formulas, whence we will be able to derive less by the second logic in the superposition.

The second example deserves a bit more explanation for the case of $\mathbf{SK2^m}$. Note that since $\Gamma_{\mathsf{e2}} \vdash_{\mathbf{K}} !^1p \vee !^1q$, neither $p$ nor $q$ can be finally $\mathbf{K_1^m}$-derived from $\Gamma_{\mathsf{e2}}$. Only $p \vee q$ is a $\mathbf{K_1^m}$-consequence of this premise set, since it can be derived on the conditions $\{!^1p\}$ and $\{!^1q\}$. As we have seen, $\neg p$ is a $\mathbf{K_2^m}$-consequence of $Cn_{\mathbf{K_1^m}}(\Gamma_{\mathsf{e2}})$. Since $\mathbf{K_2^m}$ is closed under $\mathbf{K}$, it follows that $q \in Cn_{\mathbf{K_2^m}}(Cn_{\mathbf{K_1^m}}(\Gamma_{\mathsf{e2}})) = Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{e2}})$.

This fact may appear to some as counterintuitive: with the aid of a less plausible belief $r$, we are suddenly able to decide that $q$ is the case, and that $p$ is false, although it is just as plausible as $q$. Compare this also to $\Gamma_4 = \{\Diamond p, \Diamond\neg p, \Diamond\Diamond p\}$. By the same reasoning as for $\Gamma_{\mathsf{e2}}$, $p \in Cn_{\mathbf{SK2^x}}(\Gamma_4)$ for $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$. So eventually, less plausible beliefs may allow us to choose sides in a conflict between more plausible ones. This can be explained by the fact that the second logic in the superposition does not take into account abnormalities of the first logic: $!^1p \vee !^2r$ is not a Dab-formula for $\mathbf{K_2^x}$, since $!^1p \notin \Omega_2^{\mathbf{K}}$.

Similar examples can be constructed for any logic $\mathbf{SAL} = \langle \mathbf{LLL}, \langle\Omega_i\rangle_{i\in I}, \mathbf{x}\rangle$, where for some $i, i+1 \in I$: $\Omega_i \not\subseteq \Omega_{i+1}$. Of course, it depends on the specific application context of a logic $\mathbf{SAL}$ whether this is seen as a problem or rather as an advantage for the system. If it is the aim of $\mathbf{SAL}$ to make maximal use of the (prioritized) information that is available, then it may be justified that information with a lower priority degree allows us to choose sides between two higher ranked sources of information. If conflicts between high authorities are considered a sufficient reason to drop a certain belief or obligation – regardless of what lower ranked authorities say –, then we should make our logic behave accordingly.

In any case, if one does consider this behavior as problematic, there is a straightforward solution, i.e. to replace the logics $\mathbf{SK2^r}$ and $\mathbf{SK2^m}$ by their two weaker nephews $\mathbf{SK2^r_{(2)}}$ and $\mathbf{SK2^m_{(2)}}$.[5]

Let me briefly show what happens to $\Gamma_{\mathsf{e2}}$ in the case of $\mathbf{SK2^r_{(2)}}$. Since the first logic in the superposition is the same as that of $\mathbf{SK2^r}$, it suffices to look at the second. Note that $!^1p \,\check{\vee}\, !^2r$ is a minimal Dab-consequence of $\Gamma$, and both $!^1p$ and $!^2r$ are in $\Omega_1^{\mathbf{K}} \cup \Omega_2^{\mathbf{K}}$. Hence $!^2r$ is an unreliable abnormality for the second logic. As a result, we are no longer able to derive $r$, whence we also cannot derive $\neg p$ and $q$.

### 3.1.3 Basic Facts and Theorems

Before I present the semantics of sequential superpositions, let me discuss a number of properties that follow from Definition 3.1 and the metatheory of flat

[5] It will be shown in Chapter 6 that indeed, every logic $\mathbf{SAL^r}$ is always at least as strong as the corresponding logic $\mathbf{SAL^r_{(I)}}$, and that every logic $\mathbf{SAL^m}$ is at least as strong as the corresponding logic $\mathbf{SAL^m_{(I)}}$ whenever $\Sigma(\Gamma)$ has only finitely many minimal choice sets.

ALs. As will become clear, their proofs are but a matter of routine. Nevertheless, each of these properties will be used in the proof of other, less obvious theorems of this and subsequent chapters.

The following is immediate in view of Definition 3.1, Theorem 2.16, and the reflexivity of each logic $\mathbf{AL_i}$:

**Fact 3.1** *Each of the following holds:*

1. $Cn_{\mathbf{SAL_1}}(\Gamma) = Cn_{\mathbf{AL_1}}(\Gamma)$
2. *for all* $i \in I$, $Cn_{\mathbf{SAL_{i-1}}}(\Gamma) \subseteq Cn_{\mathbf{SAL_i}}(\Gamma)$
3. $\Gamma \subseteq Cn_{\mathbf{SAL}}(\Gamma)$
4. *for all* $i \in I$, $Cn_{\mathbf{SAL_i}}(\Gamma) \subseteq Cn_{\mathbf{SAL}}(\Gamma)$
5. *for all* $i \in I$, $\Gamma \subseteq Cn_{\mathbf{SAL_i}}(\Gamma)$
6. *for all* $i \in I$, $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{SAL_i}}(\Gamma)$
7. $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{SAL}}(\Gamma)$

Recall that every flat adaptive logic **AL** is closed under **LLL** (see Theorem 2.18). I now prove a similar result for **SAL**:

**Theorem 3.1** *Where* $\Gamma \subseteq \mathcal{W}_s$, *each of the following holds:*

1. *for every* $i \in I$, $Cn_{\mathbf{SAL_i}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$.
2. $Cn_{\mathbf{SAL}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$. *(LLL-closure)*

*Proof. Ad 1.* ($i = 1$) Immediate in view of Theorem 2.18 and Fact 3.1.1.

($i \Rightarrow i+1$) By Definition 3.1, $Cn_{\mathbf{SAL_{i+1}}}(\Gamma) = Cn_{\mathbf{AL_{i+1}}}(Cn_{\mathbf{SAL_i}}(\Gamma))$. Hence by the induction hypothesis and Lemma 2.6,

$$Cn_{\mathbf{SAL_{i+1}}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL_{i+1}}}(Cn_{\mathbf{SAL_i}}(\Gamma))) \tag{3.1}$$

By (3.1) and Definition 3.1, $Cn_{\mathbf{SAL_{i+1}}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_{i+1}}}(\Gamma))$.

*Ad 2.* That $Cn_{\mathbf{SAL}}(\Gamma) \subseteq Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$ is immediate in view of the reflexivity of **LLL**. Suppose that $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$. By Definition 3.1 and the compactness of **LLL**, there is an $i \in I$ such that $A \in Cn_{\mathbf{LLL}}(\bigcup_{j\leq i} Cn_{\mathbf{SAL_j}}(\Gamma))$, whence by Fact 3.1.3, $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$. By item 1., $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$, whence by Definition 3.1, $A \in Cn_{\mathbf{SAL}}(\Gamma)$. ■

Another property that can easily be transferred from **AL** to **SAL** is the property of LLL-invariance:

**Theorem 3.2** *Where* $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{SAL}}(\Gamma) = Cn_{\mathbf{SAL}}(Cn_{\mathbf{LLL}}(\Gamma))$

*Proof.* By the LLL-invariance of $\mathbf{AL_1}$ (see Theorem 2.19), $Cn_{\mathbf{AL_1}}(\Gamma) = Cn_{\mathbf{AL_1}}(Cn_{\mathbf{LLL}}(\Gamma))$. The rest is immediate in view of Definition 3.1. ■

The following property follows almost immediately for **SAL**, in view of the reassurance that is guaranteed for each of the logics $\mathbf{AL_i}$:

**Theorem 3.3** *If* $\Gamma$ *is not* **LLL**-*trivial, then* $Cn_{\mathbf{SAL}}(\Gamma)$ *is not* **LLL**-*trivial. (Syntactic Reassurance)*

*Proof.* I prove that for every $i \in I$, $Cn_{\mathbf{SAL_i}}(\Gamma)$ is not **LLL**-trivial – the rest is immediate in view of Definition 3.1 and the compactness of **LLL**. ($i = 1$) Immediate in view of the Reassurance of $\mathbf{AL_1}$. ($i \Rightarrow i+1$) Immediate in view of the induction hypothesis and the Reassurance of $\mathbf{AL_{i+1}}$. ■

Finally, we can prove a simple equivalence criterion for all superpositions of ALs:

**Theorem 3.4** *If $\Gamma, \Gamma' \subseteq \mathcal{W}_s$ and $\Gamma$ and $\Gamma'$ are* **LLL***-equivalent, then they are* **SAL***-equivalent.*

*Proof.* Suppose the antecedent holds. Then by Theorem 2.22, $Cn_{\mathbf{AL_1}}(\Gamma) = Cn_{\mathbf{AL_1}}(\Gamma')$. The rest is immediate in view of Definition 3.1. ■

## 3.2 Some Crucial Lemmas

Several proofs in the remainder of this thesis rely on one crucial lemma and a number of other lemmas that can easily be derived from it. To facilitate the reading of this chapter, their proofs will be presented in this separate section. To prepare for the crucial lemma, one lemma about minimal choice sets needs to be established first:

**Lemma 3.1** *Where (1) $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ is a set of sets and (2) $\varphi$ is a choice set of $\Sigma$: (3) for every $A \in \varphi$, there is a $\Delta \in \Sigma$ for which $\Delta \cap \varphi = \{A\}$ iff (4) $\varphi$ is a minimal choice set of $\Sigma$.*

*Proof.* Suppose (1) and (2) hold. ($\Rightarrow$) Suppose (3) holds, and consider a $\varphi' \subset \varphi$ and a $B \in \varphi, B \notin \varphi'$. By (3), there is a $\Delta \in \Sigma$ for which $\Delta \cap \varphi = \{B\}$ and hence $\Delta \cap \varphi' = \emptyset$. This implies that $\varphi'$ is not a choice set of $\Sigma$. As a result, $\varphi$ is a minimal choice set of $\Sigma$. ($\Leftarrow$) Suppose (3) is false, whence there is a $B \in \varphi$ such that, for no $\Delta \in \Sigma$, $\varphi \cap \Delta = \{B\}$. In that case for every $\Delta$ for which $B \in \Delta$, there is a $C \in \varphi - \{B\}$ such that $C \in \Delta$. Hence $\varphi - \{B\}$ is a choice set of $\Sigma$, hence $\varphi$ is not a minimal choice set of $\Sigma$. ■

Recall that $\Sigma^{(i)}(\Gamma)$ denotes the set of all sets $\Delta$ for which $Dab(\Delta)$ is a minimal $\text{Dab}_{(i)}$-consequence of $\Gamma$, and that $\Phi^{(i)}(\Gamma)$ is the set of minimal choice sets of $\Sigma^{(i)}(\Gamma)$. In the remainder, I will establish a specific relation between the sets $\Phi^{(i)}(\Gamma)$ and $\Phi^{(i+1)}(\Gamma)$ for all $i, i+1 \in I$; this will also enable me to show the relation between each of the sets $\Phi^{(i)}(\Gamma)$ and $\Phi(\Gamma)$.

Note that for each $\Delta \in \Sigma^{(i+1)}(\Gamma) - \Sigma^{(i)}(\Gamma)$, $\Delta \cap \Omega_{[i+1]} \neq \emptyset$, i.e. $\Delta$ contains abnormalities of priority level $i+1$ that do not belong to any lower priority level. Where $i \in I$ and $\varphi \in \Phi^{(i)}(\Gamma)$, let $\Phi^{[i+1]}_{\varphi}(\Gamma)$ be the set of minimal choice sets of $\{\Delta \cap \Omega_{[i+1]} \mid \Delta \in \Sigma^{(i+1)}(\Gamma) - \Sigma^{(i)}(\Gamma), \varphi \cap \Delta = \emptyset\}$.[6]

**Lemma 3.2** *Where $i \in I$: for all $\varphi \in \Phi^{(i)}(\Gamma)$ and all $\varphi' \in \Phi^{[i+1]}_{\varphi}(\Gamma)$, $\varphi \cup \varphi' \in \Phi^{(i+1)}(\Gamma)$.*

[6] The precise definition of this set greatly benefited from some comments by Christian Straßer. He also contributed to the proof for Lemma 3.2, which originally appeared in Section 4 of [149].

*Proof.* Let $\Delta \in \Sigma^{(i+1)}(\Gamma)$. Suppose $\Delta \cap \varphi = \emptyset$. Then $\Delta \notin \Sigma^{(i)}(\Gamma)$ since $\varphi \in \Phi^{(i)}(\Gamma)$. Hence, $\Delta \in \Sigma^{(i+1)}(\Gamma) - \Sigma^{(i)}(\Gamma)$. In this case $\Delta \cap \Omega_{[i+1]} \neq \emptyset$. Hence $\varphi' \cap \Delta \neq \emptyset$, since $\varphi' \in \Phi_{\varphi}^{[i+1]}$. Hence $\varphi \cup \varphi'$ is a choice set of $\Sigma^{(i+1)}(\Gamma)$.

By the right-left direction of Lemma 3.1 and the fact that $\varphi \in \Phi^{(i)}(\Gamma)$, for every $A \in \varphi$ there is a $\Delta \in \Sigma^{(i)}(\Gamma)$ such that $\Delta \cap \varphi = \{A\}$. Moreover, for all these $\Delta$, $\varphi' \cap \Delta = \emptyset$, since $\varphi' \subseteq \Omega_{[i+1]}$. Finally, $\Sigma^{(i)}(\Gamma) \subseteq \Sigma^{(i+1)}(\Gamma)$, which gives us:

(1) for every $A \in \varphi$ there is a $\Delta \in \Sigma^{(i+1)}(\Gamma)$ such that $\Delta \cap (\varphi \cup \varphi') = \{A\}$.

From the right-left direction of Lemma 3.1: for every $A \in \varphi'$, there is a $\Theta$ such that $\Theta \cap \varphi' = \{A\}$, where $\Theta = \Delta \cap \Omega_{[i+1]}$ for a $\Delta \in \Sigma^{(i+1)}(\Gamma)$. Since $\varphi' \subseteq \Omega_{[i+1]}$, $\Delta \cap \varphi' = \{A\}$. Moreover, in view of the definition of $\Phi_{\varphi}^{[i+1]}$, $\Delta \cap \varphi = \emptyset$. Hence we have:

(2) for every $A \in \varphi'$, there is a $\Delta \in \Sigma^{(i+1)}(\Gamma)$ such that $\Delta \cap (\varphi \cup \varphi') = \{A\}$.

By (1) and (2): for every $A \in \varphi \cup \varphi'$, there is a $\Delta \in \Sigma^{(i+1)}(\Gamma)$ such that $\Delta \cap (\varphi \cup \varphi') = \{A\}$. By the left-right direction of Lemma 3.1, $\varphi \cup \varphi'$ is a minimal choice set of $\Sigma^{(i+1)}(\Gamma)$, hence $\varphi \cup \varphi' \in \Phi^{(i+1)}(\Gamma)$. ■

**Lemma 3.3** *Where $i, i+1 \in I$: for every $\varphi \in \Phi^{(i)}(\Gamma)$, there is a $\psi \in \Phi^{(i+1)}(\Gamma)$ for which $\psi \cap \Omega_{(i)} = \varphi$.*

*Proof.* Suppose $\varphi \in \Phi^{(i)}(\Gamma)$. Let $\varphi'$ be an arbitrary element in $\Phi_{\varphi}^{[i+1]}$. Note that $\varphi' \subseteq \Omega_{[i+1]}$. Define $\psi = \varphi \cup \varphi'$. The lemma follows immediately in view of Lemma 3.2. ■

It is important to observe that Lemma 3.3 holds for all sequences of sets of abnormalities $\langle \Omega_j \rangle_{j \in J}$. That is, we may also apply it to sequences of the form $\langle \Omega_i, \Omega \rangle$ and to sequences of the form $\langle \Omega_i, \Omega_{i+k} \rangle$ . Hence we can derive:

**Lemma 3.4** *For all $i \in I$ and for every $\varphi \in \Phi^{(i)}(\Gamma)$, there is a $\psi \in \Phi(\Gamma)$ for which $\psi \cap \Omega_{(i)} = \varphi$.*

**Lemma 3.5** *For all $i, i+k \in I$ and for every $\varphi \in \Phi^{(i)}(\Gamma)$, there is a $\psi \in \Phi^{(i+k)}(\Gamma)$ for which $\psi \cap \Omega_{(i)} = \varphi$.*

From the preceding, we can derive the following lemma which concerns the cardinality of $\Phi^{(i)}(\Gamma)$ and $\Phi(\Gamma)$:

**Lemma 3.6** *For every $i \in I$, the cardinality of $\Phi^{(i)}(\Gamma)$ is never greater than that of $\Phi(\Gamma)$.*

*Proof.* Let $i \in I$ and consider $\varphi, \psi \in \Phi^{(i)}(\Gamma)$, with $\varphi \neq \psi$. By Lemma 3.4, there is a $\varphi' \in \Phi(\Gamma) : \varphi' \cap \Omega_{(i)} = \varphi$ and a $\psi' \in \Phi(\Gamma) : \psi' \cap \Omega_{(i)} = \psi$. As a result, $\varphi' \neq \psi'$. ■

## 3.3 A Semantics for SAL

In this section, the semantics for sequential superpositions is discussed. As will become clear, most of the metatheoretic results I will present here follow almost immediately from properties that were established by Christian Straßer in his [137]. The only real novelty is that I include superpositions of infinitely many flat ALs, relying on the last lemma of the preceding section.

### 3.3.1 Sequential Superposition of Selections

The general idea of the **SAL**-semantics was already described in [11]. Just as the syntactic consequence relation of **SAL** is defined in terms of a superposition of consequence relations of flat ALs – whence we apply different standards of normality sequentially –, its semantics is defined in terms of a superposition of selections imposed on the set of **LLL**-models of $\Gamma$. We first select models in view of $\mathbf{AL_1}$, from the resulting set, we select a subset in view of $\mathbf{AL_2}$, etc. Of course, how the selection is performed exactly, depends on the strategy.

The selection procedure is most easily understood when applied to superpositions of ALs that use the Minimal Abnormality Strategy. Where $\mathbf{SAL^m}$ is characterized by the sequence of flat adaptive logics $\langle \mathbf{AL_i^m} \rangle_{i \in I}$, we first select the **LLL**-models of $\Gamma$ that verify a minimal set of abnormalities from $\Omega_1$. From the resulting set, we select those that verify a minimal set of abnormalities from $\Omega_2$, etc.

To include superpositions where some or all of the flat ALs use Reliability, we need a few definitions. First of all, we define the set of $i$-minimally abnormal models in a set $\mathcal{M}$ of **LLL**-models of $\Gamma$, as follows:

**Definition 3.2** *Where* $\mathcal{M} \subseteq \mathcal{M}_{\mathbf{LLL}}(\Gamma)$*:* $Min_i(\mathcal{M}) =_{\mathsf{df}} \{M \in \mathcal{M} \mid$ *there is no* $M' \in \mathcal{M} : Ab(M') \cap \Omega_i \subset Ab(M) \cap \Omega_i\}$.

Recall that the selection of the $\mathbf{AL^r}$-models of $\Gamma$ proceeds in terms of the set $U(\Gamma)$. If we want to carry forward such a selection within the sequential procedure, we have to define a set of unreliable formulas after a given step. As explained in Section 2.2, the set of unreliable formulas in view of $\Gamma$ is identical to the set of abnormalities that are verified by at least one model $M \in \mathcal{M}_{\mathbf{AL^m}}(\Gamma)$ (see page 21). Similarly, after the $(i-1)$th selection step in the recursive procedure, we may define unreliable formulas for the logic $\mathbf{AL_i}$ as those abnormalities that are are verified by at least one $i$-minimally abnormal model in $\mathcal{M}_{\mathbf{SAL_{i-1}}}(\Gamma)$:

**Definition 3.3** *For every* $i \in I$*:* $\Psi_i^{\mathbf{S}}(\Gamma) =_{\mathsf{df}} \{A \in \Omega_i \mid M \Vdash A$ *for an* $M \in Min_i(\mathcal{M}_{\mathbf{SAL_{i-1}}}(\Gamma))\}$.

The preceding definitions finally allow us to define the set of **SAL**-models of $\Gamma \subseteq \check{\mathcal{W}}_s$:

**Definition 3.4** $\mathcal{M}_{\mathbf{SAL_0}}(\Gamma) =_{\mathsf{df}} \mathcal{M}_{\mathbf{LLL}}(\Gamma)$*. For every* $i \in I$*:*

*(i)* *if* $\mathbf{x_i} = \mathbf{r}$*:* $\mathcal{M}_{\mathbf{SAL_i}}(\Gamma) =_{\mathsf{df}} \{M \in \mathcal{M}_{\mathbf{SAL_{i-1}}}(\Gamma) \mid Ab(M) \subseteq \Psi_i^{\mathbf{S}}(\Gamma)\}$
*(ii)* *if* $\mathbf{x_i} = \mathbf{m}$*:* $\mathcal{M}_{\mathbf{SAL_i}}(\Gamma) =_{\mathsf{df}} Min_i(\mathcal{M}_{\mathbf{SAL_{i-1}}}(\Gamma))$.

$\mathcal{M}_{\mathbf{SAL}}(\Gamma) =_{\mathsf{df}} \liminf_{i \to \bar{I}} \mathcal{M}_{\mathbf{SAL_i}}(\Gamma) = \bigcap_{i \in I} \mathcal{M}_{\mathbf{SAL_i}}(\Gamma)$.[7]

In view of Definitions 2.2 and 3.4, we can immediately infer:

**Fact 3.2** $\mathcal{M}_{\mathbf{SAL_1^m}}(\Gamma) = \mathcal{M}_{\mathbf{AL_1^m}}(\Gamma) = \mathcal{M}_{\mathbf{AL_{(1)}^m}}(\Gamma)$.

**Example 3.4** *Consider again* $\Gamma_{\mathsf{e3}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg p \vee \neg r, \neg q \vee \neg r\}$ *from Example 3.3. Recall that* $\Gamma_{\mathsf{e3}}$ *has the following minimal Dab-consequences:*

$!^1 p \check{\vee} !^1 q$
$!^1 p \check{\vee} !^2 r$
$!^1 q \check{\vee} !^2 r$

*Let us first take a look at how the* $\mathbf{SK2^m}$*-semantics deals with this case. After the first selection, only those models* $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{e3}})$ *are selected for which either* $Ab(M) \cap \Omega_1^{\mathbf{K}} = \{!^1 p\}$ *or* $Ab(M) \cap \Omega_1^{\mathbf{K}} = \{!^1 q\}$*. This implies that each of these models verifies* $!^2 r$*. As a result, after the second selection, we obtain* $\mathcal{M}_{\mathbf{SK2^m}}(\Gamma_{\mathsf{e3}}) = \{M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{e3}}) \mid Ab(M) = \{!^1 p, !^2 r\}$ *or* $Ab(M) = \{!^1 q, !^2 r\}\}$*.*

*Compare this to the* $\mathbf{SK2^r}$*-semantics. In this case, some models* $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{e3}})$ *for which* $Ab(M) = \{!^1 p, !^1 q\}$ *are selected after the first round. Among the latter, there are models which falsify every member of* $\Omega_2^{\mathbf{K}}$*. As a result,* $Min_2(\mathcal{M}_{\mathbf{K_1^r}}(\Gamma_{\mathsf{e3}})) = \{M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{e3}}) \mid Ab(M) = \{!^1 p, !^1 q\}\}$*. This implies that* $\Psi_2^{\mathbf{S}}(\Gamma) = \emptyset$*. But then also* $\mathcal{M}_{\mathbf{SK2^r}}(\Gamma_{\mathsf{e3}}) = \{M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{e3}}) \mid Ab(M) = \{!^1 p, !^1 q\}\}$*.*

In the remainder, I will show that the syntactic $\mathbf{SAL^r}$-consequence relation is adequate with respect to the set of $\mathbf{SAL^r}$-models. For the more generic format **SAL**, I will establish a restricted kind of adequacy. Both proofs are easily obtained through generalizations of results from [137], where these properties were proven for all cases where $I = \{1, \ldots, n\}$ for an $n \in \mathbb{N}$.

### 3.3.2 Adequacy With Respect to $Cn_{\mathbf{SAL}}(\Gamma)$

Note that the following holds:

**Theorem 3.5** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if*

$$\text{for every } i \in I,\ \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL_i}}(\Gamma) \tag{3.2}$$

*then* $\Gamma \models_{\mathbf{SAL}} A$ *iff* $A \in Cn_{\mathbf{SAL}}(\Gamma)$

*Proof.* Suppose (3.2) holds. Relying on Definition 3.1, (3.2) and Definition 3.4 consecutively, we have: $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma)) = \mathcal{M}_{\mathbf{LLL}}(\bigcup_{i \in I} Cn_{\mathbf{SAL_i}}(\Gamma)) = \bigcap_{i \in I} \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \bigcap_{i \in I} \mathcal{M}_{\mathbf{SAL_i}}(\Gamma) = \mathcal{M}_{\mathbf{SAL}}(\Gamma)$. Hence:

$$\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL}}(\Gamma) \tag{3.3}$$

($\Rightarrow$) Suppose $\Gamma \models_{\mathbf{SAL}} A$. By (3.3), $A$ is true in every $M \in \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$. Hence $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$, whence by Theorem 3.1.2, $A \in Cn_{\mathbf{SAL}}(\Gamma)$.

[7] Note that the sequence $\langle \mathcal{M}_{\mathbf{SAL_i}}(\Gamma) \rangle_{i \in I}$ converges to its limes inferior due to the fact that the sequence is (by definition) monotonic ($\mathcal{M}_{\mathbf{SAL_{i+1}}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{SAL_i}}(\Gamma)$ for all $i \in I$).

($\Leftarrow$) Suppose $A \in Cn_{\mathbf{SAL}}(\Gamma)$, whence by the reflexivity of **LLL**, $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$. This implies that $A$ is true in every $M \in \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$. By (3.3), $\Gamma \models_{\mathbf{SAL}} A$. ■

Recall that every logic $\mathbf{SAL^r}$ is obtained by the sequential superposition of logics $\langle \mathbf{AL^r_i} \rangle_{i \in I}$.

**Lemma 3.7** *Where $\Gamma \subseteq \mathcal{W}_s$: for every $i \in I$, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^r_i}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL^r_i}}(\Gamma)$.* [137, Lemma 3.2.3.]

**Corollary 3.1** *Where $\Gamma \subseteq \mathcal{W}_s$: $\Gamma \models_{\mathbf{SAL^r}} A$ iff $A \in Cn_{\mathbf{SAL^r}}(\Gamma)$.*

**Lemma 3.8** *Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$: If $\Phi^{(i)}(\Gamma)$ is finite, then $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL_i}}(\Gamma)$.* [137, Theorem 3.2.2, item (i)]

**Lemma 3.9** *Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$: if $\Phi(\Gamma)$ is finite, then $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL_i}}(\Gamma)$.*

*Proof.* Suppose $\Phi(\Gamma)$ is finite. By Lemma 3.6, for every $i \in I$, $\Phi^{(i)}(\Gamma)$ is finite. Hence by Lemma 3.8, for every $i \in I$, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL_i}}(\Gamma)$. ■

From the preceding lemmas, we can show that whenever $\Phi(\Gamma)$ is finite, then soundness and completeness (restricted to premise sets in $\mathcal{W}_s$) holds for all logics **SAL**. Thus soundness and completeness of **SAL** depends on the Dab-formulas that can be derived from $\Gamma$. In the subsequent section, a counterexample is given for unrestricted soundness and completeness of **SAL** with respect to its semantics.

**Corollary 3.2** *Where $\Gamma \subseteq \mathcal{W}_s$ and $\Phi(\Gamma)$ is finite, each of the following holds:*

1. *for every $i \in I$, $\Gamma \models_{\mathbf{SAL_i}} A$ iff $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$*
2. *$\Gamma \models_{\mathbf{SAL}} A$ iff $A \in Cn_{\mathbf{SAL}}(\Gamma)$*

### 3.3.3 No Unrestricted Adequacy for SAL

I will now briefly show why we cannot generalize the above result and prove that for every premise set $\Gamma \subseteq \mathcal{W}_s$, the **SAL**-semantics is adequate with respect to $Cn_{\mathbf{SAL}}(\Gamma)$. Although my argument uses a logic in the format $\mathbf{SAL^m}$, it can easily be generalized to all superpositions where a logic $\mathbf{AL^m_i}$ is applied before another logic $\mathbf{AL^x_{i+1}}$.

Let the logic $\mathbf{SK2^m}$ be defined as before, i.e. by the superposition of the logics $\mathbf{K^m_1} = \langle \mathbf{K}, \Omega^{\mathbf{K}}_1, \mathbf{m} \rangle$ and $\mathbf{K^m_2} = \langle \mathbf{K}, \Omega^{\mathbf{K}}_2, \mathbf{m} \rangle$. I will now show that $\mathbf{SK2^m}$ is not in general sound or complete with respect to its semantics.

**Proposition 3.1** *There are $\Gamma, A$ such that $\Gamma \models_{\mathbf{SK2^m}} A$, but $A \notin Cn_{\mathbf{SK2^m}}(\Gamma)$.*

*Proof.*[8] Let $\Gamma_{\mathsf{c}} = \{!^1 p_i \vee !^1 p_j \mid i, j \in \mathbb{N}, i \neq j\} \cup \{!^1 p_i \vee !^2 q_i \vee r \mid i \in \mathbb{N}\}$. Note that every $\mathbf{K^m_1}$-model of $\Gamma_{\mathsf{c}}$ falisifies exactly one abnormality $!^1 p_i$. After the second

[8] The original version of this example is given by Diderik Batens in his [25, Chapter 6].

selection, every selected model will moreover falsify all the abnormalities $!^2 q_i$. It follows that for every $M \in \mathcal{M}_{\mathbf{SK2^m}}(\Gamma_{\mathsf{c}})$, $M \Vdash r$.

However, $r \notin Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{c}})$. To see why, suppose $r \in Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{c}})$. Hence $r \in Cn_{\mathbf{K_2^m}}(Cn_{\mathbf{K_1^m}}(\Gamma_{\mathsf{c}}))$. By Theorem 2.6.1, there is a $\Delta \subset \Omega_2^{\mathbf{K}}$ such that $Cn_{\mathbf{K_1^m}}(\Gamma_{\mathsf{c}}) \vdash_{\mathbf{K}} r \check{\vee} Dab(\Delta)$, whence by the LLL-closure of $\mathbf{K_1^m}$, $r \check{\vee} Dab(\Delta) \in Cn_{\mathbf{K_1^m}}(\Gamma_{\mathsf{c}})$.

Since $\Delta$ is finite, there is an $i \in \mathbb{N}$: for every $j \geq i$, $!^2 q_j \notin \Delta$. Let $M$ be a $\mathbf{K}$-model of $\Gamma_{\mathsf{c}}$ for which each of the following holds:

- $M \nVdash !^1 p_i$
- for every $k \neq i$: $M \Vdash !^1 p_k$
- for every $A \in \Omega_2^{\mathbf{K}} - \{!^2 q_i\}$: $M \nVdash A$
- $M \nVdash r$
- $M \Vdash !^2 q_i$

I leave it to the reader to prove that such a model $M$ exists, that it is a $\mathbf{K_1^m}$-model of $\Gamma_{\mathsf{c}}$, and that it does not verify $r \check{\vee} Dab(\Delta)$. By the soundness of $\mathbf{K_1^m}$, it follows that $r \check{\vee} Dab(\Delta) \notin Cn_{\mathbf{K_1^m}}(\Gamma_{\mathsf{c}})$ — a contradiction. ■

**Proposition 3.2** *There are* $\Gamma, A$ *such that* $A \in Cn_{\mathbf{SK2^m}}(\Gamma)$*, but* $\Gamma \not\models_{\mathbf{SK2^m}} A$*.*

*Proof.* Let $\Gamma_{\mathsf{s}} = \{!^1 p_i \vee !^1 p_j \mid i, j \in \mathbb{N}, i \neq j\} \cup \{!^1 p_i \vee !^2 q_i \mid i \in \mathbb{N}\} \cup \{!^2 q_1 \vee r\}$. Since $Cn_{\mathbf{K_1^m}}(\Gamma_{\mathsf{s}})$ has no minimal Dab$_2$-consequences, $\check{\neg} !^2 q_k \in Cn_{\mathbf{K_2^m}}(Cn_{\mathbf{K_1^m}}(\Gamma_{\mathsf{s}}))$ for every $k \in \mathbb{N}$. By the reflexivity of $\mathbf{SK2^m}$, $!^2 q_1 \vee r \in Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{s}})$. By the LLL-closure of $\mathbf{SK2^m}$, it follows that $r \in Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{s}})$.

However, $\Gamma_{\mathsf{s}} \not\models_{\mathbf{SK2^m}} r$. To see why, note that every $\mathbf{K_1^m}$-model $M$ of $\Gamma_{\mathsf{s}}$ falsifies exactly one abnormality $!^1 p_i$, whence it verifies one abnormality $!^1 q_i$. In particular, there is a model $M_1 \in \mathcal{M}_{\mathbf{K_1^m}}(\Gamma_{\mathsf{s}})$ such that each of the following holds:

- $M_1 \Vdash !^2 q_1$
- $M_1 \nVdash r$
- $M_1 \nVdash A$ for all $A \in \Omega^{\mathbf{K}} - \{!^2 q_1\}$

Assume that there is an $M \in \mathcal{M}_{\mathbf{K_1^m}}(\Gamma_{\mathsf{s}})$ such that $Ab(M) \cap \Omega_2^{\mathbf{K}} \subset Ab(M_1) \cap \Omega_2^{\mathbf{K}}$. It follows that $M \nVdash !^2 q_i$ for every $i \in \mathbb{N}$. But then $M \Vdash !^1 p_i$ for every $i \in \mathbb{N}$, whence $M \notin \mathcal{M}_{\mathbf{K_1^m}}(\Gamma_{\mathsf{s}})$ — a contradiction.

It follows that $M_1 \in \mathcal{M}_{\mathbf{SK2^m}}(\Gamma_{\mathsf{s}})$. As a result, $\Gamma_{\mathsf{s}} \not\models_{\mathbf{SK2^m}} r$. ■

### 3.3.4 Semantic Reassurance

Before closing this section, let me briefly consider the question whether the superposition of semantic selections for **SAL** satisfies the Semantic Reassurance property: if $\Gamma$ has **LLL**-models, then it has adaptive models. As shown in Chapter 2, *flat* adaptive logics have this property – this follows immediately from the theorem of Strong Reassurance (see Theorem 2.12).

In Section 3.6.1, I will discuss the notion of Strong Reassurance in the context of sequential selections of models. But let us first consider the much more basic property Reassurance. This property does not hold for **SAL** in general. Consider the following example:

**Example 3.5** *Let* $\Gamma_{\mathsf{r}} = \{!^1 p_i \vee !^1 p_j \mid i, j \in \mathbb{N}, i \neq j\} \cup \{!^1 p_i \vee !^2 q_j \mid i, j \in \mathbb{N}, j \geq i\}$. *Note that* $\mathcal{M}_{\mathbf{SK^m_1}}(\Gamma_{\mathsf{r}}) = \{M_k \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{r}}) \mid k \in \mathbb{N}, Ab(M_k) \cap \Omega^{\mathbf{K}}_1 = \{!^1 p_i \mid i \in \mathbb{N} - \{k\}\}\}$. *Note that the following holds:*

$$
\begin{array}{l}
Ab(M_1) \cap \Omega^{\mathbf{K}}_2 = \{!^2 q_i \mid i \in \mathbb{N}\} \\
Ab(M_2) \cap \Omega^{\mathbf{K}}_2 = \{!^2 q_i \mid i \in \mathbb{N} - \{1\}\} \\
Ab(M_3) \cap \Omega^{\mathbf{K}}_2 = \{!^2 q_i \mid i \in \mathbb{N} - \{1, 2\}\} \\
Ab(M_4) \cap \Omega^{\mathbf{K}}_2 = \{!^2 q_i \mid i \in \mathbb{N} - \{1, 2, 3\}\} \\
\vdots
\end{array}
$$

*Hence for every* $i \in \mathbb{N}$, $Ab(M_i) \cap \Omega^{\mathbf{K}}_2 \subset Ab(M_{i+1}) \cap \Omega^{\mathbf{K}}_2$

So the problem is that, once we superpose semantic selections according to the Minimal Abnormality Strategy, we can encounter infinite sequences of models $M_1, M_2, \ldots$, where each model $M_{i+1}$ is less abnormal than $M_i$ with respect to one of the sets $\Omega_k$. In the above example, $\mathcal{M}_{\mathbf{SK2^m}}(\Gamma_{\mathsf{r}}) = \emptyset$.

As will be shown in Chapter 6, logics in the $\mathbf{SAL^m_{(I)}}$-format do have the Semantic Reassurance property. Also, as I will now briefly show, **SAL**-logics satisfy Semantic Reassurance in exactly those cases where they are sound and complete with respect to their semantics.

**Theorem 3.6** *If* $\Gamma$ *is* **LLL**-*satisfiable, and*

1. *for every* $i \in I$, $\mathbf{x_i} = \mathbf{r}$, *or*
2. $\Phi(\Gamma)$ *is finite*

*then* $\mathcal{M}_{\mathbf{SAL}}(\Gamma) \neq \emptyset$.

*Proof.* Suppose $\Gamma$ is **LLL**-satisfiable, and either 1 or 2 holds. By the soundness of **LLL**, $\Gamma$ is not **LLL**-trivial. By Theorem 3.3, $Cn_{\mathbf{SAL}}(\Gamma)$ is not **LLL**-trivial, whence $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma)) \neq \emptyset$. By Corollary 3.1 and 3.2 respectively, $\mathcal{M}_{\mathbf{SAL}}(\Gamma) = \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$. Putting everything together, $\mathcal{M}_{\mathbf{SAL}}(\Gamma) \neq \emptyset$. ■

## 3.4 A Proof Theory for SAL

### 3.4.1 The Quest for a Proof Theory

In [11], an attractive proof theory is proposed for a specific class of superpositions of ALs, i.e. the logics $\mathbf{SAL^r}$ and $\mathbf{SAL^m}$ where (†) for all $i, j \in I$, $i \neq j$: $\Omega_i \cap \Omega_j = \emptyset$. This proof theory is very similar to that of flat adaptive logics: the same generic rules are used, with a conditional rule that allows one to push abnormalities to the condition; a marking definition determines which lines are in and which are out at a given stage $s$ of the proof; the notions of derivability at a stage and final derivability are exactly the same as for flat ALs.

The proof theory from [11] has a certain intuitive appeal. Whether or not a line is marked is defined recursively. If the user of a logic wants to find out whether or not a line is marked or not at stage $s$, she can follow a sequential marking procedure. Roughly speaking, such a procedure goes as follows: at a stage $s$, mark lines according to a first marking criterion. This criterion solely depends on the lines that have been derived on the empty condition. In view of the lines that remain unmarked after this first step, we obtain a new marking criterion, which then allows us to determine a third marking criterion, etc. Lines that remain unmarked at the end of the whole procedure are said to be unmarked at stage $s$.

For logics of the format $\mathbf{SAL^r}$ that obey restriction (†), this proof theory is sound and complete with respect to both the semantic and the syntactic consequence relation of $\mathbf{SAL^r}$. However, for the $\mathbf{SAL^m}$-logics of this specific class, soundness with respect to the consequence relation fails for certain (fairly simple) premise sets – I will return to this point in Section 3.5.1.

If we remove restriction (†), several difficulties arise even in the case of $\mathbf{SAL^r}$. In [25], Diderik Batens proposes a generic proof theory for these two classes of sequential superpositions. For $\mathbf{SAL^r}$, this proof theory is adequate with respect to $Cn_{\mathbf{SAL^r}}(\Gamma)$. However, for $\mathbf{SAL^m}$, there are $\Gamma, A$ such that $A$ is derivable from $\Gamma$ in a proof, whereas $A \notin Cn_{\mathbf{SAL^m}}(\Gamma)$.[9]

Christian Straßer made a different attempt to characterize some sequential superpositions by a dynamic proof theory in his [137]. On the one hand, Straßer broadens the scope to include superpositions of ALs with mixed strategies. On the other hand, Straßer restricts himself again to logics that obey (†), and only considers the case in which $I = \{1, \ldots, n\}$. Again, for all logics $\mathbf{SAL^r}$, this proof theory is adequate, whereas for the Minimal Abnormality-variants and those with mixed strategies, Straßers proposal faces the same problem as Batens' older proposal.

In this and the next section – which is based on joint work with Christian Straßer –, I will present two proposals of generic proof theories for *all* logics **SAL**, and prove them to be adequate with respect to the syntactic consequence relation of **SAL**. Although I consider this an important achievement, a small warning is in place. In order to obtain full adequacy for the general case of superpositions, it turns out that some traditional conceptions have to be overthrown. In the first proposal, these conceptions have to do with the reasons for marking lines in an adaptive proof. In the second proposal, the concept of an adaptive proof itself is changed, be it in a rather conservative way. Since both proof theories have their own advantages and disadvantages, I decided to present both here.

### 3.4.2 Recursive Unmarking

The generic rules for the first proposal are exactly those of the flat adaptive logic $\mathbf{AL^x} = \langle \mathbf{LLL}, \bigcup_{i \in I} \Omega_i, \mathbf{x} \rangle$ – see page 21. For the rule RC, this implies that we may push abnormalities of any priority level to the condition. So for instance, where each $B_i \in \Omega_i$, and $A \check{\vee} B_5 \check{\vee} B_2$ is derived on the condition $\{B_3\}$, we may apply RC and obtain a line on which $A$ is derived, on the condition $\{B_3, B_5, B_2\}$.

Hence, although **SAL** is the result of the sequential application of each of the logics $\mathbf{AL_i}$, in an **SAL**-proof, we can apply all these logics at the same time

[9] An example is $\Gamma_c$ from Section 3.3.3. Let $\vdash^B_{\mathbf{SK2^m}}$ denote the derivability relation defined by Batens, when applied to the specific logic $\mathbf{SK2^m}$. It can be shown that $\Gamma_c \vdash^B_{\mathbf{SK2^m}} r$, whereas $r \notin Cn_{\mathbf{SK2^m}}(\Gamma_c)$.

or in whatever order we want to apply them. It is the marking definition and the definition of final derivability that warrant that this seemingly very liberal proof theory still renders the right outcome, viz. a derivability relation that is adequate with respect to $Cn_{\mathbf{SAL}}(\Gamma)$. As the definition of final derivability is the same as for flat ALs, I will mainly focus on the marking definition for **SAL** in this section, or rather, the sequential procedure of *unmarking* that allows us to determine which formulas can be considered derived at a given stage, and which not.

The overall idea behind this procedure is the following: whenever our best insights at stage $s$ indicate that $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$, then we are allowed to $i$-unmark at least one line at which $A$ is derived. However, recall that

$$Cn_{\mathbf{SAL_i}}(\Gamma) = Cn_{\mathbf{AL_i}}(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$$

Thus, having good reasons to assume that $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$ means having good reasons to assume certain things about $Cn_{\mathbf{SAL_{i-1}}}(\Gamma)$. So if we want to obtain a reasonable estimate of $Cn_{\mathbf{SAL}}(\Gamma)$, we should start with $Cn_{\mathbf{AL_1}}(\Gamma)$, next consider $Cn_{\mathbf{AL_1}}(Cn_{\mathbf{AL_2}}(\Gamma))$, and so on. Put differently, we should start by unmarking lines in view of the first logic in the superposition, next in view of the second logic and the lines that have been unmarked so far, and so on.

Before we turn over to the exact marking definitions, let us try to give an idea of what they look like, by means of an object-level proof. Recall that $\mathbf{SK2^r}$ was defined as the superposition of the logics $\langle \mathbf{K}, \Omega_i^{\mathbf{K}}, \mathbf{r}\rangle_{i\leq 2}$. The following is a $\mathbf{SK2^r}$-proof from $\Gamma_{\mathsf{p1}} = \{\Diamond p, \Diamond\Diamond q, \Diamond\Diamond r, \neg p \vee \neg r\}$:

| | | | | |
|---|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\emptyset$ | $-_0$ |
| 2 | $\Diamond\Diamond q$ | PREM | $\emptyset$ | $-_0$ |
| 3 | $(p \wedge q) \check{\vee} !^1 p \check{\vee} !^2 q$ | 1,2;RU | $\emptyset$ | $-_0$ |
| 4 | $(p \wedge q) \check{\vee} !^2 q$ | 3;RC | $\{!^1 p\}$ | $-_1$ |
| 5 | $p \wedge q$ | 4;RC | $\{!^1 p, !^2 q\}$ | $-_2$ |
| 6 | $\Diamond\Diamond r$ | PREM | $\emptyset$ | $-_0$ |
| 7 | $\neg p \vee \neg r$ | PREM | $\emptyset$ | $-_0$ |
| 8 | $!^1 p \check{\vee} !^2 r$ | 1,6,7;RU | $\emptyset$ | $-_0$ |
| 9 | $!^2 r$ | 8;RC | $\{!^1 p\}$ | $-_1$ |
| 10 | $r \check{\vee} !^2 r$ | 6;RU | $\emptyset$ | $-_0$ |
| 11 | $r$ | 10;RC | $\{!^2 r\}$ | ✓ |

The symbol $-_i$ indicates that at the current stage, a line is $i$-unmarked. The above result is obtained as follows. We start by marking all lines with a non-empty condition, i.e. lines 4, 5, 9 and 11. All other lines contain **K**-consequences of the premise set, whence they also follow by the prioritized adaptive logic. Lines with an empty condition are said to be 0-unmarked.

In the second step of the procedure, we look at all the minimal $\mathrm{Dab}_1$-formulas, i.e. disjunctions of abnormalities $A \in \Omega_1$, that have been derived on the empty condition. This gives us the set ${}^{\mathbf{S}}U^1_{11}(\Gamma) = \emptyset$ — note that the formula on line 8 is not a $\mathrm{Dab}_1$-formula since it contains an abnormality of level two, viz. $!^2 r$. We *1-unmark* all lines with a condition $\Theta \subset \Omega_1 - {}^{\mathbf{S}}U^1_{11}(\Gamma)$. This means that lines 4 and 9 are 1-unmarked, indicating that at this stage, we have sufficient reasons to assume e.g. that $p \in Cn_{\mathbf{K^r_1}}(\Gamma_{\mathsf{p1}})$ and that $!^2 \neg r \in Cn_{\mathbf{K^r_1}}(\Gamma_{\mathsf{p1}})$.

To prepare for the third and final step of the procedure, we consider all Dab$_2$-formulas that are derived on a line that is unmarked after the preceding step – recall that $Dab(\Delta)$ is a Dab$_2$-formula iff $\Delta \subset \Omega_2$. In the current case, there is only one such formula, viz. the one on line 10. From this, we obtain a second set of unreliable formulas: ${}^{\mathbf{S}}U^2_{11}(\Gamma) = \{!^2 r\}$. Finally, we *2-unmark* all lines on which a formula $B$ is derived on a condition $\Delta \subset \Omega$, such that for a $\Theta \subset \Delta \cap \Omega^{\mathbf{K}}_2$, each of the following holds:

(a) $B \check{\vee} Dab(\Theta)$ has been derived on a line that was 1-unmarked, with condition $\Delta - \Theta$
(b) $\Theta \cap {}^{\mathbf{S}}U^2_{11}(\Gamma) = \emptyset$

As is clear from requirement (b), in order for a line with condition $\Delta$ to become 2-unmarked, not only $\Delta \cap \Omega_2$ has to be reliable, but also other formulas have to be derived on an unmarked line in the proof. This requirement reflects the recursive character of $Cn_{\mathbf{SK2^r}} = Cn_{\mathbf{K^r_2}}(Cn_{\mathbf{K^r_1}}(\Gamma))$.

Now consider line 5. Let $\Delta = \{!^1 p, !^2 q\}$ and let $\Theta = \{!^2 q\}$. Note that (a) is fulfilled in view of line 4 and that (b) is fulfilled since $!^2 q \notin {}^{\mathbf{S}}U^2_{11}(\Gamma_{\mathsf{p1}})$. This means that line 5 is 2-unmarked, indicating that according to our best insights at stage 11, $p \wedge q \in Cn_{\mathbf{SK2^r}}(\Gamma)$. On the other hand, line 11 cannot be 2-unmarked, since its condition has a non-empty intersection with ${}^{\mathbf{S}}U^2_{11}(\Gamma_{\mathsf{p1}})$. Hence line 11 is marked at stage 11.

Note that a line can be 2-unmarked without being 1-unmarked. However, as we will see below, whenever a line is $i$-unmarked for an $i \in I$, then it is also $j$-unmarked for all $j \in I$ such that $j > i$.

In order to derive $p \wedge q$ on an unmarked line, it is crucial that we proceed stepwise. That is, we first have to derive $(p \wedge q) \check{\vee} !^1 p \check{\vee} !^2 q$ on the empty condition. Then we have to derive $(p \wedge q) \check{\vee} !^2 q$ on the condition $\{!^1 p\}$, and only after that we should push $!^2 q$ to the condition. If we would e.g. push both abnormalities $!^1 p$ and $!^2 q$ to the condition at once, the resulting line would become marked as long as there is no 1-unmarked line $l$ with formula $(p \wedge q) \check{\vee} !^2 q$ and condition $\{!^1 p\}$. I will return to this point below.

Suppose now that we add the premise $\neg p \vee \neg q$. Obviously, there has to be a way to render line 5 marked, once this additional premise is introduced. Consider the following continuation of the proof:

| | | | | |
|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| 4 | $(p \wedge q) \check{\vee} !^2 q$ | 3;RC | $\{!^1 p\}$ | $\__1$ |
| 5 | $p \wedge q$ | 4;RC | $\{!^1 p, !^2 q\}$ | ✓ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| 12 | $\neg p \vee \neg q$ | PREM | $\emptyset$ | $\__0$ |
| 13 | $!^1 p \check{\vee} !^2 q$ | 1,2,12;RU | $\emptyset$ | $\__0$ |
| 14 | $!^2 q$ | 13;RC | $\{!^1 p\}$ | $\__1$ |

At this stage of the proof, $!^2 q$ becomes unreliable. More formally, ${}^{\mathbf{S}}U^2_{14}(\Gamma) = \{!^2 r, !^2 q\}$, whence line 5 cannot be 2-unmarked at stage 14. Again, it is very important to note that in the **SAL**-proof theory, Dab-formulas can (and often have to be) derived on a non-empty condition – this will be the case in the second proposal as well (see infra).

Enough with the example, let us now turn to the definitions for the first proposal. For every $i \in I$, we define prioritized counterparts of the sets $\Sigma_s(\Gamma)$, $U_s(\Gamma)$ and $\Phi_s(\Gamma)$ from the standard format:

**Definition 3.5** *A line is 0-marked iff it has a non-empty condition. Otherwise we say it is 0-unmarked. For every $i \in I$:*

- *$Dab(\Delta)$ is a $Dab_i$-formula at stage $s$ iff $\Delta \subset \Omega_i$ and $Dab(\Delta)$ is derived at an $(i-1)$-unmarked line at stage $s$ (see below).*
- *$Dab(\Delta)$ is a minimal $Dab_i$-formula at stage $s$ iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is a $Dab_i$-formula at stage $s$.*
- *Where $\langle Dab(\Delta)\rangle_{j\in J}$ are the minimal $Dab_i$-formulas at stage $s$, ${}^{\mathbf{S}}\Sigma^i_s(\Gamma) =_{\mathsf{df}} \{\Delta_j \mid j \in J\}$.*
- ${}^{\mathbf{S}}U^i_s(\Gamma) =_{\mathsf{df}} \bigcup {}^{\mathbf{S}}\Sigma^i_s(\Gamma)$.
- *${}^{\mathbf{S}}\Phi^i_s(\Gamma)$ is the set of minimal choice sets of ${}^{\mathbf{S}}\Sigma^i_s(\Gamma)$.*

The preceding definition goes hand in hand with two marking definitions, one for each strategy:

**Definition 3.6 ($i$-unmarking for $\mathbf{x_i} = \mathbf{r}$)** *A line with formula $A$ and condition $\Delta \subset \Omega_{(i)}$ is $i$-unmarked at stage $s$ iff there is a $\Theta \subseteq \Delta \cap \Omega_i$ such that (a) $A \check{\vee} Dab(\Theta)$ is derived on an $(i-1)$-unmarked line with condition $\Delta - \Theta$ at stage $s$, and (b) $\Theta \cap {}^{\mathbf{S}}U^i_s(\Gamma) = \emptyset$.*

**Definition 3.7 ($i$-unmarking for $\mathbf{x_i} = \mathbf{m}$)** *A line with formula $A$ and condition $\Delta \subset \Omega_{(i)}$ is $i$-unmarked at stage $s$ iff each of the following holds:*

*(i)* *there is a $\Theta \subseteq \Delta \cap \Omega_i$ such that*

  *(i.a)* *$A \check{\vee} Dab(\Theta)$ is derived on an $(i-1)$-unmarked line with condition $\Delta - \Theta$ at stage $s$, and*

  *(i.b)* *for a $\varphi \in {}^{\mathbf{S}}\Phi^i_s(\Gamma)$, $\varphi \cap \Theta = \emptyset$*

*(ii)* *for every $\varphi' \in {}^{\mathbf{S}}\Phi^i_s(\Gamma)$, $A$ is derived on a condition $\Delta'$ at stage $s$ such that there is a $\Theta' \subset \Delta' \cap \Omega_i$ for which*

  *(ii.a)* *$A \check{\vee} Dab(\Theta')$ is derived on an $(i-1)$-unmarked line with condition $\Delta' - \Theta'$ at stage $s$, and*

  *(ii.b)* *$\varphi' \cap \Theta' = \emptyset$.*

A line is $i$-marked iff it is not $i$-unmarked. We say that line $l$ is *unmarked* at stage $s$ iff there is an $i \in I$ such that $l$ is $i$-unmarked at stage $s$. Note that although I introduced the above definitions in terms of an unmarking-procedure, whether or not a line is marked at stage $s$ is determined by a recursive definition and does in no way depend on choices made by the user of the logic.

The following lemma mirrors the fact that where $i, i+1 \in I$, $Cn_{\mathbf{SAL_i}}(\Gamma) \subseteq Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$:

**Lemma 3.10** *For every $i \in I$: if line $l$ is $(i-1)$-unmarked at stage $s$, then line $l$ is $i$-unmarked at stage $s$.*

*Proof.* Immediate in view of the Definitions 3.6, resp. 3.7 — let $\Theta = \emptyset$ and for $\mathbf{x_i} = \mathbf{m}$, let $\Theta' = \Theta$. ■

In view of this lemma, we may alternatively stipulate that a line $l$ is unmarked at stage $s$ iff there is an $i \in I$, such that for all $j \in I$ with $j \geq i$, line $l$ is $j$-unmarked.

To find out which lines are marked at stage $s$ and which not, one starts by 0-marking all lines with a non-empty condition, next 1-unmarking lines in view of the minimal $\text{Dab}_1$-formulas at stage $s$ (and in view of the appropriate strategy). After that, one can 2-unmark lines in view of the minimal $\text{Dab}_2$-formulas at stage $s$, and of other formulas that are derived on 1-unmarked lines. And so on, until it is no longer possible to unmark any further line. It can easily be verified that for a finite stage $s$, this procedure stops at a finite point.

To complete the proof theory, we still have to define final derivability for **SAL**. In the remainder of this thesis, the definition of the derivability relation will be taken from the standard format. Hence where **PAL** is any prioritized adaptive logic from this thesis, we have:

**Definition 3.8** *$A$ is* finally derived *from $\Gamma$ on line $l$ of a finite stage $s$ of a* **PAL**-*proof iff (i) $A$ is the second element of line $l$, (ii) line $l$ is unmarked at stage $s$, and (iii) every extension of the proof at stage $s$, in which line $l$ is marked may be further extended in such a way that line $l$ is unmarked again.*

**Definition 3.9** $\Gamma \vdash_{\mathbf{PAL}} A$ *iff $A$ is finally derived on a line of a* **PAL**-*proof from $\Gamma$.*

Together with the format-specific marking definition, Definitions 3.8 and 3.9 yield the derivability relation $\vdash_{\mathbf{SAL}}$. In the next section, I show that $\vdash_{\mathbf{SAL}}$ is sound and complete with respect to $Cn_{\mathbf{SAL}}(\Gamma)$. To finish this section, let me illustrate the above definitions with one example – this time with a superposition of logics that have the Minimal Abnormality Strategy.

**Example 3.6** *Let $\Gamma = \{\neg p \vee \neg q, \neg p \vee \neg s, \Diamond(p \wedge q), \Diamond\Diamond(p \supset r), \Diamond\Diamond s\}$. Recall that the logic $\mathbf{SK2^m_{(2)}}$ is defined as the superposition of $\mathbf{K^m_1} = \langle \mathbf{K}, \Omega^{\mathbf{K}}_1, \mathbf{m}\rangle$ and $\mathbf{K^m_2} = \langle \mathbf{K}, \Omega^{\mathbf{K}}_1 \cup \Omega^{\mathbf{K}}_2, \mathbf{m}\rangle$. By the first logic in the superposition, we can finally derive $p \vee q$. By the second logic, we can finally derive $p \vee s$ and $p \supset r$. Since $Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma)$ is closed under* **K**, *it follows that e.g. also $q \vee r \in Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma)$. I will now show how this formula can be finally derived in an $\mathbf{SK2^m_{(2)}}$-proof from $\Gamma$:*

| | | | | |
|---|---|---|---|---|
| *1* | $\neg p \vee \neg q$ | *PREM* | $\emptyset$ | $^{-}0$ |
| *2* | $\neg p \vee \neg s$ | *PREM* | $\emptyset$ | $^{-}0$ |
| *3* | $\Diamond(p \wedge q)$ | *PREM* | $\emptyset$ | $^{-}0$ |
| *4* | $\Diamond\Diamond(q \supset r)$ | *PREM* | $\emptyset$ | $^{-}0$ |
| *5* | $\Diamond\Diamond s$ | *PREM* | $\emptyset$ | $^{-}0$ |
| *6* | $(p \wedge q) \check{\vee} !^1 p \check{\vee} !^1 q$ | *3;RU* | $\emptyset$ | $^{-}0$ |
| *7* | $p \wedge q$ | *6;RC* | $\{!^1 p, !^1 q\}$ | ✓ |
| *8* | $\Diamond p$ | *3;RU* | $\emptyset$ | $^{-}0$ |
| *9* | $\Diamond q$ | *3;RU* | $\emptyset$ | $^{-}0$ |

| | | | | |
|---|---|---|---|---|
| 10 | $!^1p \check{\vee} !^1q$ | 1,3;RU | $\emptyset$ | $\neg_0$ |
| 11 | $(p \vee q) \check{\vee} !^1p$ | 8;RU | $\emptyset$ | $\neg_0$ |
| 12 | $p \vee q$ | 11;RC | $\{!^1p\}$ | $\neg_1$ |
| 13 | $(p \vee q) \check{\vee} !^1q$ | 8;RU | $\emptyset$ | $\neg_0$ |
| 14 | $p \vee q$ | 13;RC | $\{!^1q\}$ | $\neg_1$ |

*At stage 14, only line 7 is marked. Note that line 7 is only marked from stage 10 on, in view of the formula on line 10. Let me briefly explain why lines 12 is 1-unmarked – the reasoning is analogous for line 14. Note that* ${}^{\mathbf{S}}\Sigma^1_{14}(\Gamma) = \{\{!^1p, !^1q\}\}$*, whence* ${}^{\mathbf{S}}\Phi^1_{14}(\Gamma) = \{\{!^1p\}, \{!^1q\}\}$*. Hence, in order for line 12 to be unmarked, Definition 3.7 stipulates that, first of all, the following must hold:*

*(i)* *there is a* $\Theta \subseteq \{!^1p\}$ *such that (i.a)* $(p \vee q) \check{\vee} Dab(\Theta)$ *is derived on a 0-unmarked line with condition* $\{!^1p\} - \Theta$ *at stage 14, and (i.b) either* $\Theta \cap \varphi = \emptyset$ *for a* $\varphi \in {}^{\mathbf{S}}\Phi^1_{14}(\Gamma)$.

*This holds trivially for* $\Theta = \{!^1p\}$ *and* $\varphi = \{!^1q\}$*, in view of line 11. The second requirement for line 12 to be unmarked reads as follows: for every* $\varphi' \in {}^{\mathbf{S}}\Phi^1_{14}(\Gamma)$*,* $p \vee q$ *should derived on a condition* $\Delta' \subset \Omega_1$ *such that, for a* $\Theta' \subseteq \Delta'$*:*

*(ii.a)* $\Theta' \cap \varphi = \emptyset$

*(ii.b)* $(p \vee q) \check{\vee} Dab(\Delta' - \Theta')$ *is derived on a 0-unmarked line at stage 14*

*This requirement too is fulfilled – where* $\varphi' = \{!^1p\}$*, let* $\Delta' = \Theta' = \{!^1q\}$*. Note that lines 11 and 13 are crucial for line 12 to be 1-unmarked.*

*Let us now extend the above proof in order to derive* $q \vee r$*:*

| | | | | |
|---|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ | |
| 15 | $\Diamond\Diamond(\neg p \vee r)$ | 4;RU | $\emptyset$ | $\neg_0$ |
| 16 | $\Diamond^2\neg p \vee \Diamond^2 r$ | 15;RU | $\emptyset$ | $\neg_0$ |
| 17 | $(\neg p \vee r) \check{\vee} !^2\neg p \check{\vee} !^2 r$ | 16;RU | $\emptyset$ | $\neg_0$ |
| 18 | $\neg p \vee r$ | $\{!^2\neg p, !^2 r\}$ | $\emptyset$ | $\neg_1$ |
| 19 | $q \vee r$ | 12,18;RU | $\{!^1p, !^2\neg p, !^2 r\}$ | ✓ |
| 20 | $q \vee r$ | 14,18;RU | $\{!^1q, !^2\neg p, !^2 r\}$ | ✓ |

*One could think that at stage 20, lines 19 and 20 should be unmarked. It can easily be verified that* ${}^{\mathbf{S}}\Phi^2_{20}(\Gamma) = {}^{\mathbf{S}}\Phi^1_{20}(\Gamma) = {}^{\mathbf{S}}\Phi^1_{14}(\Gamma) = \{\{!^1p\}, \{!^1q\}\}$*. However, condition (b) in the definition of unmarking for Minimal Abnormality is not fulfilled. It is only fulfilled in the following extension of the proof:*

| | | | | |
|---|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ | |
| 19 | $q \vee r$ | 12,18;RU | $\{!^1p, !^2\neg p, !^2 r\}$ | $\neg_2$ |
| 20 | $q \vee r$ | 14,18;RU | $\{!^1q, !^2\neg p, !^2 r\}$ | $\neg_2$ |
| 21 | $(q \vee r) \check{\vee} !^1p \check{\vee} !^2\neg p \check{\vee} !^2 r$ | 3,4;RU | $\emptyset$ | $\neg_0$ |
| 22 | $(q \vee r) \check{\vee} !^1q \check{\vee} !^2\neg p \check{\vee} !^2 r$ | 3,4;RU | $\emptyset$ | $\neg_0$ |
| 23 | $(q \vee r) \check{\vee} !^2\neg p \check{\vee} !^2 r$ | 21;RC | $\{!^1p\}$ | $\neg_1$ |
| 24 | $(q \vee r) \check{\vee} !^2\neg p \check{\vee} !^2 r$ | 22;RC | $\{!^1q\}$ | $\neg_1$ |

The above example illustrates that, in order to finally derive a formula $A$ in an **SAL**-proof, a number of intermediate and seemingly redundant steps are required. However, this requirement is indispensable for the soundness and completeness of the **SAL**-proof theory with respect to $Cn_{\mathbf{SAL}}(\Gamma)$.[10] I will return to this point at the end of this section.

### 3.4.3 Adequecy of $\vdash_{\mathbf{SAL}}$

**Some Useful Properties.** I start with a theorem and lemma that follow immediately from the definition of **SAL** and properties of the standard format, but are of vital importance in the current context.

**Theorem 3.7** *Where $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$, each of the following holds:*

1. $\Gamma \vdash_{\mathbf{AL^r}} A$ *iff there is a* $\Delta \subset \Omega$ *such that* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *and* $\Delta \cap U(\Gamma) = \emptyset$
2. $\Gamma \vdash_{\mathbf{AL^m}} A$ *iff for every* $\varphi \in \Phi(\Gamma)$*, there is a* $\Delta \subset \Omega$ *such that* $\varphi \cap \Delta = \emptyset$ *and* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$

*Proof.* Immediate in view of the (unrestricted) soundness and completeness of $\mathbf{AL^x}$ whenever $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$, Theorems 2.6.1 and 2.7.1, and Lemma 2.5. ■

**Lemma 3.11** *Each of the following holds:*

1. *Where* $i \in I$ *and* $\mathbf{x_i} = \mathbf{r}$*: if* $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$*, then there is a* $\Delta \subset \Omega_{(i)}$ *such that* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$*, and for a* $\Theta \subseteq \Delta \cap \Omega_i$*, each of the following holds:*
   (a) $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL_{i-1}}}(\Gamma)$
   (b) $\Theta \cap U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma)) = \emptyset$
2. *Where* $i \in I$ *and* $\mathbf{x_i} = \mathbf{m}$*: if* $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$*, then for every* $\varphi \in \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$*, there is a* $\Delta \subset \Omega_{(i)}$ *such that* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$*, and for a* $\Theta \subseteq \Delta \cap \Omega_i$*, each of the following holds:*
   (a) $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL_{i-1}}}(\Gamma)$
   (b) $\Theta \cap \varphi = \emptyset$
3. *If* $A \in Cn_{\mathbf{SAL}}(\Gamma)$*, then there is a* $\Delta \subset \Omega$ *such that* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$*.*

*Proof. Ad 1.* ($i = 1$) Immediate in view of Theorem 2.6.1— let $\Theta = \Delta$.

($i \Rightarrow i+1$) Suppose $A \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$ and $\mathbf{x_{i+1}} = \mathbf{r}$. By Theorem 2.6.1, $Cn_{\mathbf{SAL_i}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta)$ for a $\Theta \subseteq \Omega_{i+1} - U^{i+1}(Cn_{\mathbf{SAL}i}(\Gamma))$. By the **LLL**-closure of $Cn_{\mathbf{SAL_i}}(\Gamma)$, $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$. But then by the induction hypothesis, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta) \check{\vee} Dab(\Theta')$ for a $\Theta' \subset \Omega_{(i)}$, whence $(\Theta \cup \Theta') \subset \Omega_{(i+1)}$.

*Ad 2.* Analogous to the reasoning for item 1 – replace Theorem 2.6.1 by Theorem 2.7.1.

*Ad 3.* Suppose $A \in Cn_{\mathbf{SAL}}(\Gamma)$. By Definition 3.1, there is an $i \in I$ such that $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$. The rest is immediate in view of items 1 and 2. ■

[10] Of course, one cannot argue that it is impossible to develop a generic proof theory for **SAL** without this requirement, using exactly the same format of proofs as flat ALs. However, a significant number of previous proposals all turned out to be either not sound or incomplete, in view of examples such as $\Gamma_{\mathbf{c}}$ and $\Gamma_{\mathbf{s}}$ from Section 3.3.3.

**The stage g.** Before we turn to the metaproof for the adequacy of $\vdash_{\mathbf{SAL}}$, it is worthwhile to take a closer look at two crucial concepts that are used in it. The first is the infinite stage $\mathsf{g}$. For every $A$ and $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, $A$ is derived on the condition $\Delta$ at stage $\mathsf{g}$.

To see how the stage $\mathsf{g}$ can be reached for any $\Gamma \subseteq \mathcal{W}_s$, let $Cn_{\mathbf{LLL}}(\Gamma) = \{A_1, A_2, \ldots\}$. Note that every $A_i \in Cn_{\mathbf{LLL}}(\Gamma)$ corresponds to finitely many formulas of the form $B \check{\vee} Dab(\Delta)$: $B_i^1 \check{\vee} Dab(\Delta_i^1), \ldots, B_i^{n_i} \check{\vee} Dab(\Delta_i^{n_i})$. For instance, the formula $(p \wedge q) \check{\vee} !^1p \check{\vee} !^2q$ corresponds to exactly three formulas of the form $B \check{\vee} Dab(\Delta)$: $(p \wedge q) \check{\vee} Dab(!^1p, !^2q)$, $(p \wedge q) \check{\vee} !^1p \check{\vee} Dab(\{!^2q\})$, and $((p \wedge q) \check{\vee} !^1p \check{\vee} !^2q) \check{\vee} Dab(\emptyset)$ — recall that "$\check{\vee} Dab(\emptyset)$" denotes the empty string.

By the compactness of **LLL**, for every such $A_i$, there are $C_i^1, \ldots, C_i^{m_i} \in \Gamma$ such that $\{C_i^1, \ldots, C_i^{m_i}\} \vdash_{\mathbf{LLL}} A_i$. Hence $\{C_i^1, \ldots, C_i^{m_i}\} \vdash_{\mathbf{LLL}} B_i^1 \check{\vee} Dab(\Delta_i^1)$ $\ldots$, and $\{C_i^1, \ldots, C_i^{m_i}\} \vdash_{\mathbf{LLL}} B_i^{n_i} \check{\vee} Dab(\Delta_i^{n_i})$. This means we can arrive at stage $\mathsf{g}$ as follows:

| | | | |
|---|---|---|---|
| 1 | $C_1^1$ | PREM | $\emptyset$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\mathrm{m}_1$ | $C_1^{m_1}$ | PREM | $\emptyset$ |
| $\mathrm{m}_1+1$ | $B_1^1$ | $1,\ldots,\mathrm{m}_1$;RC | $\{\Delta_1^1\}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\mathrm{m}_1+\mathrm{n}_1$ | $B_1^{n_1}$ | $1,\ldots,\mathrm{m}_1$;RC | $\{\Delta_1^{n_1}\}$ |
| $\mathrm{m}_1+\mathrm{n}_1+1$ | $C_2^1$ | PREM | $\emptyset$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\mathrm{m}_1+\mathrm{n}_1+\mathrm{m}_2$ | $C_2^{m_2}$ | PREM | $\emptyset$ |
| $\mathrm{m}_1+\mathrm{n}_1+\mathrm{m}_2+1$ | $B_2^1$ | $\mathrm{m}_1+\mathrm{n}_1+1,\ldots,\mathrm{m}_1+\mathrm{n}_1+\mathrm{m}_2$;RC | $\{\Delta_2^1\}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\mathrm{m}_1+\mathrm{n}_1+\mathrm{m}_2+\mathrm{n}_2$ | $B_2^{n_2}$ | $\mathrm{m}_1+\mathrm{n}_1+1,\ldots,\mathrm{m}_1+\mathrm{n}_1+\mathrm{m}_2$;RC | $\{\Delta_2^{n_2}\}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

As we will see below, a formula $A$ is derived on an unmarked line at stage $\mathsf{g}$ iff $A \in Cn_{\mathbf{SAL}}(\Gamma)$. This point can be made more precise if we first define the set of $i$-derived formulas at stage $s$. Where $s$ is the stage of an **SAL**-proof from $\Gamma$, ${}^{\mathbf{S}}\Lambda_s^i(\Gamma)$ is the set of all formulas $A$ such that $A$ is derived on an $i$-unmarked line at stage $s$.

Below, it is shown that ${}^{\mathbf{S}}\Lambda_{\mathsf{g}}^i(\Gamma) = Cn_{\mathbf{SAL_i}}(\Gamma)$ for all $i \in I$ – see Lemma 3.12. In other words, the set of formulas that are derived on an $i$-unmarked line converges towards $Cn_{\mathbf{SAL_i}}(\Gamma)$ as we derive more and more formulas.[11] In view of Definition 3.1, the set of formulas that are derived on an unmarked line converges towards $Cn_{\mathbf{SAL}}(\Gamma)$ as we derive more and more formulas in a proof.

I end this paragraph with a number of facts about $\mathsf{g}$, which will be used in the remainder:

**Fact 3.3** *For every $i \in I$: If a line $l$ is $i$-(un)marked in a proof at stage* $\mathsf{g}$*, then it is $i$-(un)marked in every further extension of this proof.*

[11] Of course, the speed by which we converge towards $Cn_{\mathbf{SAL_i}}(\Gamma)$ depends on the specific moves we make throughout the proof, which may be optimized by specific heuristic devices.

**Fact 3.4** *Each of the following holds:*

1. $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *iff* $A$ *is derived on the condition* $\Delta$ *at stage* $\mathsf{g}$.
2. *Where* $\Theta \subseteq \Delta$: $A$ *is derived on the condition* $\Delta$ *at stage* $\mathsf{g}$ *iff* $A \check{\vee} Dab(\Theta)$ *is derived on the condition* $\Delta - \Theta$ *at stage* $\mathsf{g}$

**Fact 3.5** ${}^{\mathbf{S}}\Sigma^1_{\mathsf{g}}(\Gamma) = \Sigma^1(\Gamma) = \Sigma^1(Cn_{\mathbf{LLL}}(\Gamma))$.

**Fact 3.6** ${}^{\mathbf{S}}\Sigma^i_s(\Gamma) = \{\Delta \subset \Omega_i \mid Dab(\Delta) \in {}^{\mathbf{S}}\Lambda^{i-1}_s(\Gamma)$ *and for no* $\Delta' \subset \Delta : Dab(\Delta') \in {}^{\mathbf{S}}\Lambda^{i-1}_s(\Gamma)\}$.

**Adequacy of the Stage** $\mathsf{g}$ The following lemma states that at stage $\mathsf{g}$, all formulas derived on an unmarked line are **SAL**-consequences of $\Gamma$, and vice versa.

**Lemma 3.12** *Where* $\Gamma \subseteq \mathcal{W}_s$, *each of the following holds for every* $i \in I$:

*1a.* ${}^{\mathbf{S}}\Sigma^i_{\mathsf{g}}(\Gamma) = \Sigma^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$, *whence also*
*1b.* ${}^{\mathbf{S}}U^i_{\mathsf{g}}(\Gamma) = U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$ *and*
*1c.* ${}^{\mathbf{S}}\Phi^i_{\mathsf{g}}(\Gamma) = \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$
*2.* ${}^{\mathbf{S}}\Lambda^i_{\mathsf{g}}(\Gamma) = Cn_{\mathbf{SAL_i}}(\Gamma)$

*Proof.* ($i = 1$) *Ad 1.* This is Fact 3.5.

*Ad 2.* ($\mathbf{x_1} = \mathbf{r}$) The following are equivalent in view of (1) Definition 3.6, (2) item 1b and Fact 3.4.1, (3) Theorem 2.6 and the fact that $\mathbf{SAL_0} = \mathbf{LLL}$:

- $A \in {}^{\mathbf{S}}\Lambda^1_{\mathsf{g}}(\Gamma)$
- $A$ is derived on the condition $\Delta \subseteq \Omega_1 - {}^{\mathbf{S}}U^1_{\mathsf{g}}(\Gamma)$ at stage $\mathsf{g}$
- there is a $\Delta \subseteq \Omega_1 - U^1(Cn_{\mathbf{SAL_0}}(\Gamma))$ such that $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL_0}}(\Gamma)$
- $A \in Cn_{\mathbf{SAL_1}}(\Gamma)$

($\mathbf{x_1} = \mathbf{m}$) The following are equivalent in view of (1) Definition 3.7, (2) item 1c and Fact 3.4.1, (3) Theorem 2.7 and the fact that $\mathbf{SAL_0} = \mathbf{LLL}$:

- $A \in {}^{\mathbf{S}}\Lambda^1_{\mathsf{g}}(\Gamma)$
- for every $\varphi \in {}^{\mathbf{S}}\Phi^1_{\mathsf{g}}(\Gamma)$, $A$ is derived on a condition $\Delta \subseteq \Omega_1 - \varphi$ at stage $\mathsf{g}$, and $A \check{\vee} Dab(\Delta)$ is derived on a 0-unmarked line at stage $\mathsf{g}$
- for every $\varphi \in \Phi^1(Cn_{\mathbf{SAL_0}}(\Gamma))$, there is a $\Delta \subseteq \Omega_1 - \varphi$ such that $A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{SAL_0}}(\Gamma)$
- $A \in Cn_{\mathbf{SAL_1}}(\Gamma)$

($i \Rightarrow i+1$) *Ad 1.* In view of (1) Fact 3.6, (2) item 2 of the induction hypothesis and (3) the LLL-closure of $Cn_{\mathbf{SAL_i}}(\Gamma)$, the following are equivalent for every $\Delta \subset \Omega_{i+1}$:

- $\Delta \in {}^{\mathbf{S}}\Sigma^{i+1}_{\mathsf{g}}(\Gamma)$
- $Dab(\Delta) \in {}^{\mathbf{S}}\Lambda^i_{\mathsf{g}}(\Gamma)$, and for no $\Delta' \subset \Delta$: $Dab(\Delta') \in {}^{\mathbf{S}}\Lambda^i_{\mathsf{g}}(\Gamma)$
- $Dab(\Delta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$, and for no $\Delta' \subset \Delta$: $Dab(\Delta') \in Cn_{\mathbf{SAL_i}}(\Gamma)$
- $\Delta \in \Sigma^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$

*Ad 2.* ($\mathbf{x_{i+1}} = \mathbf{r}$) The following are equivalent in view of (1) Definition 3.6, (2) Fact 3.4.2 and item 2 of the induction hypothesis, (3) item 1b, (4) Lemma 3.11.1 and Theorem 3.7.1:

- $A \in {}^{\mathbf{S}}\Lambda^{i+1}_{\mathsf{g}}(\Gamma)$
- $A$ is derived on a condition $\Delta \subset \Omega_{(i+1)}$ at stage $\mathsf{g}$, and there is a $\Theta \subseteq \Delta \cap \Omega_{i+1}$ such that (a) $A \check{\vee} Dab(\Theta)$ is derived on an $i$-unmarked line with condition $\Delta - \Theta$ at stage $\mathsf{g}$, and (b) $\Theta \cap {}^{\mathbf{S}}U^{i+1}_{\mathsf{g}}(\Gamma) = \emptyset$.
- $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega_{(i+1)}$, and there is a $\Theta \subseteq \Delta \cap \Omega_{i+1}$ such that (a) $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$, and (b) $\Theta \cap {}^{\mathbf{S}}U^{i+1}_{\mathsf{g}}(\Gamma) = \emptyset$.
- $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega_{(i+1)}$, and there is a $\Theta \subseteq \Delta \cap \Omega_{i+1}$ such that (a) $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$, and (b) $\Theta \cap U^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \emptyset$.
- $A \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$

($\mathbf{x_{i+1}} = \mathbf{m}$) The following are equivalent in view of (1) Definition 3.7, (2) Fact 3.4.2 and item 2 of the induction hypothesis, (3) item 1c, (4) Lemma 3.11.2 and Theorem 3.7.2:

- $A \in {}^{\mathbf{S}}\Lambda^{i+1}_{\mathsf{g}}(\Gamma)$
- for every $\varphi \in {}^{\mathbf{S}}\Phi^{i+1}_{\mathsf{g}}(\Gamma)$: $A$ is derived on a condition $\Delta \subset \Omega_{(i+1)}$ at stage $\mathsf{g}$, and there is a $\Theta \subseteq \Delta \cap \Omega_{i+1}$ such that (a) $A \check{\vee} Dab(\Theta)$ is derived on an $i$-unmarked line with condition $\Delta - \Theta$ at stage $\mathsf{g}$, and (b) $\Theta \cap \varphi = \emptyset$.
- for every $\varphi \in {}^{\mathbf{S}}\Phi^{i+1}_{\mathsf{g}}(\Gamma)$: $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega_{(i)}$, and there is a $\Theta \subseteq \Delta \cap \Omega_{i+1}$ such that (a) $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$, and (b) $\Theta \cap \varphi = \emptyset$.
- for every $\varphi \in \Phi^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega_{(i)}$, and there is a $\Theta \subseteq \Delta \cap \Omega_{i+1}$ such that (a) $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$, and (b) $\Theta \cap \varphi = \emptyset$.
- $A \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$

■

**The Finite Stage Lemma for SAL.** In order to obtain a proof of adequacy, it does not suffice to just prove that the infinite proof at stage $\mathsf{g}$ corresponds to $Cn_{\mathbf{SAL}}(\Gamma)$. As for flat ALs, we also need to show that whenever $A \in Cn_{\mathbf{SAL}}(\Gamma)$, then the proponent can derive $A$ in a *finite* proof from $\Gamma$, on an unmarked line. For the **SAL**-proof theory, this is a rather complex matter, in view of the recursive character of the marking definitions.

**Lemma 3.13** *Where $\Gamma \subseteq \mathcal{W}_s$: if $A \in Cn_{\mathbf{SAL}}(\Gamma)$, then there is a finite* **SAL***-proof from $\Gamma$ in which $A$ is derived on an unmarked line.*

*Proof.* Suppose that $A \in Cn_{\mathbf{SAL}}(\Gamma)$. If $A \in Cn_{\mathbf{LLL}}(\Gamma)$, let $n = 0$. Otherwise, let $i_1 \in I$ be such that ($\star_1$) $A \in Cn_{\mathbf{SAL_{i_1}}}(\Gamma) - Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma)$. By Theorem 2.6.1, resp. Theorem 2.7.1, the following holds:

($\star_1^{\mathbf{r}}$) Where $\mathbf{x_{i_1}} = \mathbf{r}$: $Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_1)$, for a $\Delta_1 \subseteq \Omega_{i_1} - U^{i_1}(Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma))$

($\star_1^{\mathbf{m}}$) Where $\mathbf{x_{i_1}} = \mathbf{m}$: $Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_1)$, for a $\Delta_1 \subseteq \Omega_{i_1} - \varphi$, for a $\varphi \in \Phi^{i_1}(Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma))$

If $A \check{\vee} Dab(\Delta_1) \in Cn_{\mathbf{LLL}}(\Gamma)$, let $n = 1$. Otherwise, let $i_2 < i_1$ be such that $(\star_2)$ $A \check{\vee} Dab(\Delta_1) \in Cn_{\mathbf{SAL_{i_2}}}(\Gamma) - Cn_{\mathbf{SAL_{i_2-1}}}(\Gamma)$. By Theorem 2.6.1, resp. Theorem 2.7.1, the following holds:

$(\star_2^{\mathbf{r}})$ Where $\mathbf{x_{i_2}} = \mathbf{r}$: $Cn_{\mathbf{SAL_{i_2-1}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_1) \check{\vee} Dab(\Delta_2)$, for a $\Delta_2 \subseteq \Omega_{i_2} - U^{i_2}(Cn_{\mathbf{SAL_{i_2-1}}}(\Gamma))$

$(\star_2^{\mathbf{m}})$ Where $\mathbf{x_{i_2}} = \mathbf{m}$: $Cn_{\mathbf{SAL_{i_2-1}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_1) \check{\vee} Dab(\Delta_2)$, for a $\Delta_2 \subseteq \Omega_{i_2} - \varphi$, for a $\varphi \in \Phi^{i_2}(Cn_{\mathbf{SAL_{i_2-1}}}(\Gamma))$

Repeating the same reasoning finitely many times, we obtain that there are $i_1, \ldots, i_n \in I$ and $\Delta_1 \subset \Omega_{i_1}, \ldots, \Delta_n \subset \Omega_{i_n}$, such that (i) $A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_n) \in Cn_{\mathbf{SAL_0}}(\Gamma)$, and (ii) for every $k \leq n$:

$(\star_k)$ $A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_{k-1}) \in Cn_{\mathbf{SAL_{i_k}}}(\Gamma) - Cn_{\mathbf{SAL_{i_k-1}}}(\Gamma)$

$(\star_k^{\mathbf{m}})$ Where $\mathbf{x_{i_k}} = \mathbf{r}$: $Cn_{\mathbf{SAL_{i_k-1}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_k)$, for a $\Delta_k \subseteq \Omega_{i_k} - U^{i_k}(Cn_{\mathbf{SAL_{i_k-1}}}(\Gamma))$

$(\star_k^{\mathbf{r}})$ Where $\mathbf{x_{i_k}} = \mathbf{m}$: $Cn_{\mathbf{SAL_{i_k-1}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_1)$, for a $\Delta_k \subseteq \Omega_{i_k} - \varphi$, for a $\varphi \in \Phi^{i_k}(Cn_{\mathbf{SAL_{i_k-1}}}(\Gamma))$

In view of (i) and the compactness of **LLL**, there is a $\Gamma' = \{B_1, \ldots, B_m\} \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_n)$. Hence we may start an **SAL**-proof P from $\Gamma$ as follows:

| | | | |
|---|---|---|---|
| 1 | $B_1$ | PREM | $\emptyset$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| m | $B_m$ | PREM | $\emptyset$ |
| m+1 | $A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_n)$ | 1,…,m;RU | $\emptyset$ |
| m+2 | $A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_{n-1})$ | m+1;RC | $\Delta_n$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| m+n | $A \check{\vee} Dab(\Delta_1)$ | m+n-1;RC | $\Delta_2 \cup \ldots \cup \Delta_n$ |
| m+n+1 | $A$ | m+n;RC | $\Delta_1 \cup \ldots \cup \Delta_n$ |

Let $s$ be the stage consisting of line 1 up to line m+n+1. In the remainder of this proof, I will show by a reductio that line m+n+1 is unmarked at stage $s$. So assume that (†) line $m+n+1$ is marked at stage $s$. By Lemma 3.10, (†$_1$) for every $i \leq i_1$, line $m+n+1$ is $i$-marked at stage $s$.

**Proposition 3.1** *There is no $k \leq n$ such that $A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_k)$ is a $Dab_{i_1}$-formula.*

*Subproof.* Suppose that for a $k \leq n$, $A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_k)$ is a Dab$_{i_1}$-formula. It follows that $A$ is also a Dab$_{i_1}$-formula. But then, since $A \in Cn_{\mathbf{SAL_{i_1}}}(\Gamma)$ and by the Dab-conservativity of $\mathbf{AL_{i_1}}$ (see Lemma 2.4.2), we can derive that $A \in Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma)$, which contradicts $(\star_1)$. ∎

**Proposition 3.2** *Let $\Theta_1 = \{B_1, \ldots, B_m\} \cap \Omega_{i_1}$. Then each of the following holds:*[12]

1. ${}^{\mathbf{S}}U^{i_1}_s(\Gamma) = \Theta_1$
2. $\Theta_1 \subseteq U^{i_1}(Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma))$
3. ${}^{\mathbf{S}}\Phi^{i_1}_s(\Gamma) = \{\Theta_1\}$
4. *For every* $\varphi \in \Phi^{i_1}(Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma))$*:* $\Theta_1 \subseteq \varphi$

*Subproof.* By $(\dagger_1)$, Proposition 3.1 and Definition 3.5, the only minimal $\mathrm{Dab}_{i_1}$-formulas at stage $s$ (if any) are $B_i$ with $i \leq m$. Since these are members of $\mathcal{W}_s$, they are Dab-singletons $Dab(\{D\})$, with $D \in \Gamma$. From this, we can immediately derive items 1 and 3. Items 2 and 4 are immediate in view of the fact that $\{B_1, \ldots, B_m\} \in \Gamma$ and the reflexivity of $\mathbf{SAL_{i_1-1}}$. ∎

By $(\dagger_1)$, line $m+n+1$ is $i_1$-marked at stage $s$. Hence in view of Definitions 3.6 and 3.7, one of the following holds:

$(\ddagger_1)$ line $m+n$ is $(i_1-1)$-marked
$(\ddagger_1^{\mathbf{r}})$ $\mathbf{x_{i_1}} = \mathbf{r}$ and $\Delta_1 \cap {}^{\mathbf{S}}U^{i_1}_s(\Gamma) \neq \emptyset$
$(\ddagger_1^{\mathbf{m}})$ $\mathbf{x_{i_1}} = \mathbf{m}$ and there is a $\varphi \in {}^{\mathbf{S}}\Phi^{i_1}_s(\Gamma)$ such that $\Delta_1 \cap \varphi \neq \emptyset$

However, $(\ddagger_1^{\mathbf{r}})$ is excluded in view of Proposition 3.2.1-2 and $(\star_1^{\mathbf{r}})$. Likewise, $(\ddagger_1^{\mathbf{m}})$ is excluded in view of Proposition 3.2.3-4 and $(\star_1^{\mathbf{m}})$. It follows that line $m+n$ is $(i_1-1)$-marked at stage $s$. Hence, since $i_2 < i_1$ and by Lemma 3.10:

$(\dagger_2)$ for every $i \leq i_1 - 1$: line $m+n$ is $i$-marked at stage $s$.

By the same reasoning as in the proof of Proposition 3.1, relying on the Dab-conservativity of $\mathbf{AL_{i_2}}$ and $(\star_2)$, we can derive:

**Proposition 3.3** *There is no $k \leq n-1$ such that $A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_k)$ is a $Dab_{i_2}$-formula.*

Hence by the same reasoning as in the proof of Proposition 3.2, relying on $(\dagger_1)$, $(\dagger_2)$ and Proposition 3.3, we can derive:

**Proposition 3.4** *Let $\Theta_2 = \{B_1, \ldots, B_m\} \cap \Omega_{i_2}$. Then each of the following holds:*

1. ${}^{\mathbf{S}}U^{i_2}_s(\Gamma) = \Theta_2$
2. $\Theta_2 \subseteq U^{i_2}(Cn_{\mathbf{SAL_{i_2-1}}}(\Gamma))$
3. ${}^{\mathbf{S}}\Phi^{i_2}_s(\Gamma) = \{\Theta_2\}$
4. *For every* $\varphi \in \Phi^{i_2}(Cn_{\mathbf{SAL_{i_2-1}}}(\Gamma))$*:* $\Theta_2 \subseteq \varphi$

By $(\dagger_2)$ and since $i_2 < i_1$, line $m+n$ is $i_2$-marked. In view of Definitions 3.6 and 3.7, one of the following holds:

$(\ddagger_2)$ line $m+n-1$ is $(i_2-1)$-marked
$(\ddagger_2^{\mathbf{r}})$ $\mathbf{x_{i_2}} = \mathbf{r}$ and $\Delta_2 \cap {}^{\mathbf{S}}U^{i_2}_s(\Gamma) \neq \emptyset$
$(\ddagger_2^{\mathbf{m}})$ $\mathbf{x_{i_2}} = \mathbf{m}$ and there is a $\varphi \in {}^{\mathbf{S}}\Phi^{i_2}_s(\Gamma)$ such that $\Delta_2 \cap \varphi \neq \emptyset$

[12] Items 2 and 4 of this proposition are used below, in the proof of Theorem 3.9 below.

However, ($\ddagger_2^{\mathbf{r}}$) is excluded in view of Proposition 3.4.1 and ($\star_2^{\mathbf{r}}$). Likewise, ($\ddagger_2^{\mathbf{m}}$) is excluded in view of Proposition 3.4.3 and ($\star_2^{\mathbf{m}}$). It follows that line $m+n-1$ is $(i_2-1)$-marked at stage $s$. Hence also:

($\dagger_3$) for every $i \leq i_2 - 1$: line $m+n-1$ is $i_3$-marked at stage $s$.

Repeating this reasoning $n$ times, we can derive that line $m+1$ is $i_n$-marked at stage $s$. But this contradicts the fact that the condition of line $m+1$ is empty. ■

**Soundness and Completeness** The preceding results finally allow us to prove the adequacy of $\vdash_{\mathbf{SAL}}$. We first prove soundness, next completeness.

**Theorem 3.8** *Where $\Gamma \subseteq \mathcal{W}_s$: if $\Gamma \vdash_{\mathbf{SAL}} A$, then $A \in Cn_{\mathbf{SAL}}(\Gamma)$.*

*Proof.* Suppose $\Gamma \vdash_{\mathbf{SAL}} A$. By Definitions 3.8 and 3.9, $A$ is derived on an unmarked line $l$ of a finite **SAL**-proof P from $\Gamma$. Suppose we extend P up to stage $\mathsf{g}$. If $l$ is marked in this extension, then by Fact 3.3, $l$ is marked in every further extension of the proof, which contradicts the fact that $A$ is finally derived on line $l$. Hence line $l$ is unmarked at stage $\mathsf{g}$. It follows that for an $i \in I$, line $l$ is $i$-unmarked at stage $\mathsf{g}$, whence $A \in {}^{\mathbf{S}}\Lambda^i_{\mathsf{g}}(\Gamma)$. By Lemma 3.12.2, $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$, whence by Definition 3.1, $A \in Cn_{\mathbf{SAL}}(\Gamma)$. ■

**Theorem 3.9** *Where $\Gamma \subseteq \mathcal{W}_s$: if $A \in Cn_{\mathbf{SAL}}(\Gamma)$, then $\Gamma \vdash_{\mathbf{SAL}} A$.*

*Proof.* Suppose $A \in Cn_{\mathbf{SAL}}(\Gamma)$. Let P be the same proof as the one constructed in the proof of Lemma 3.13. It follows that $A$ is derived on the $i_1$-unmarked line $m+n+1$ at stage $s$ of that proof. Also, $A \in Cn_{\mathbf{SAL_{i_1}}}(\Gamma)$. Let $\Delta = \Delta_1 \cup \ldots \cup \Delta_n$.

Suppose line m+n+1 is marked in an extension of the proof. We further extend the proof up to stage $\mathsf{g}$. By ($\star_1^{\mathbf{r}}$), resp. ($\star_1^{\mathbf{m}}$), $A \check{\vee} Dab(\Delta_1) \in Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma)$, whence by Lemma 3.12.2, $A \check{\vee} Dab(\Delta_1) \in {}^{\mathbf{S}}\Lambda^{i_1-1}_{\mathsf{g}}(\Gamma)$. Hence ($\dagger$) $A \check{\vee} Dab(\Delta_1)$ is derived on an $(i_1-1)$-unmarked line at stage $\mathsf{g}$, on the condition $\Delta - \Delta_1$.

($\mathbf{x_{i_1}} = \mathbf{r}$) By ($\star_1^{\mathbf{r}}$), $\Delta_1 \cap U^{i_1}(Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma)) = \emptyset$. By Lemma 3.12.1b, $\Delta_1 \cap {}^{\mathbf{S}}U^{i_1}_{\mathsf{g}}(\Gamma) = \emptyset$. By ($\dagger$) and Definition 3.6, line $m+n+1$ is $i_1$-unmarked at stage $\mathsf{g}$. By Definitions 3.8 and 3.9, $\Gamma \vdash_{\mathbf{SAL}} A$.

($\mathbf{x_{i_1}} = \mathbf{m}$) By ($\star_1^{\mathbf{m}}$), $\Delta_1 \cap \varphi = \emptyset$ for a $\varphi \in \Phi^{i_1}(Cn_{\mathbf{SAL_{i_1-1}}}(\Gamma))$. By Lemma 3.12.1c, $\varphi \in {}^{\mathbf{S}}\Phi^{i_1}_{\mathsf{g}}(\Gamma)$.

Since $A \in Cn_{\mathbf{SAL_{i_1}}}(\Gamma)$, also $A \in {}^{\mathbf{S}}\Lambda^i_{\mathsf{g}}(\Gamma)$ by Lemma 3.12.2. Hence by Definition 3.7, for every $\varphi' \in {}^{\mathbf{S}}\Phi^i_{\mathsf{g}}(\Gamma)$, $A$ is derived on a condition $\Delta'$ at stage $\mathsf{g}$ such that there is a $\Theta' \subset \Delta' \cap \Omega_i$ for which

(ii.a) $A \check{\vee} Dab(\Theta')$ is derived on an $(i-1)$-unmarked line with condition $\Delta' - \Theta'$ at stage $\mathsf{g}$, and

(ii.b) $\varphi' \cap \Theta' = \emptyset$.

By Definition 3.7, line $m+n+1$ is $i_1$-unmarked at stage $\mathsf{g}$. By Definitions 3.8 and 3.9, $\Gamma \vdash_{\mathbf{SAL}} A$. ■

### 3.4.4 Towards the Second Proposal

As already announced in Section 3.4.1, the first proposal has a certain non-standard aspect, even though the format of its proof is very similar to that of flat ALs. That is, the marking definition sometimes requires us to retract certain inferences, seemingly without there being a reason to do so. Consider the following $\mathbf{SK2^r}$-proof from $\Gamma_{\mathsf{p1}}$:

| | | | | |
|---|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\emptyset$ | $-_0$ |
| 2 | $\Diamond\Diamond q$ | PREM | $\emptyset$ | $-_0$ |
| 3 | $p \wedge q$ | 5; RU | $\{!^1 p, !^2 q\}$ | ✓ |

Note that the inference to line 3 is perfectly valid: since $\{\Diamond p, \Diamond\Diamond q\} \vdash_{\mathbf{K}} (p \wedge q) \check{\vee} !^1 p \check{\vee} !^2 q$, we can apply RC and push both $!^1 p$ and $!^2 q$ to the condition in one fell swoop. Note also that no Dab-consequences have been derived so far. Nevertheless, line 3 is marked, since requirement (a) of the definition of 2-unmarking for Reliability is violated. Not so in the following continuation of the proof:

| | | | | |
|---|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\emptyset$ | $-_0$ |
| 2 | $\Diamond\Diamond q$ | PREM | $\emptyset$ | $-_0$ |
| 3 | $p \wedge q$ | 5; RU | $\{!^1 p, !^2 q\}$ | $-_2$ |
| 4 | $(p \wedge q) \check{\vee} !^1 p \check{\vee} !^2 q$ | 1,2;RU | $\emptyset$ | $-_0$ |
| 5 | $(p \wedge q) \check{\vee} !^2 q$ | 4;RC | $\{!^1 p\}$ | $-_1$ |

That is, by adding lines 4 and 5, we have ensured that $(p \wedge q) \check{\vee} !^2 q$ is derived on a 1-unmarked line. This way, requirement (a) is fulfilled.

This aspect of the **SAL**-proof theory may strike some as counterintuitive. Indeed, in most approaches to defeasible reasoning, an assumption is only withdrawn as soon as we have good reasons to think that upholding the assumption will result in triviality or the violation of certain constraints. But if we have not yet derived any Dab-consequence (as in the proof at stage 3), then how can we have reasons to think so? Is the first proposal not fairly unnatural from this viewpoint? And does this make it unfit to explicate prioritized reasoning, or should we at least try to remedy this shortcoming?

I will not answer these (very general) questions here. In my opinion, what is most disadvantageous about the first proposal is the fact that it requires the user to make more inferences than at first sight seems to be necessary (see also Example 3.6), and that the unmarking procedure is rather demanding: to find out whether a formula is derived or not at stage $s$, one has to take not only the sets of unreliable formulas resp. sets of minimal choice sets into account, but also whether other formulas have been derived on unmarked lines.[13] It was this

[13] To some extent, this is already the case for the marking in flat ALs that use the Minimal Abnormality Strategy. However, in that case we only need to consider the question whether *the same* formula $A$ has been derived on other conditions, in order to see whether a line $l$ is marked. In Definitions 3.6 and 3.7, one has to take into account the derivation of the same formula $A$ *in disjunction* with some abnormalities, which cannot be fully determined in view of the condition of line $l$.

tiresome procedure in particular that motivated the transition to a new proof format, which allows for much simpler marking definitions.

Nevertheless, the **SAL**-proof theory does have a great advantage, i.e. that it uses the same proof format as flat adaptive logics. To see why this is important, consider the relation between logics $\mathbf{AL^x} = \langle \mathbf{LLL}, \bigcup_{i \in I} \Omega_i, \mathbf{x} \rangle$ and logics $\mathbf{SAL^x}$. It is very plausible that in some contexts, we make a *transition* from $\mathbf{AL^x}$ to $\mathbf{SAL^x}$. For instance, starting with a set of beliefs, we may first reason with these beliefs without taking into account the specific degree of plausibility of each belief. After a while, we may see that certain beliefs are problematic in view of the facts at hand. Hence at that point, we have to retract some of these beliefs. However, we can also start taking into account the priority degrees of our beliefs, hoping that this will allow us to uphold some of the beliefs that are involved in a conflict.

In this and similar cases, we do not start to reason again from scratch; rather, we build further on what has been derived so far, but *interpret* it differently. This shift in interpretation of the same evidence corresponds to a shift in the marking definitions that are applied to (extensions of) the same proof. As we will see below, such a transition cannot always be easily modeled in the format that is spelled out in the next section.

## 3.5 Another Proof Theory for SAL

*This section is based on the paper "Proof Theories for Superpositions of Adaptive Logics" (in preparation), which is co-authored by Christian Straßer.*

### 3.5.1 Conditions As Sequences

**The proof format** The proof format of the second proposal is nearly identical to the one of flat ALs. Again, a line is a quadruple consisting of a line number, a formula, a justification and a condition. The only difference concerns the last element. A condition is not just a finite set of abnormalities, but instead a *sequence* of sets of abnormalities $\langle \Delta_i \rangle_{i \in I}$ where (i) each $\Delta_i$ is a subset of $\Omega_i$, and (ii) $\bigcup_{i \in I} \Delta_i$ is a finite set. Note that since $\bigcup_{i \in I} \Delta_i$ is finite, we can always represent $\langle \Delta_i \rangle_{i \in I}$ by a finite string, e.g. by $\langle \Delta_1, \ldots, \Delta_n, \emptyset, \ldots \rangle$, where the second "..." denotes a sequence of finitely or infinitely many times $\emptyset$, depending on the cardinality of $I$. In the following, we write $\mathbf{\Delta}$ for $\langle \Delta_i \rangle_{i \in I}$, $\boldsymbol{\emptyset}$ for the sequence $\langle \emptyset, \emptyset, \ldots \rangle$, $\bigcup \mathbf{\Delta}$ for $\bigcup_{i \in I} \Delta_i$ and $Dab(\mathbf{\Delta})$ for $Dab(\bigcup_{i \in I} \Delta_i)$.[14]

Suppose we have the following line in a proof[15]

| | | | |
|---|---|---|---|
| l | $A$ | $k_1, \ldots, k_n$; R | $\langle \Delta_1, \Delta_2, \emptyset, \ldots \rangle$ |

where $\Delta_1 \neq \emptyset \neq \Delta_2$. Suppose moreover that line $l$ is unmarked. The idea is that $A$ is derived on the assumption that no abnormality in $\Delta_1 \cup \Delta_2$ is true. Hence, we make use of the defeasible reasoning forms represented by both $\mathbf{AL_1}$ and $\mathbf{AL_2}$. Moreover, in case $A$ is finally derived at line $l$ (see the definition

[14] The number of members in $\boldsymbol{\emptyset}$ will of course depend on the cardinality of $I$.

[15] We use R as a metavariable for the generic inference rules.

below), then $A$ is a consequence of the superposition of $\mathbf{AL_2}$ on $\mathbf{AL_1}$, since no defeasible assumptions were made that correspond to ALs higher in the sequence of **SAL**.

**The generic inference rules** In order to realize this idea we will again make use of three generic rules and marking definitions.

Similar as in flat adaptive proofs we will need to merge the conditions of two or more lines. In the flat case we could just take the union of the respective sets of abnormalities. This idea can easily be generalized to the sequential case in the following way: $\mathbf{\Delta} \Cup \mathbf{\Theta} =_{\mathsf{df}} \langle \Delta_i \cup \Theta_i \rangle_{i \in I}$. For instance,

$$\langle \{A, B\}, \{C\}, \emptyset \rangle \Cup \langle \emptyset, \{D\}, \{E\} \rangle = \langle \{A, B\}, \{C, D\}, \{E\} \rangle$$

As in the proof theory of flat ALs, we make use of three generic rules: a premise introduction rule PREM, an unconditional rule RU, and a conditional rule RC. Let us start with the first two:

PREM If $A \in \Gamma$:
$$\begin{array}{ll} \vdots & \vdots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:
$$\begin{array}{ll} A_1 & \mathbf{\Delta}_1 \\ \vdots & \vdots \\ A_n & \mathbf{\Delta}_n \\ \hline B & \mathbf{\Delta}_1 \Cup \ldots \Cup \mathbf{\Delta}_n \end{array}$$

As in the flat case, by the rule PREM premises can be introduced on the empty condition (which is now a sequence of empty sets). Also, the unconditional rule RU is analogous to the flat case. In case $B$ is derivable from $A_1, \ldots, A_n$ in the lower limit logic, we may derive $B$ also in an adaptive proof from $A_1, \ldots, A_n$ whereby the conditions $\mathbf{\Delta}_i$ on which the $A_i$'s were derived are carried forward and merged to $\mathbf{\Delta}_1 \Cup \ldots \Cup \mathbf{\Delta}_n$.

The generic conditional rule for our proof theory also closely resembles the conditional rule of Section 2.3:

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\mathbf{\Theta})$:
$$\begin{array}{ll} A_1 & \mathbf{\Delta}_1 \\ \vdots & \vdots \\ A_n & \mathbf{\Delta}_n \\ \hline B & \mathbf{\Delta}_1 \Cup \ldots \Cup \mathbf{\Delta}_n \Cup \mathbf{\Theta} \end{array}$$

Suppose we are able to derive $B \check{\vee} Dab(\Theta_1 \cup \ldots \cup \Theta_n)$ in **LLL** from $A_1, \ldots, A_n$, where each $\Theta_i \subset \Omega_i$. In that case the proof theory allows us to defeasibly derive $B$ from $A_1, \ldots, A_n$, namely on the assumption that none of the abnormalities in $\Theta_1 \cup \ldots \cup \Theta_n$ is true. This is realized by merging $\mathbf{\Theta} = \langle \Theta_1, \ldots, \Theta_n, \emptyset, \ldots \rangle$ with all the conditions on which the $A_i$'s were derived.

In case some $\Omega_i$'s are intersecting, this can have an interesting consequence. Suppose for instance that $C_1 \in \Omega_1 \cap \Omega_2$ and that $C_2 \in \Omega_2 \setminus \Omega_1$. Suppose furthermore that $A_1, A_2 \vdash_{\mathbf{LLL}} B \check{\vee} (C_1 \check{\vee} C_2)$. Then the following lines can occur in a proof:

| | | | |
|---|---|---|---|
| $l_1$ | $A_1$ | ... | $\mathbf{\Delta}_1$ |
| $l_2$ | $A_2$ | ... | $\mathbf{\Delta}_2$ |
| $l_3$ | $B$ | $l_1, l_2$; RC | $\mathbf{\Delta}_1 \Cup \mathbf{\Delta}_2 \Cup \langle \{C_1\}, \{C_2\}, \emptyset, \ldots \rangle$ |
| $l_4$ | $B$ | $l_1, l_2$; RC | $\mathbf{\Delta}_1 \Cup \mathbf{\Delta}_2 \Cup \langle \emptyset, \{C_1, C_2\}, \emptyset, \ldots \rangle$ |

Note that RC allows for both inferences, the one at line $l_3$ and the one at line $l_4$, and hence leaves room for a choice. We will return to this point at the end of this section, and show that in some cases, it is crucial to warrant the completeness of the proof theory with respect to the syntactic consequence relation of **SAL**.

This relates to another important aspect of the new proof format. On the one hand, we can easily translate every proof in this format into a proof in the standard format, simply replacing each sequential condition $\mathbf{\Delta}$ with the set $\bigcup \mathbf{\Delta}$. This can easily be seen in view of the above definition of the generic rules. However, for a given proof P in the standard format, there is not always just one unique proof P′ in the new format which corresponds to P. This is only the case if for all $i, j \in I$ such that $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$ — we will consider this special case below. As a result, the transition from a "flat interpretation" of a proof to a "prioritized interpretation" (see also Section 3.4.4), is not always as straightforward as was the case with the first proposal for an **SAL**-proof theory.

To disambiguate between the two proposals, we will call every proof that is the result of applications of the above three generic rules an **SAL**′-proof. In the remainder, we will spell out the marking definitions for **SAL**′-proofs, and show how this gives us the derivability relation $\vdash_{\mathbf{SAL}'}$.

**Preparing for the marking definitions** As in the case of flat ALs, lines in an **SAL**′-proof are marked at a certain stage of the proof in order to signify that the corresponding inference is retracted at that stage.

For each level $i \in I$ we will state $i$-marking definitions. If a line in an **SAL**′-proof is $i$-marked for an $i \in I$, then this means the line is retracted at the given stage of the proof.

As before, we introduce some conventions to simplify the marking definitions:

**Definition 3.10** *Let $s$ be the stage of an* **SAL**′*-proof from* $\Gamma$.

- *We call $Dab(\Delta)$ a* minimal $Dab_1$-formula at stage $s$ *in case (i) $Dab(\Delta)$ has been derived on the condition* $\boldsymbol{\emptyset}$ *at stage $s$, and (ii) there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ has been derived on the condition* $\boldsymbol{\emptyset}$ *at stage $s$.*

  *Where $\Delta \subseteq \Omega_{i+1}$, $Dab(\Delta)$ is a* minimal $Dab_{i+1}$-formula at stage $s$ *in case (i) $Dab(\Delta)$ has been derived at some line $l$ on a condition $\langle \Theta_1, \ldots, \Theta_i, \emptyset, \ldots \rangle$ at stage $s$, (ii) line $l$ is not $i$-marked at stage $s$ (see below for the marking definition), and (iii) for no $\Delta' \subset \Delta$, $Dab(\Delta')$ has been derived at an $i$-unmarked line on a condition $\langle \Theta'_1, \ldots, \Theta'_i, \emptyset, \ldots \rangle$ at stage $s$.*

- *Where $\langle Dab(\Delta) \rangle_{j \in J}$ are the minimal $Dab_i$-formulas at stage $s$,* ${}^{\mathbf{C}}\Sigma^i_s(\Gamma) =_{\mathsf{df}} \{\Delta_j \mid j \in J\}$.

- ${}^{\mathbf{C}}U^i_s(\Gamma) =_{\mathsf{df}} \bigcup {}^{\mathbf{C}}\Sigma^i_s(\Gamma)$
- ${}^{\mathbf{C}}\Phi^i_s(\Gamma)$ *is the set of all minimal choice sets of* ${}^{\mathbf{C}}\Sigma^i_s(\Gamma)$

**$i$-Marking for Reliability** Now we are able to the define the $i$-marking at a stage $s$. Let us begin with the marking definition for the Reliability Strategy.

**Definition 3.11 ($i$-marking for Reliability)** *A line $l$ with condition $\mathbf{\Delta}$ is $i$-marked at stage $s$ iff (a) $l$ is $(i-1)$-marked at stage $s$, or (b) $\Delta_i \cap {}^{\mathbf{C}}U^i_s(\Gamma) \neq \emptyset$.*

In case $i = 1$, the above marking definition refers to 0-marking: we stipulate for this case that no line is 0-marked.

Before we turn to the $i$-marking definition for Minimal Abnormality, let us illustrate the generic inference rules and the above marking definition by means of a simple example. Recall that the logic $\mathbf{SK2^r}$ is defined as the superposition of the logic $\mathbf{K^r_2}$ on the logic $\mathbf{K^r_1}$. Now consider the premise set $\Gamma_{\mathsf{p1}} = \{\Diamond p, \Diamond\Diamond q, \Diamond\Diamond r, \neg p \vee \neg r\}$. According to this premise set, $p$, $q$ and $r$ are all three plausible, but $p$ is more plausible than the other two propositions. However, we also know that either $p$ or $r$ is false. Hence we can expect that the prioritized logic will only allow us to finally derive $p$, and hence by disjunctive syllogism $\neg r$. Also, since $q$ is not involved in the conflict, we expect it to be finally derivable. This can be done as follows.

We start by introducing the premises on the condition $\langle \emptyset, \emptyset \rangle$:

| | | | |
|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\langle \emptyset, \emptyset \rangle$ |
| 2 | $\Diamond\Diamond q$ | PREM | $\langle \emptyset, \emptyset \rangle$ |
| 3 | $\Diamond\Diamond r$ | PREM | $\langle \emptyset, \emptyset \rangle$ |
| 4 | $\neg p \vee \neg r$ | PREM | $\langle \emptyset, \emptyset \rangle$ |

By the rule RC, we may subsequently derive $p$, $q$ and $r$ from the first three premises – note that $\Gamma_{\mathsf{p1}} \vdash_{\mathbf{K}} p \check{\vee} !^1 p$, $\Gamma_{\mathsf{p1}} \vdash_{\mathbf{K}} q \check{\vee} !^2 q$ and $\Gamma_{\mathsf{p1}} \vdash_{\mathbf{K}} r \check{\vee} !^2 r$:

| | | | |
|---|---|---|---|
| 5 | $p$ | 1;RC | $\langle \{!^1 p\}, \emptyset \rangle$ |
| 6 | $q$ | 2;RC | $\langle \emptyset, \{!^2 q\} \rangle$ |
| 7 | $r$ | 3;RC | $\langle \emptyset, \{!^2 r\} \rangle$ |

At line 5, we "pushed" the abnormality $!^1 p$ to the first member of the condition; at lines 6 and 7, we "pushed" the abnormalities $!^2 q$ resp. $!^2 r$ to the second member of the condition. To understand the rule RU, consider the following continuation of the proof, in which the conditions of line 5 and 6 are merged:[16]

| | | | |
|---|---|---|---|
| 8 | $p \wedge q$ | 5,6;RU | $\langle \{!^1 p\}, \{!^2 q\} \rangle$ |

Let us now turn to the marking definitions. We use $\checkmark_i$ to denote that a line is $j$-marked for all $j \geq i$. To avoid clutter, we will only represent the marks at one stage; where $k$ is the last line in the example proof, the displayed marks represent marking at stage $k$.

In order to render line 7 marked, we first have to derive the $\mathrm{Dab}_2$-formula $!^2 r$. This is done as follows:

[16] Note that it is also possible to derive $p \wedge q$ from lines 1 and 2, using the rule RC.

| | | | | |
|---|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\langle\emptyset, \emptyset\rangle$ | |
| 2 | $\Diamond\Diamond q$ | PREM | $\langle\emptyset, \emptyset\rangle$ | |
| 3 | $\Diamond\Diamond r$ | PREM | $\langle\emptyset, \emptyset\rangle$ | |
| 4 | $\neg p \vee \neg r$ | PREM | $\langle\emptyset, \emptyset\rangle$ | |
| 5 | $p$ | 1;RC | $\langle\{!^1 p\}, \emptyset\rangle$ | |
| 6 | $q$ | 2;RC | $\langle\emptyset, \{!^2 q\}\rangle$ | |
| 7 | $r$ | 3;RC | $\langle\emptyset, \{!^2 r\}\rangle$ | $\checkmark_2$ |
| 8 | $p \wedge q$ | 5,7;RU | $\langle\{!^1 p\}, \{!^2 q\}\rangle$ | |
| 9 | $!^1 p \check{\vee} !^2 r$ | 1,3,4;RU | $\langle\emptyset, \emptyset\rangle$ | |
| 10 | $!^2 r$ | 9;RC | $\langle\{!^1 p\}, \emptyset\rangle$ | |

Let us discuss the marking procedure at stage 10 step by step. First of all, note that at stage 10, no Dab$_1$-formula has been derived on the condition $\langle\emptyset, \emptyset\rangle$.[17] This means that ${}^{\mathbf{C}}\Sigma^1_{10}(\Gamma_{\mathsf{p1}}) = \emptyset$, whence also ${}^{\mathbf{C}}U^1_{10}(\Gamma_{\mathsf{p1}}) = \emptyset$. As a result, no line is 1-marked at stage 10.

Now consider line 10 and its formula $!^2 r$. This is a Dab$_2$-formula, derived on a condition of the form $\langle\Delta, \emptyset\rangle$. Moreover, line 10 is not 1-marked. As a result, $!^2 r$ is a minimal Dab$_2$-formula at stage 10. This implies that ${}^{\mathbf{C}}\Sigma^2_{10}(\Gamma_{\mathsf{p1}}) = \{\{!^2 r\}\}$, whence ${}^{\mathbf{C}}U^2_{10}(\Gamma_{\mathsf{p1}}) = \{!^2 r\}$. As a result, line 7 is 2-marked at stage 10, as indicated by the symbol $\checkmark_2$.

As a matter of fact, $p$, $q$ and $p \wedge q$ are finally derived in this proof from $\Gamma_{\mathsf{p1}}$. That is, no Dab$_1$-formula is derivable from this premise set, and the only minimal Dab$_2$-formula that can be derived from $\Gamma_{\mathsf{p1}}$ is $!^2 r$. This means that in every extension of the proof, the marking of lines 1-10 remains unchanged.

**$i$-Marking for Minimal Abnormality** The $i$-marking for Minimal Abnormality is slightly more complicated:

**Definition 3.12 ($i$-marking for Minimal Abnormality)** *A line $l$ with formula $A$ and condition $\mathbf{\Delta}$ is $i$-marked at stage $s$ iff (a) $l$ is $(i-1)$-marked at stage $s$, or (b) one of the following conditions hold:*

*(i) there is no $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$ such that $\Delta_i \cap \varphi \neq \emptyset$*

*(ii) for a $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$: there is no line $l'$ that is not $(i-1)$-marked at stage $s$, with formula $A$ and condition $\langle\Theta_1, \ldots, \Theta_i, \Delta_{i+1}, \Delta_{i+2}, \ldots\rangle$, and $\Theta_i \cap \varphi = \emptyset$.*

Requirement (ii) may strike some as surprising. The marking condition has a prospective character since it also takes into account sets of abnormalities in $\mathbf{\Delta}$ that are of higher levels than $i$. Naively it may be expected that requirement (ii) reads as follows:

(ii') for a $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$: there is no line $l'$ that is not $(i-1)$-marked at stage $s$, with formula $A$ and condition $\mathbf{\Theta}$ such that $\Theta_i \cap \varphi = \emptyset$.

Let us interpret Definition 3.12 in terms of an argumentation game. Suppose our proponent derives formula $A$ on the condition $\mathbf{\Delta}$ at stage $s$. The $i$-marking

[17] The formula on line 9 is not a Dab$_1$-formula, since it contains the abnormality $!^2 r$ which is not a member of $\Omega^{\mathbf{K}}_1$.

concerns the question whether the defeasible assumption that corresponds to level $i$ in the superposition is feasible. The minimal choice sets of ${}^{\mathbf{C}}\Sigma_s^i(\Gamma)$ offer minimally abnormal interpretations (in terms of abnormalities in $\Omega_i$) of the premises at the given stage $s$. That is, they offer possible counter-arguments against the defeasible assumption $\mathbf{\Delta}$ of line $l$. However, there is a slight complication involved.

The assumptions used in order to derive $A$ may involve abnormalities of lower and higher levels than $i$. Concerning the lower levels we adopt a bottom-up approach. In case one of the defeasible assumptions at a lower level is not feasible we rely on the marking corresponding to the lower level to retract the line. In this sense the $i$-marking procedure safely ignores the defeasible assumptions belonging to lower levels. However, the $i$-marking is sensitive with respect to the defeasible assumptions that belong to higher levels.

The idea is as follows. According to point (i) there should be at least one minimally abnormal interpretation $\varphi \in {}^{\mathbf{C}}\Phi_s^i(\Gamma)$ in which the $i$th defeasible assumption is valid, i.e., $\Delta_i \cap \varphi = \emptyset$. Moreover, for each counter-argument, i.e., for each $\varphi \in {}^{\mathbf{C}}\Phi_s^i(\Gamma)$ for which $\Delta_i \cap \varphi \neq \emptyset$, our proponent should be able to defend herself in the following way. She should be able to produce an argument such that the $i$th defeasible assumption is valid in $\varphi$ and such that all the higher level defeasible assumptions are the same as in her original argument at line $l$ (see point (ii)).

It is crucial that in her defense, the proponent uses the same higher level defeasible assumptions as in her original argument. Let us demonstrate this by a simple example. As before, we use a **K**-based prioritized logic with only two levels of abnormalities. This time however, we consider the Minimal Abnormality-variant, i.e. $\mathbf{SK2^m}$.

Let $\Gamma_{\mathsf{p2}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \Diamond\Diamond s, \neg p \vee \neg q, \neg p \vee \neg r, \neg q \vee \neg s\}$. Note that the following disjunctions of abnormalities are **K**-derivable from $\Gamma_{\mathsf{p2}}$:

(i) $!^1 p \check{\vee} !^1 q$
(ii) $!^1 p \check{\vee} !^2 r$
(iii) $!^1 q \check{\vee} !^2 s$

However, (ii) and (iii) are neither $\mathrm{Dab}_1$-formulas nor $\mathrm{Dab}_2$-formulas. The following $\mathbf{SK2^m}$-proof shows how we can derive $\mathrm{Dab}_2$-formulas from $\Gamma_{\mathsf{p2}}$:

| | | | |
|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 2 | $\Diamond q$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 3 | $\Diamond\Diamond r$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 4 | $\Diamond\Diamond s$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 5 | $\neg p \vee \neg q$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 6 | $\neg p \vee \neg r$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 7 | $\neg q \vee \neg s$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 8 | $!^1 p \check{\vee} !^1 q$ | 1,2,5;RU | $\langle\emptyset,\emptyset\rangle$ |
| 9 | $!^1 p \check{\vee} !^2 r$ | 1,3,6;RU | $\langle\emptyset,\emptyset\rangle$ |
| 10 | $!^1 q \check{\vee} !^2 s$ | 2,4,7;RU | $\langle\emptyset,\emptyset\rangle$ |
| 11 | $!^2 r \check{\vee} !^2 s$ | 9;RC | $\langle\{!^1 p\},\emptyset\rangle$ |
| 12 | $!^2 r \check{\vee} !^2 s$ | 10;RC | $\langle\{!^1 q\},\emptyset\rangle$ |

Note that ${}^{\mathbf{C}}\Sigma^1_{12}(\Gamma_{\mathsf{p2}}) = \{\{!^1p, !^1q\}\}$, whence ${}^{\mathbf{C}}\Phi^1_{12}(\Gamma_{\mathsf{p2}}) = \{\{!^1p\}, \{!^1q\}\}$. This means that we cannot finally derive $!^2r$ on the condition $\langle\{!^1p\}, \emptyset\rangle$, since we cannot exclude the case where $!^1p$ is the only true abnormality of level 1. For the same reason, we cannot finally derive $!^2s$. However, the disjunction of both level 2-abnormalities is finally derived at stage 12. This follows immediately from the fact that $!^1p \check{\vee} !^1q$ is the only minimal Dab$_1$-consequence of $\Gamma_{\mathsf{p2}}$. Also, it can easily be verified that $!^2r \check{\vee} !^2s$ is the *only* minimal Dab$_2$-consequence of $Cn_{\mathbf{K}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma_{\mathsf{p2}})$.

In view of the preceding, it is easy to see that the sets ${}^{\mathbf{C}}\Sigma^1_s(\Gamma_{\mathsf{p2}})$ and ${}^{\mathbf{C}}\Sigma^2_s(\Gamma_{\mathsf{p2}})$ remain stable from stage 12 on. Put differently, (†) in every further stage $s$ of the proof,

($\dagger^1$) ${}^{\mathbf{C}}\Phi^1_s(\Gamma_{\mathsf{p2}}) = {}^{\mathbf{C}}\Phi^1_{12}(\Gamma_{\mathsf{p2}}) = \{\{!^1p\}, \{!^1q\}\}$
($\dagger^2$) ${}^{\mathbf{C}}\Phi^2_s(\Gamma_{\mathsf{p2}}) = {}^{\mathbf{C}}\Phi^2_{12}(\Gamma_{\mathsf{p2}}) = \{\{!^2r\}, \{!^2s\}\}$

Let us now return to the prospective character of clause (ii) in Definition 3.12. Consider the following extension, in which the (arbitrarily chosen) formula $t$ is derived:

| | | | | |
|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| 9 | $!^1p \check{\vee} !^2r$ | 1,3,6;RU | $\langle\emptyset, \emptyset\rangle$ | |
| 10 | $!^1q \check{\vee} !^2s$ | 2,4,7;RU | $\langle\emptyset, \emptyset\rangle$ | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| 13 | $t \check{\vee} !^1p \check{\vee} !^2r$ | 9;RU | $\langle\emptyset, \emptyset\rangle$ | |
| 14 | $t$ | 13;RC | $\langle\{!^1p\}, \{!^2r\}\rangle$ | $\checkmark_1$ |
| 15 | $t \check{\vee} !^1q \check{\vee} !^2s$ | 10;RU | $\langle\emptyset, \emptyset\rangle$ | |
| 16 | $t$ | 15;RC | $\langle\{!^1q\}, \{!^2s\}\rangle$ | $\checkmark_1$ |

Since we obtained lines 13 and 15 by the rule of addition, we can make a similar move with any formula $A \in \check{\mathcal{W}}_s$ instead of $t$. Let $\boldsymbol{\Theta}$ be the condition of line 14. In view of lines 14 and 16 and (†), the following facts hold:

(i.$t_1$) there is a $\varphi \in {}^{\mathbf{S}}\Phi^1_{16}(\Gamma_{\mathsf{p2}})$ such that $\Theta_1 \cap \varphi = \emptyset$ (viz. $\psi_1 = \{!^1q\}$)
(ii.$t_1$)' for every $\varphi \in {}^{\mathbf{S}}\Phi^1_{16}(\Gamma_{\mathsf{p2}})$, $A$ is derived on a condition $\boldsymbol{\Theta}'$ such that $\Theta'_1 \cap \varphi = \emptyset$ at stage 16
(i.$t_2$) there is a $\varphi \in {}^{\mathbf{S}}\Phi^2_{16}(\Gamma_{\mathsf{p2}})$ such that $\Theta_2 \cap \varphi = \emptyset$ (viz. $\psi_2 = \{!^2s\}$)
(ii.$t_2$)' for every $\varphi \in {}^{\mathbf{S}}\Phi^2_{16}(\Gamma_{\mathsf{p2}})$, $A$ is derived on a condition $\boldsymbol{\Theta}'$ such that $\Theta'_2 \cap \varphi = \emptyset$ at stage 16

In other words, replacing clause (ii) with (ii)' in the definition of $i$-marking for Minimal Abnormality, would imply that lines 14 and 16 are not marked at stage 16 of the proof. Moreover, in view of (†), these lines would not be marked in any further extension of the proof.

This is where the prospective character of Definition 3.12 comes in play. That is, it is not the case that for every $\varphi \in {}^{\mathbf{S}}\Phi^1_{16}(\Gamma_{\mathsf{p2}})$, $t$ is derived on a condition $\langle\Delta, \{!^2r\}\rangle$ such that $\Delta \cap \varphi = \emptyset$ – this requirement fails for $\{!^1p\}$, which is a minimal choice set of level 1. Similarly, $t$ is not derived on a condition $\langle\Delta, \{!^2s\}\rangle$ such that $\Delta \cap \{!^1q\} = \emptyset$. As a result, lines 14 and 16 are 1-marked at stage

16. Moreover, there is no way to extend this proof such that these lines are not 1-marked.

Recall the remark at the beginning of this section that the proof theories proposed in [11] and [137] are not sound with respect to $\mathbf{SAL^m}$, even in very simple (finite) cases and under the assumption that for every $i, j \in I$ such that $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$. The above example is one of those cases. What was lacking in those earlier proposals, is precisely the prospective character of marking for Minimal Abnormality.

The following continuation of the proof shows how the formula $(p \wedge s) \vee (q \wedge r)$ can be finally $\mathbf{SK2^m}$-derived from $\Gamma_{\mathsf{p2}}$. In this case, requirement (ii) of Definition 3.12 is fulfilled for both $i = 1$ and $i = 2$, whence lines 19-22 are neither 1-marked nor 2-marked.

| | | | | |
|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| 17 | $p \wedge s$ | 1,4;RC | $\langle\{!^1p\}, \{!^2s\}\rangle$ | $\checkmark_1$ |
| 18 | $q \wedge r$ | 2,3;RC | $\langle\{!^1q\}, \{!^2r\}\rangle$ | $\checkmark_1$ |
| 19 | $(p \wedge s) \vee (q \wedge r)$ | 17;RU | $\langle\{!^1p\}, \{!^2s\}\rangle$ | |
| 20 | $(p \wedge s) \vee (q \wedge r)$ | 18;RU | $\langle\{!^1q\}, \{!^2r\}\rangle$ | |
| 21 | $(p \wedge s) \vee (q \wedge r)$ | 9;RU | $\langle\{!^1p\}, \{!^2r\}\rangle$ | |
| 22 | $(p \wedge s) \vee (q \wedge r)$ | 10;RU | $\langle\{!^1q\}, \{!^2s\}\rangle$ | |

Of course, to finish this second proposal for a generic proof theory, we have to define final derivability. As before, this is given by Definitions 3.8 and 3.9 (see page 59). In the remainder, we use $\vdash_{\mathbf{SAL}'}$ to denote the resulting derivability relation.

**A Special Case** To end this section, we consider the special case in which for all $i, j \in I$ for which $i \neq j$ we have $\Omega_i \neq \Omega_j$.[18] In this case the logical form of an abnormality $A$ unambiguously determines an $i \in I$ such that $A \in \Omega_i$. This means in turn that we do not need to represent the condition of lines in the proof in terms of sequences of sets of abnormalities but can instead just represent them by means of sets of abnormalities in $\bigcup_{i \in I} \Omega_i$.

We can then adjust the $i$-marking in the following way. Where $\Delta \subset \Omega_i$, we say that $Dab(\Delta)$ is a Dab$_i$-formula at stage $s$ iff (i) $Dab(\Delta)$ is derived at a line $l$ with condition $\Theta \subset \Omega_1 \cup \ldots \cup \Omega_{i-1}$ and (ii) line $l$ is not $(i-1)$-marked at stage $s$. $Dab(\Delta)$ is a minimal Dab$_i$-formula at stage $s$ iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is a Dab$_i$-formula at stage $s$. Then we adjust the marking definitions as follows:

**Definition 3.13 ($i$-marking for Reliability, special case)** *A line $l$ with condition $\Delta$ is $i$-marked at stage $s$ iff (a) it is $(i-1)$-marked, or (b) $\Delta \cap {}^{\mathbf{C}}U^i_s(\Gamma) \neq \emptyset$.*

**Definition 3.14 ($i$-marking for Minimal Abnormality, special case)** *A line $l$ with formula $A$ and condition $\Delta$ is $i$-marked at stage $s$ iff (a) it is $(i-1)$-marked, or (b) one of the following conditions hold:*

[18] Most superpositions in the literature fall within this class. The only exceptions I know of can be found in [24]. However, as we will see in Section 3.6, there are several meta-theoretic reasons to prefer other superpositions, e.g. those in which $\Omega_i \subset \Omega_{i+1}$ for every $i, i+1 \in I$.

*(i)* *there is no* $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$ *such that* $\Delta \cap \varphi \neq \emptyset$, *or*
*(ii)* *there is a* $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$ *such that there is no line* $l'$ *that is not* $(i-1)$-*unmarked at stage* $s$, *with formula* $A$ *and condition* $\Theta$ *such that* $\Theta \cap \varphi = \emptyset$, *and* $\Theta \cap (\Omega_{i+1} \cup \Omega_{i+2} \cup \ldots) = \Delta \cap (\Omega_{i+1} \cup \Omega_{i+2} \cup \ldots)$.

Note that even in this special case, we cannot do without the prospective character of the marking definition for Minimal Abnormality – this follows immediately from the example $\Gamma_{\mathsf{p2}}$ which we discussed above.

Let us now briefly show by an example why we need to represent conditions as sequences in the more general case. Consider the superposition-logic **SKP**, which defined as follows:

$$Cn_{\mathbf{SKP}}(\Gamma) = Cn_{\mathbf{K^r_1}}(Cn_{\mathbf{K^r_2}}(Cn_{\mathbf{K^r_1}}(\Gamma)))$$

Note that in this specific superposition, $\Omega_1 = \Omega_3 = \Omega^{\mathbf{K}}_1$. Let $\Gamma_{\mathsf{p3}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg p \vee \neg r\}$. Note that the following are minimal Dab-consequences of $\Gamma_{\mathsf{p3}}$:

(i) $!^1 p \check{\vee} !^1 q$
(ii) $!^1 p \check{\vee} !^2 r$

In view of (i), both $!^1 p$ and $!^1 q$ are unreliable for the first logic in the superposition. This means that we cannot finally derive $!^2 r$ on the condition $\{!^1 p\}$ in a $\mathbf{K^r_1}$-proof from $\Gamma_{\mathsf{p3}}$. More generally, $!^2 r \notin Cn_{\mathbf{K^r_1}}(\Gamma_{\mathsf{p3}})$. Hence this is a reliable abnormality in view of the second logic in the superposition. Since also $!^1 p \check{\vee} !^2 r \in Cn_{\mathbf{K^r_1}}(\Gamma)$, it follows that we can derive $!^1 p$ on the condition $!^2 r$ in a $\mathbf{K^r_2}$-proof from $Cn_{\mathbf{K^r_1}}(\Gamma_{\mathsf{p3}})$. But then $!^1 p \check{\vee} !^1 q$ is no longer a minimal Dab-formula for the *third* logic in the superposition, whence $q$ is finally $\mathbf{K^r_1}$-derivable from $\Diamond q$ on the condition $\{!^1 q\}$, and hence $q \in Cn_{\mathbf{SKP}}(\Gamma_{\mathsf{p3}})$.

The following proof illustrates the fact that $q$ is not $\mathbf{K^r_1}$-derivable from $\Gamma_{\mathsf{p3}}$, but only from $Cn_{\mathbf{K^r_2}}(Cn_{\mathbf{K^r_1}}((\Gamma_{\mathsf{p3}}))$, whence it is **SKP**-derivable from $\Gamma_{\mathsf{p3}}$:

| | | | | |
|---|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\langle\emptyset,\emptyset,\emptyset\rangle$ | |
| 2 | $\Diamond q$ | PREM | $\langle\emptyset,\emptyset,\emptyset\rangle$ | |
| 3 | $\Diamond\Diamond r$ | PREM | $\langle\emptyset,\emptyset,\emptyset\rangle$ | |
| 4 | $\neg p \vee \neg q$ | PREM | $\langle\emptyset,\emptyset,\emptyset\rangle$ | |
| 5 | $\neg p \vee \neg r$ | PREM | $\langle\emptyset,\emptyset,\emptyset\rangle$ | |
| 6 | $!^1 p \check{\vee} !^1 q$ | 1,2,4;RU | $\langle\emptyset,\emptyset,\emptyset\rangle$ | |
| 7 | $!^1 p \check{\vee} !^2 r$ | 1,3,5;RU | $\langle\emptyset,\emptyset,\emptyset\rangle$ | |
| 8 | $!^1 p$ | 7;RC | $\langle\emptyset,\{!^2 r\},\emptyset\rangle$ | |
| 9 | $q$ | 2;RC | $\langle\{!^1 q\},\emptyset,\emptyset\rangle$ | $\checkmark_1$ |
| 10 | $r$ | 3;RC | $\langle\emptyset,\{!^2 r\},\emptyset\rangle$ | |
| 11 | $q$ | 2;RC | $\langle\emptyset,\emptyset,\{!^1 q\}\rangle$ | |

Note that ${}^{\mathbf{C}}U^1_{11}(\Gamma_{\mathsf{p3}}) = \{!^1 p, !^1 q\}$. This explains why line 9 is 1-marked: the first member of its condition contains the abnormality $\{!^1 q\}$, which is unreliable at level 1. Since ${}^{\mathbf{C}}\Sigma^2_{11}(\Gamma_{\mathsf{p3}}) = \emptyset$, lines 8 and 10 are not 1- or 2-marked. But this means that $!^1 p$, the formula derived on line 8, is a $\mathrm{Dab}_3$-formula at stage 11 of the proof. Hence, $!^1 p \check{\vee} !^1 q$ is no longer a minimal $\mathrm{Dab}_3$-formula at stage

11, whence ${}^{\mathbf{C}}U^3_{11}(\Gamma_{\mathsf{p3}}) = \{!^1 p\}$. The last crucial move takes place at line 11. Here, $q$ is derived, but this time by pushing $!^1 q$ to the *third* set in the condition – note that this is perfectly in line with the generic rule RC, which leaves room for choice in this case. Since $\{!^1 q\} \cap {}^{\mathbf{C}}U^3_{11}(\Gamma_{\mathsf{p3}}) = \emptyset$, line 11 is unmarked and will remain so in every further extension of the proof.

### 3.5.2 Metatheory for the Second Proposal

First of all, note that the following facts hold:

**Fact 3.7** $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\mathbf{\Delta})$ *iff there is an* $\mathbf{SAL}'$*-proof from* $\Gamma$ *in which* $A$ *is derived on the condition* $\mathbf{\Delta}$*.*

**Fact 3.8** *If a line* $l$ *with condition* $\langle \Delta_1, \ldots, \Delta_i, \emptyset, \ldots \rangle$ *is not* $i$*-marked, then it is not* $j$*-marked for any* $j > i$*.*

**Adequacy of the infinite stage** $\mathsf{d}$ As was the case in the metatheory of the first proposal, we will refer to the stage at which everything is derived that can be derived from $\Gamma$, on every possible condition. Let us call this stage $\mathsf{d}$. By the same reasoning as for the stage $\mathsf{g}$, it can easily be inferred that the stage $\mathsf{d}$ exists and that every proof can be extended up to this stage. Note that the following holds:

(†) If there is an $\mathbf{SAL}'$-proof in which $A$ is derived on the condition $\langle \Delta_i \rangle_{i \in I}$, then $A$ is derived on the condition $\langle \Delta_i \rangle_{i \in I}$ at stage $\mathsf{d}$

**Fact 3.9** ${}^{\mathbf{C}}\Sigma^i_{\mathsf{d}}(\Gamma) = \Sigma^i(Cn_{\mathbf{LLL}}(\Gamma))$

**Lemma 3.14** *Where* $\Gamma \subseteq \mathcal{W}_s$*, each of the following holds for every* $i \in I$*:*

*1a.* ${}^{\mathbf{C}}\Sigma^i_{\mathsf{d}}(\Gamma) = \Sigma^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$*, whence also*
*1b.* ${}^{\mathbf{C}}U^i_{\mathsf{d}}(\Gamma) = U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$ *and*
*1c.* ${}^{\mathbf{C}}\Phi^i_{\mathsf{d}}(\Gamma) = \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$
*2.* *there is a line* $l$ *with formula* $A$ *and condition* $\langle \Delta_j \rangle_{j \in I}$ *that is not* $i$*-marked at stage* $\mathsf{d}$ *iff* $A \check{\vee} Dab(\Delta_{i+1} \cup \Delta_{i+2} \cup \ldots) \in Cn_{\mathbf{SAL_i}}(\Gamma)$*.*

*Proof.* $(i = 1)$ *Ad 1.* Immediate in view of Fact 3.9.

*Ad 2.* Immediate in view of item 1 and the construction of $\mathsf{d}$.

$(i \Rightarrow i+1)$ *Ad 1.* Where $\Delta \subset \Omega_{i+1}$, the following are equivalent in view of (1) the definition of ${}^{\mathbf{C}}\Sigma^i_{\mathsf{d}}(\Gamma)$, (2) item 2 of the induction hypothesis, (3) the fact that $Cn_{\mathbf{SAL_i}}(\Gamma)$ is $\mathbf{LLL}$-closed and (4) the definition of $\Sigma^i(\Gamma)$:

- $\Delta \in {}^{\mathbf{C}}\Sigma^{i+1}_{\mathsf{d}}(\Gamma)$
- $Dab(\Delta)$ is derived on an $i$-unmarked line with condition $\langle \Theta_1, \ldots, \Theta_i, \emptyset, \emptyset, \ldots \rangle$ at stage $\mathsf{d}$, and for no $\Delta' \subset \Delta$: $Dab(\Delta')$ is derived on an $i$-unmarked line with condition $\langle \Theta'_1, \ldots, \Theta'_i, \emptyset, \emptyset, \ldots \rangle$ at stage $\mathsf{d}$
- $Dab(\Delta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$, and for no $\Delta' \subset \Delta$: $Dab(\Delta') \in Cn_{\mathbf{SAL_i}}(\Gamma)$
- $Dab(\Delta) \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$, and there is no $\Delta' \subset \Delta$, such that $Dab(\Delta') \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$

- $\Delta \in \Sigma^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$.

*Ad 2.* $\mathbf{x_{i+1}} = \mathbf{r}$. At stage $\mathsf{d}$, each of the following are equivalent in view of (1) Definition 3.11, (2) item 1b, (3) item 2 of the induction hypothesis and (4) Theorem 3.7.1 and the fact that $Cn_{\mathbf{SAL_i}}(\Gamma)$ is **LLL**-closed:

- there is a line $l$ with formula $A$ and condition $\langle\Delta_j\rangle_{j\in I}$ that is not $(i+1)$-marked
- there is a line $l$ with formula $A$ and condition $\langle\Delta_j\rangle_{j\in I}$ that is not $i$-marked, and $\Delta_{i+1} \cap {}^{\mathbf{C}}U^{i+1}_{\mathsf{d}}(\Gamma) = \emptyset$
- there is a line $l$ with formula $A$ and condition $\langle\Delta_j\rangle_{j\in I}$ that is not $i$-marked, and $\Delta_{i+1} \cap U^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \emptyset$
- There are $\Delta_{i+1} \subset \Omega_{i+1}, \Delta_{i+2} \subset \Omega_{i+2}, \ldots$, such that $A \check{\vee} Dab(\Delta_{i+1} \cup \Delta_{i+2} \cup \ldots) \in Cn_{\mathbf{SAL_i}}(\Gamma)$ and $\Delta_{i+1} \cap U^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \emptyset$
- There are $\Delta_{i+2} \subset \Omega_{i+2}, \Delta_{i+3} \subset \Omega_{i+3}, \ldots$, such that $A \check{\vee} Dab(\Delta_{i+2} \cup \Delta_{i+3} \cup \ldots) \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$

$\mathbf{x_{i+1}} = \mathbf{m}$. At stage $\mathsf{d}$, each of the following are equivalent in view of (1) Definition 3.12, (2) item 1c, (3) item 2 of the induction hypothesis and (4) Theorem 3.7.2 and the fact that $Cn_{\mathbf{SAL_i}}(\Gamma)$ is **LLL**-closed:

- there is a line $l$ with formula $A$ and condition $\langle\Delta_j\rangle_{j\in I}$ that is not $(i+1)$-marked
- there is a line $l$ with formula $A$ and condition $\langle\Delta_j\rangle_{j\in I}$ such that
  - (a) $l$ is not $i$-marked,
  - (b) $\Delta_{i+1} \cap \varphi = \emptyset$ for a $\varphi \in {}^{\mathbf{C}}\Phi^{i+1}_{\mathsf{d}}(\Gamma)$, and
  - (c) for every $\varphi \in {}^{\mathbf{C}}\Phi^{i+1}_{\mathsf{d}}(\Gamma)$: $A$ is derived on a line $l_\varphi$ with condition $\langle\Theta_1, \ldots, \Theta_{i+1}, \Delta_{i+2}, \Delta_{i+3}, \ldots\rangle$ such that $\Theta_{i+1} \cap \varphi = \emptyset$, and each line $l_\varphi$ is not $i$-marked
- there is a line $l$ with formula $A$ and condition $\langle\Delta_i\rangle_{i\in I}$ such that
  - (a) $l$ is not $i$-marked,
  - (b) $\Delta_{i+1} \cap \varphi = \emptyset$ for a $\varphi \in \Phi^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$, and
  - (c) for every $\varphi \in \Phi^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$: $A$ is derived on a line $l_\varphi$ with condition $\langle\Theta_1, \ldots, \Theta_{i+1}, \Delta_{i+2}, \Delta_{i+3}, \ldots\rangle$ such that $\Theta_{i+1} \cap \varphi = \emptyset$, and each line $l_\varphi$ is not $i$-marked
- There are $\Delta_{i+2} \subset \Omega_{i+2}, \Delta_{i+3} \subset \Omega_{i+3}, \ldots$, such that for every $\varphi \in \Phi^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$, $A \check{\vee} Dab(\Theta_{i+1} \cup \Delta_{i+2} \cup \Delta_{i+3} \cup \ldots) \in Cn_{\mathbf{SAL_i}}(\Gamma)$ for a $\Theta_{i+1} \subseteq \Omega_{i+1} - \varphi$
- $A \check{\vee} Dab(\Delta_{i+2} \cup \Delta_{i+3} \cup \ldots) \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$

■

**Lemma 3.15** *$A \in Cn_{\mathbf{SAL}}(\Gamma)$ iff $A$ is derived on an unmarked line at stage* $\mathsf{d}$.

*Proof.* ($\Rightarrow$) Suppose $A \in Cn_{\mathbf{SAL}}(\Gamma)$, whence there is an $i \in I$: $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$. By Lemma 3.14, $A$ is derived on a line $l$ at stage $\mathsf{d}$, with condition $\boldsymbol{\Delta} = \langle\Delta_1, \ldots, \Delta_i, \emptyset, \emptyset, \ldots\rangle$, and line $l$ is not $i$-marked at stage $\mathsf{d}$. It follows that there is no $j \leq i$ such that line $l$ is $j$-marked at stage $\mathsf{d}$. Also, by Fact 3.8, line $l$ is not $k$-marked for any $k > i$. As a result, line $l$ is not marked at stage $\mathsf{d}$.

($\Leftarrow$) Suppose $A$ is derived on an unmarked line at stage $\mathsf{d}$. Let $\langle \Delta_1, \ldots, \Delta_i, \emptyset, \emptyset, \ldots \rangle$ be the condition of line $l$. Then since line $l$ is not $i$-marked at stage $\mathsf{d}$, we can derive by item 2 from Lemma 3.14 that $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$, whence $A \in Cn_{\mathbf{SAL}}(\Gamma)$. ■

**Soundness and Completeness**

**Theorem 3.10** *Where $\Gamma \subseteq \mathcal{W}_s$: if $\Gamma \vdash_{\mathbf{SAL}'} A$, then $A \in Cn_{\mathbf{SAL}}(\Gamma)$.*

*Proof.* Suppose $\Gamma \vdash_{\mathbf{SAL}'} A$. By Definitions 3.8 and 3.9, $A$ is derived on an unmarked line $l$ of a finite $\mathbf{SAL}'$-proof $\mathrm{P}$ from $\Gamma$. Suppose we extend $\mathrm{P}$ up to stage $\mathsf{d}$. If $l$ is marked in this extension, then $l$ is marked in every further extension of the proof, which contradicts the fact that $A$ is finally derived on line $l$. Hence line $l$ is unmarked at stage $\mathsf{d}$. By Lemma 3.15, $A \in Cn_{\mathbf{SAL}}(\Gamma)$. ■

**Theorem 3.11** *Where $\Gamma \subseteq \mathcal{W}_s$: if $A \in Cn_{\mathbf{SAL}}(\Gamma)$, then $\Gamma \vdash_{\mathbf{SAL}'} A$.*

*Proof.* Suppose $A \in Cn_{\mathbf{SAL}}(\Gamma)$. By Lemma 3.15, (†) $A$ is derived on an unmarked line $l$ with condition $\langle \Delta_i \rangle_{i \in I}$ at stage $\mathsf{d}$. In view of Lemma 3.15 and the marking definitions, we can infer that for all $i \in I$:

($\dagger^{\mathbf{r}}$) where $\mathbf{x_i} = \mathbf{r}$: $\Delta_i \cap U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma)) \neq \emptyset$
($\dagger^{\mathbf{m}}$) where $\mathbf{x_i} = \mathbf{m}$: $\Delta_i \cap \varphi \neq \emptyset$, for a $\varphi \in \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$

By Fact 3.7, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\bigcup_{i \in I} \Delta_i)$, whence by the compactness of $\mathbf{LLL}$, there is a $\Gamma' = \{B_1, \ldots, B_m\} \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\bigcup_{i \in I} \Delta_i)$.

Let the $\mathbf{SAL}'$-proof $\mathrm{P}$ be constructed as follows. At line 1 we introduce the premise $B_1$ by PREM, ..., and at line $m$ we introduce the premise $B_m$ by PREM. At line $m+1$ we derive $A$ by RC, on the condition $\langle \Delta_i \rangle_{i \in I}$. Let $s$ be the stage consisting of lines 1 up to $m+1$.

Since $\Gamma' \subseteq \Gamma \subseteq \mathcal{W}_s$, for every $i \in I$, all Dab$_i$-formulas that are derived at stage $s$ (if any) are singletons $C \in \Omega_i$. Moreover, by the reflexivity of each logic $\mathbf{SAL_i}$, for every such $C$, $C \in Cn_{\mathbf{SAL_{i-1}}}(\Gamma)$, whence also $C \in U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$ and $C \in \varphi$ for every $\varphi \in \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$. Hence for every $i \in I$:

($\ddagger^{\mathbf{r}}$) ${}^{\mathbf{C}}U^i_s(\Gamma) \subseteq U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$
($\ddagger^{\mathbf{m}}$) $\bigcup {}^{\mathbf{C}}\Phi^i_s(\Gamma) \subseteq \varphi$ for every $\varphi \in \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$

By ($\dagger^{\mathbf{r}}$), ($\dagger^{\mathbf{m}}$), ($\ddagger^{\mathbf{r}}$) and ($\ddagger^{\mathbf{m}}$), we can infer that there is no $i \in I$ such that line $m+1$ is $i$-marked at stage $s$.

Suppose that line $m+1$ is marked in an extension of the proof. In that case, we may further extend the proof up to stage $\mathsf{d}$, whence in view of (†), line $m+1$ is again unmarked in the second extension. By Definitions 3.8 and 3.9, $A$ is finally derived at stage $s$. ■

## 3.6 Some More Metatheoretic Properties

To end this chapter, I will discuss some additional meta-theoretic properties of **SAL**, each of which shall also be considered for the two alternative formats of prioritized ALs in Chapters 4 and 5. In Section 3.6.1, a specific property of the **SAL**-semantics is discussed. In the next two sections, I consider the syntactic consequence relation, as given by Definition 3.1.

### 3.6.1 Strong Reassurance

In Section 3.3.4, we saw that logics in $\mathbf{SAL^m}$-format do not in general have the Semantic Reassurance property, but that this property holds whenever $\Phi(\Gamma)$ is finite. So how about *Strong* Reassurance? In fact, this is a rather problematic notion in the context of prioritized consequence relations. That is, there are several different ways to justify the fact that a given model $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ is not selected in the semantics of a given prioritized logic **PAL**, which give rise to distinct variants of the Strong Reassurance property:

(SR1) If $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{PAL}}(\Gamma)$, then there is an $M' \in \mathcal{M}_{\mathbf{PAL}}(\Gamma)$ such that $Ab(M') \subset Ab(M)$.

(SR2) If $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{PAL}}(\Gamma)$, then there is an $M' \in \mathcal{M}_{\mathbf{PAL}}(\Gamma)$ such that $Ab(M') \cap \Omega_i \subset Ab(M) \cap \Omega_i$ for an $i \in I$.

(SR3) If $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{PAL}}(\Gamma)$, then there is an $M' \in \mathcal{M}_{\mathbf{PAL}}(\Gamma)$ such that $Ab(M') \cap \Omega_{(i)} \subset Ab(M) \cap \Omega_{(i)}$, for an $i \in I$.

I will discuss each of these properties for the logics $\mathbf{SK2^r}$ and $\mathbf{SK2^m}$ introduced in Section 3.1.2 – my observations can easily be generalized to a very large class of logics **SAL**.

First of all, SR1 is far too strong, since it does not take into account the priorities that are involved in the selection of models. Consider $\Gamma_{\mathsf{sr1}} = \{!^1 p \vee !^2 q\}$. Note that there is an $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{sr1}})$ such that $Ab(M) = \{!^1 p\}$. However, this model is disselected after the first round of selections by $\mathbf{SK2^r}$ and $\mathbf{SK2^m}$, in view of the models $M' \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{sr1}})$ for which $Ab(M') = \{!^2 q\}$. This is as it should be – otherwise one can hardly speak of a logic that takes priorities into account. However, for those models $M$ and $M'$, it is not the case that $Ab(M') \subset Ab(M)$. In other words, the Strong Reassurance property which is usually considered in the metatheory of flat ALs, is of little use in the prioritized context.

So let us turn to SR2. After giving it some thought, the consequent of SR2 can hardly be called a justification for the disselection of $M$. What if there is a $j < i$ such that $Ab(M) \cap \Omega_j \subset Ab(M') \cap \Omega_j$? Then clearly $M$ should be selected and not $M'$. We could therefore propose the following strengthening of SR2:

(SR2+) If $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{PAL}}(\Gamma)$, then there is an $M' \in \mathcal{M}_{\mathbf{PAL}}(\Gamma)$ such that (1) $Ab(M') \cap \Omega_i \subset Ab(M) \cap \Omega_i$ for an $i \in I$, and (2) for no $j < i$: $Ab(M) \cap \Omega_j \subset Ab(M) \cap \Omega_j$.

I first consider the selection of models for the logic $\mathbf{SK2^m}$. Let

$$\Gamma_{\mathsf{sr2}} = \{!^1 p \vee !^1 q \vee !^1 r, !^1 p \vee !^1 r \vee !^2 s, !^1 q \vee !^1 r \vee !^2 s\}$$

Note that $!^1 p \check{\vee} !^1 q \check{\vee} !^1 r$ is the only minimal $\mathrm{Dab}_1$-consequence of $\Gamma_{\mathsf{sr2}}$. Let $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{sr2}})$ be such that $Ab(M) = \{!^1 p, !^1 q\}$. Note that $M$ is not minimal abnormal with respect to $\Omega_1^{\mathbf{K}}$, whence also $M \notin \mathcal{M}_{\mathbf{SK2^m}}(\Gamma_{\mathsf{sr2}})$. That is, there are $M' \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{sr2}})$ such that $Ab(M') \cap \Omega_1^{\mathbf{K}} = \{!^1p\} \subset Ab(M) \cap \Omega_1^{\mathbf{K}}$.

However, consider such an $M'$. In view of $\Gamma_{\mathsf{sr2}}$, $M' \Vdash !^2s$. This implies that $M'$ is not minimally abnormal in $\mathcal{M}_{\mathbf{K_1^m}}(\Gamma_{\mathsf{sr2}})$ with respect to $\Omega_2^{\mathbf{K}}$: there is an $M'' \in \mathcal{M}_{\mathbf{K_1^m}}(\Gamma_{\mathsf{sr2}})$ such that $Ab(M'') = \{!^1r\}$, whence $Ab(M'') \cap \Omega_2^{\mathbf{K}} = \emptyset \subset Ab(M') \cap \Omega_2^{\mathbf{K}}$. So only these models $M''$ are selected after the second round. Note however that each of the following holds:

$$Ab(M'') \cap \Omega_1^{\mathbf{K}} \not\subset Ab(M) \cap \Omega_1^{\mathbf{K}}$$
$$Ab(M'') \cap \Omega_2^{\mathbf{K}} \not\subset Ab(M) \cap \Omega_2^{\mathbf{K}}$$

Hence, the models $M''$ cannot justify the disselection of models $M$, along the lines of SR2+.

For $\mathbf{SK2^r}$, there is an even simpler example that illustrates the failure of SR2+. Let $\Gamma_{\mathsf{sr3}} = \{!^1p \vee !^1q, !^1p \vee !^2r, !^1q \vee !^2r\}$. Note that both $!^1p$ and $!^1q$ are unreliable for the logic $\mathbf{K_1^r}$. It follows that $!^2r$ is not a minimal $\mathrm{Dab}_2$-consequence of $Cn_{\mathbf{K_1^r}}(\Gamma_{\mathsf{sr3}})$, and more generally, that $!^2r \notin U^2(Cn_{\mathbf{K_1^r}}(\Gamma))$. Let $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{sr3}})$ be such that $Ab(M) = \{!^1p, !^2r\}$. Since $M$ verifies an abnormality that is reliable for $\mathbf{SK2^r}$, viz. $!^2r$, it is not an $\mathbf{SK2^r}$-model of $\Gamma_{\mathsf{sr3}}$. The only $\mathbf{SK2^r}$-models of $\Gamma_{\mathsf{sr3}}$ are those $M'$ for which $Ab(M') = \{!^1p, !^1q\}$. However, these models $M'$ cannot justify the disselection of the model $M$ – in fact, they are *more* abnormal than $M$ with regards to abnormalities of the first level. Hence $\mathbf{SK2^r}$ violates SR2+.

The third Strong Reassurance-variant, SR3, is still stronger than SR2+. To see why, suppose that SR3 holds, and let $j \leq i$ be the smallest $j \in I$ such that $Ab(M') \cap \Omega_{(j)} \subset Ab(M) \cap \Omega_{(j)}$. It follows that (1') $Ab(M') \cap \Omega_j \subset Ab(M) \cap \Omega_j$, and (2') for all $k < j$, $Ab(M') \cap \Omega_k = Ab(M) \cap \Omega_k$. Hence, since $\mathbf{SK2^r}$ and $\mathbf{SK2^m}$ do not satisfy SR2+, they also cannot satisfy SR3.

All this should not come as a surprise: since the **SAL**-semantics is by definition sequential, the disselection of a model $M$ at a given step in the procedure can only be justified in terms of the selection of another model $M'$ at this step. At a later point in the procedure, $M'$ can itself be discarded in view of yet another model $M''$, where $M$ and $M''$ are incomparable in terms of their abnormal parts.

Whether or not these facts should be seen as a real disadvantage, depends on the application context of the logic. For example, in a deontic context, one may not want to exclude any scenario, unless there is an alternative scenario that clearly outbeats the former and is kept as one of the options to guide our actions. In inductive generalization, one might just want to obtain the strongest possible hypotheses, and thereby exclude as many alternative models as possible.

In the two subsequent chapters, I will return to the above criteria, and show that the two other formats for prioritized ALs from this thesis both satisfy SR3. In Chapter 6, it is shown that all superpositions of the form $\mathbf{SAL^m_{(I)}}$ also satisfy this specific kind of Strong Reassurance.

### 3.6.2 Cumulative Indifference

Recall that Cumulative Indifference is equivalent to the conjunction of the following two properties:

*Cumulative Transitivity*: Where $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$, $Cn_{\mathbf{L}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{L}}(\Gamma)$

*Cautious Monotonicity*: Where $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$, $Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$

In the remainder, I will show that both properties fail for $\mathbf{L} = \mathbf{SAL}$ (Sections 3.6.2 and 3.6.2). I will restrict myself to logics in $\mathbf{SAL^x}$-format — one can easily translate the counterexamples to superpositions of flat ALs with mixed strategies. The examples are spelled out in a rather loose way – the proofs for the claims I make here proceed in a similar fashion as those in Section 3.3.3 and are equally simple.

On the positive part, I will prove that the more restricted format $\mathbf{SAL^r_{(I)}}$ warrants Cumulative Indifference (Section 3.6.2). As a corollary, it follows that $Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$ is a fixed point. In Chapter 6, a similar result is obtained for $\mathbf{SAL^m_{(I)}}$, be it for a restricted class of premise sets.

**Fixed Point and Cumulative Transitivity** The counterexample for $\mathbf{SAL^r}$ is a rather straightforward one. I will present it for the concrete logic $\mathbf{SK2^r}$ – a more abstract version of it is mentioned in [25].

Let $\Gamma_{\mathsf{ctr}} = \{!^1p \check{\vee} !^1q, !^1p \check{\vee} !^2r, s \check{\vee} !^1q\}$. Note that since $!^1q \in U^1(\Gamma_{\mathsf{ctr}})$, we cannot finally $\mathbf{K^r_1}$-derive $s$ from $\Gamma_{\mathsf{ctr}}$, whence also $s \notin Cn_{\mathbf{SK2^r}}(\Gamma_{\mathsf{ctr}})$. However, we can derive $!^1p$ from $\Gamma_{\mathsf{ctr}}$, on the condition $!^2r$.

Now consider $\Gamma_{\mathsf{ctr}} \cup \{!^1p\}$. Note that $!^1p \,\check{\vee}\, !^1q$ is no longer a minimal $\mathrm{Dab}_1$-consequence of the extended premise set, whence $!^1q$ becomes a reliable abnormality at priority level 1. It follows that $s \in Cn_{\mathbf{K^r_1}}(\Gamma_{\mathsf{ctr}})$, whence also $s \in Cn_{\mathbf{SK2^r}}(\Gamma_{\mathsf{ctr}})$.

The above example does not work for $\mathbf{SK2^m}$. That is, $\check{\neg}!^1p \,\check{\vee}\, \check{\neg}!^1q$ is a $\mathbf{K^m_1}$-consequence of $\Gamma_{\mathsf{ctr}}$. Moreover, by the second logic of the superposition, we can derive $!^1p$ (just as was the case for the Reliability-variant). As a result, $\check{\neg}!^1q \in Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{ctr}})$, and by the reflexivity and $\mathbf{LLL}$-closure of $\mathbf{SK2^m}$, also $s \in Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{ctr}})$.

So we need a more complex example to show that $\mathbf{SAL^m}$-logics are not cumulatively transitive. I have already introduced this example in Section 3.3.3: $\Gamma_{\mathsf{c}} = \{!^1p_i \check{\vee} !^1p_j \mid i,j \in \mathbb{N}, i \neq j\} \cup \{!^1p_i \check{\vee} !^2q_i \check{\vee} r \mid i \in \mathbb{N}\}$. As explained in that section, $r \notin Cn_{\mathbf{SK2^m}}(\Gamma)$. However, note that for every $i \in \mathbb{N}$, $\check{\neg}!^2q_i \in Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{c}})$, since neither of these abnormalities occur in a minimal $\mathrm{Dab}_2$-consequence of $Cn_{\mathbf{K^m_1}}(\Gamma)$.

Let $\Gamma'_{\mathsf{c}} = \Gamma_{\mathsf{c}} \cup \{\check{\neg}!^2q_i \mid i \in \mathbb{N}\}$. Note that $\Gamma'_{\mathsf{c}} \vdash_{\mathbf{K}} r \check{\vee} !^1p_i$ for every $i \in \mathbb{N}$. The minimal $\mathrm{Dab}_1$-consequences of $\Gamma'_{\mathsf{c}}$ are the same as those of $\Gamma_{\mathsf{c}}$, i.e. $\{!^1p_i \,\check{\vee}\, !^1p_j \mid i,j \in \mathbb{N}, i \neq j\}$. Hence $\Phi^1(\Gamma'_{\mathsf{c}}) = \{\{!^1p_i \mid i \in \mathbb{N} - \{k\}\} \mid k \in \mathbb{N}\}$. It follows that for every $\varphi \in \Phi^1(\Gamma'_{\mathsf{c}})$, we can derive $r$ on a condition $\Delta \subseteq \Omega^{\mathbf{K}}_1 - \varphi$, whence $r \in Cn_{\mathbf{K^m_1}}(\Gamma'_{\mathsf{c}}) \subseteq Cn_{\mathbf{SK2^m}}(\Gamma'_{\mathsf{c}})$.

The above examples also illustrate that neither $\mathbf{SAL^r}$ nor $\mathbf{SAL^m}$ have the Fixed Point property. Recall that for flat ALs, this property is an immediate consequence of their reflexivity and cumulative transitivity. The following facts can easily be derived from the preceding observations:

$s \notin Cn_{\mathbf{SK2^r}}(\Gamma_{\mathsf{ctr}})$, whereas $s \in Cn_{\mathbf{SK2^r}}(Cn_{\mathbf{SK2^r}}(\Gamma_{\mathsf{ctr}}))$
$r \notin Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{ctr}})$, whereas $r \in Cn_{\mathbf{SK2^m}}(Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{ctr}}))$

**Cautious Monotonicity** For Cautious Monotonicity, it turns out that there is a single counterexample that works for both $\mathbf{SK2^r}$ and $\mathbf{SK2^m}$. I will spell out the argument for $\mathbf{SK2^r}$ – the case for $\mathbf{SK2^m}$ is completely analogous. Let $\Gamma_{\mathsf{cm}} = \{!^1p \check{\vee} !^1q, !^1r \check{\vee} !^1p \check{\vee} !^2s\}$. Note that $\check{\neg}!^1r \in Cn_{\mathbf{SK2^r}}(\Gamma_{\mathsf{cm}})$, since the abnormality $!^1r$ does not occur in a minimal Dab$_1$-consequence of $\Gamma_{\mathsf{cm}}$. Also, $\check{\neg}!^2s \in Cn_{\mathbf{SK2^r}}(\Gamma_{\mathsf{cm}})$. That is, we can only derive the Dab-formula $!^1p \check{\vee} !^2s$ from $\Gamma_{\mathsf{cm}}$ – we cannot push $!^1p$ to the condition since it is unreliable at level 1. Since $!^1p \check{\vee} !^2s$ is not a Dab$_2$-consequence, and more generally, since there are no Dab$_2$-consequences of $\Gamma_{\mathsf{cm}}$, $!^2s$ will be considered as a reliable abnormality by the second logic in the superposition.

Consider $\Gamma'_{\mathsf{cm}} = \Gamma_{\mathsf{cm}} \cup \{\check{\neg}!^2s\}$. Note that the following are minimal Dab$_1$-consequences of the extended premise set:

$!^1p \check{\vee} !^1q$
$!^1p \check{\vee} !^1r$

It follows that we cannot derive $\check{\neg}!^1r$ from $\Gamma'_{\mathsf{cm}}$ by $\mathbf{K^r_1}$. The second logic cannot come to the rescue either, whence $!^1 \check{\neg} r \notin Cn_{\mathbf{SK2^r}}(\Gamma'_{\mathsf{cm}})$. To summarize, if we add a $\mathbf{SK2^r}$-consequence of $\Gamma_{\mathsf{cm}}$ to this premise set, then the resulting premise set no longer allows us to derive another $\mathbf{SK2^r}$-consequence of $\Gamma_{\mathsf{cm}}$.

**Cumulative Indifference for $\mathbf{SAL^r_{(I)}}$** As announced, I will prove here that $\mathbf{SAL^r_{(I)}}$ is cumulatively indifferent.[19] The proof requires four lemmas and is spelled out on the next two pages. In the remainder, I will often rely on the Dab-conservativity of flat ALs – see Lemma 2.4.2: where $\Delta \subset \Omega$, $\Gamma \vdash_{\mathbf{AL}} Dab(\Delta)$ iff $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$. I will also make use of the **LLL**-closure of **SAL** – see Theorem 3.1.

**Lemma 3.16** *Where $\Gamma \subseteq \mathcal{W}_s$, $i, i+k \in I$ and $A \in \Omega_i$: if $A \notin U^i(Cn_{\mathbf{SAL^r_{i-1}}}(\Gamma))$, then $A \notin U^{i+k}(Cn_{\mathbf{SAL^r_{i+k-1}}}(\Gamma))$.*

*Proof.*[20] Suppose $A \notin U^i(Cn_{\mathbf{SAL^r_{i-1}}}(\Gamma))$. Hence $\check{\neg} A \in Cn_{\mathbf{SAL^r_i}}(\Gamma)$, whence by Fact 3.1.2, $\check{\neg} A \in Cn_{\mathbf{SAL^r_{i+k}}}(\Gamma)$.

Assume that $A \in U^{i+k}(Cn_{\mathbf{SAL^r_{i+k-1}}}(\Gamma))$. This implies that there is a $\Delta \subseteq \Omega_{i+k}$ such that $Dab(\Delta)$ is a minimal Dab$_{i+k}$-consequence of $Cn_{\mathbf{SAL^r_{i+k-1}}}(\Gamma)$ and $A \in \Delta$. However, since $\check{\neg} A \in Cn_{\mathbf{SAL^r_{i+k}}}(\Gamma)$ and by the **LLL**-closure of $Cn_{\mathbf{SAL^r_{i+k}}}$, $Dab(\Delta - \{A\}) \in Cn_{\mathbf{SAL^r_{i+k}}}(\Gamma)$. Hence $Dab(\Delta - \{A\}) \in Cn_{\mathbf{AL^r_{i+k}}}(Cn_{\mathbf{SAL^r_{i+k-1}}}(\Gamma))$. By the Dab-conservativity of $\mathbf{AL^r_{i+k}}$, $Dab(\Delta - \{A\}) \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL^r_{i+k-1}}}(\Gamma))$. But this contradicts the fact that $Dab(\Delta)$ is a minimal Dab$_{i+k}$-consequence of $Cn_{\mathbf{SAL^r_{i+k-1}}}(\Gamma)$. ■

**Lemma 3.17** *Where $\Gamma \subseteq \mathcal{W}_s$, $i, i+k \in I$ and $\Delta \subset \Omega_{(i)}$: if $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i+k)}}}(\Gamma)$, then $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$*

[19] Batens already suggests that this is the case in the last paragraphs of [25, Chapter 6, Section 6.2.3].

[20] Note that this lemma applies to the more general class of logics $\mathbf{SAL^r}$.

*Proof.* Suppose $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i+k)}}}(\Gamma)$ for a $\Delta \subset \Omega_{(i)}$. By Definition 3.1, $Dab(\Delta) \in Cn_{\mathbf{AL^r_{(i+k)}}}(Cn_{\mathbf{SAL^r_{(i+k-1)}}}(\Gamma))$. Note that $\Delta \subset \Omega_{(i+k)}$, whence by the Dab-conservativity of $\mathbf{AL^r_{(i+k)}}$, $Cn_{\mathbf{SAL^r_{(i+k-1)}}}(\Gamma) \vdash_{\mathbf{LLL}} Dab(\Delta)$. By the **LLL**-closure of $Cn_{\mathbf{SAL^r_{(i+k-1)}}}(\Gamma)$, $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i+k-1)}}}(\Gamma)$. Repeating the same reasoning for all $j < k$, $j \geq 0$, relying on (i) the Dab-conservativity of each logic $\mathbf{AL^r_{(i+j)}}$ and (ii) the fact that $\Delta \subset \Omega_{(i+j)}$, we obtain that $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$. ■

**Lemma 3.18** *Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$: $A \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$ iff there is a $\Delta \subset \Omega_{(i)} - U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma))$ such that $A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$.*

*Proof.* ($\Rightarrow$) Suppose $A \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$. By Theorem 2.6, there is a $\Theta_1 \subseteq \Omega_{(i)} - U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma))$ such that $Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta_1)$. By the **LLL**-closure of $Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$, $A \check{\vee} Dab(\Theta_1) \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$. Hence by Theorem 2.6, there is a $\Theta_2 \subseteq \Omega_{(i-1)} - U^{(i-1)}(Cn_{\mathbf{SAL^r_{(i-2)}}}(\Gamma))$ such that $Cn_{\mathbf{SAL^r_{(i-2)}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta_1) \check{\vee} Dab(\Theta_2)$. Note that $\Theta_2 \subset \Omega_{(i)}$, and by Lemma 3.16, $\Theta_2 \cap U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)) = \emptyset$. Also, by the **LLL**-closure of $Cn_{\mathbf{SAL^r_{(i-2)}}}(\Gamma)$, $A \check{\vee} Dab(\Theta_1) \check{\vee} Dab(\Theta_2) \in Cn_{\mathbf{SAL^r_{(i-2)}}}(\Gamma)$, whence by Fact 3.1.2, $A \check{\vee} Dab(\Theta_1) \check{\vee} Dab(\Theta_2) \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$.

Repeating the same reasoning $i$ times, we obtain sets $\Theta_1, \ldots, \Theta_i$ such that (i) $A \check{\vee} Dab(\Theta_1) \check{\vee} \ldots \check{\vee} Dab(\Theta_i) \in Cn_{\mathbf{LLL}}(\Gamma)$ and (ii) $(\Theta_1 \cup \ldots \cup \Theta_i) \subseteq \Omega_{(i)} - U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma))$.

($\Leftarrow$) Suppose there is an $i \in I$ such that $A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$ for a $\Delta \subset \Omega_{(i)}$ such that $\Delta \cap U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)) = \emptyset$. By the reflexivity of $\mathbf{SAL^r_{(i-1)}}$ and **LLL**, $Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By Theorem 2.6, $A \in Cn_{\mathbf{AL^r_{(i)}}}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma))$ whence by Definition 3.1, $A \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$ and $A \in Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$. ■

**Lemma 3.19** *Let $\Gamma' \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$ and $\Gamma \subseteq \mathcal{W}_s$. Then for every $i \in I$, $U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)) = U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma \cup \Gamma'))$*

*Proof.* Suppose $\Gamma' \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$. Note that (†) $\Gamma \cup \Gamma' \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$ by the reflexivity of $\mathbf{SAL^r_{(I)}}$ and the supposition. I prove that for every $i \in I$ and every $\Delta \subseteq \Omega_{(i)}$: $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$ iff $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma \cup \Gamma')$, whence the lemma follows immediately.

($i = 1$) Let $\Delta \subset \Omega_{(1)}$. ($\Rightarrow$) Suppose $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(0)}}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma)$. By the monotonicity of **LLL**, it follows immediately that $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma') = Cn_{\mathbf{SAL^r_{(0)}}}(\Gamma \cup \Gamma')$.

($\Leftarrow$) Suppose $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(0)}}}(\Gamma \cup \Gamma') = Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma')$. By (†) and the monotonicity of **LLL**, also $Dab(\Delta) \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma))$, whence by the **LLL**-closure of $\mathbf{SAL^r_{(I)}}$, $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$. Hence for a $j \in I$, $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(j)}}}(\Gamma)$. By Lemma 3.17, $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(0)}}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma)$.

($i \Rightarrow i+1$) Suppose $\Delta \subset \Omega_{(i+1)}$. ($\Rightarrow$) Let $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$. Hence by the left-right direction of Lemma 3.18, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta) \check{\vee} Dab(\Delta')$, for a $\Delta' \subseteq \Omega_{(i)} - U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma))$. By the monotonicity of **LLL**, $\Gamma \cup \Gamma' \vdash_{\mathbf{LLL}} Dab(\Delta) \check{\vee} Dab(\Delta')$. By the induction hypothesis, $\Delta' \cap U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma \cup \Gamma')) = \emptyset$. By the right-left direction of Lemma 3.18, $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma \cup \Gamma')$.

($\Leftarrow$) Suppose $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma \cup \Gamma')$. Hence by the left-right direction of Lemma 3.18, $\Gamma \cup \Gamma' \vdash_{\mathbf{LLL}} Dab(\Delta) \check{\vee} Dab(\Delta')$, for a $\Delta' \subseteq \Omega_{(i)}$ such that $\Delta' \cap U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma \cup \Gamma')) = \emptyset$. By (†) and the monotonicity of **LLL**, $Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma) \vdash_{\mathbf{LLL}} Dab(\Delta) \check{\vee} Dab(\Delta')$. Hence there is a $j \in I$: $Cn_{\mathbf{SAL^r_{(j)}}}(\Gamma) \vdash_{\mathbf{LLL}} Dab(\Delta) \check{\vee} Dab(\Delta')$. By the **LLL**-closure of $\mathbf{SAL^r_{(j)}}$, $Dab(\Delta) \check{\vee} Dab(\Delta') \in Cn_{\mathbf{SAL^r_{(j)}}}(\Gamma)$.

We can derive that (‡) $Dab(\Delta) \check{\vee} Dab(\Delta') \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$ — for $j \leq i$, see Fact 3.1.2; for $j > i$, this follows from the fact that $\Delta \cup \Delta' \subset \Omega_{(i+1)}$ and by Lemma 3.17. Also, by the induction hypothesis, $\Delta' \cap U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)) = \emptyset$, whence $\check{\neg} Dab(\Delta') \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$. By (‡) and the **LLL**-closure of $\mathbf{SAL^r_{(i)}}$, $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$. ■

**Theorem 3.12** *Let* $\Gamma' \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$ *and* $\Gamma \subseteq \mathcal{W}_s$. *Then* $Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma) = Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma \cup \Gamma')$.

*Proof.* Suppose (†) $\Gamma' \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$. Note that (‡) $\Gamma \cup \Gamma' \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$ by the reflexivity of $\mathbf{SAL^r_{(I)}}$ and the supposition.

($Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma) \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma \cup \Gamma')$) Suppose $A \in Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$. By Definition 3.1, there is an $i \in I$ such that $A \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$. By Lemma 3.18, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega_{(i)} - U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma))$. By the monotonicity of **LLL**, $\Gamma \cup \Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, whence by the reflexivity of $\mathbf{SAL^r_{(i-1)}}$ and **LLL**, $Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma \cup \Gamma') \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By Lemma 3.19 and (†), $\Delta \subseteq \Omega_{(i)} - U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma \cup \Gamma'))$. Hence by Lemma 3.18, $A \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$.

($Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$) Suppose $A \in Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma \cup \Gamma')$. By Definition 3.1, there is an $i \in I$ such that $A \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma \cup \Gamma')$. By Lemma 3.18, $\Gamma \cup \Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega_{(i)} - U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma \cup \Gamma'))$. By (‡) and the monotonicity of **LLL**, ($\star$) $Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.

By (†) and Lemma 3.19, $\Delta \cap U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)) = \emptyset$. It follows that $\check{\neg} Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$, whence also $\check{\neg} Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$. But then by ($\star$) and the **LLL**-closure of $\mathbf{SAL^r_{(I)}}$, $A \in Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$. ■

By Lemmas 2.7 and 2.8, we have:

**Corollary 3.3** *Each of the following holds:*

1. *Where* $\Gamma' \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$ *and* $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$ *(Cumulative Transitivity)*
2. *Where* $\Gamma' \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$ *and* $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma) \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma \cup \Gamma')$ *(Cautious Monotonicity)*

3. *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $Cn_{\mathbf{SAL^r_{(I)}}}(Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)) = Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$*. (Fixed Point)*
4. *Where* $\Gamma, \Gamma' \subseteq \mathcal{W}_s$*: if* $\Gamma \subseteq Cn_{\mathbf{SAL^r}}(\Gamma')$ *and* $\Gamma' \subseteq Cn_{\mathbf{SAL^r}}(\Gamma)$*, then* $Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma) = Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma')$*. (Equivalence Criterion (C1) )*

### 3.6.3 Normal Premise Sets

In Chapter 2, we saw that whenever $\Gamma \cup \Omega^{\check{\neg}}$ has **LLL**-models, then **AL** is identical to its upper limit logic **ULL** (see Theorem 2.17). As we will see below, this result can be generalized to sequential superpositions of ALs – see Theorem 3.15 below.[21]

However, for prioritized ALs in general, one may also wonder whether a slightly stronger property holds. That is, suppose that for some $i \in I$, it is possible to verify all members of $\Gamma$, yet also falsify all abnormalities up to level $i$. In that case, it seems a desirable property for a prioritized logic **PAL** that $Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(i)}) \subseteq Cn_{\mathbf{PAL}}(\Gamma)$ – in other words, that the prioritized logic indeed considers all the members of $\Omega_{(i)}$ to be false.

To formally express this property, let me introduce the concepts of normality at level $i$, resp. up to level $i$:

**Definition 3.15** $\Gamma$ *is* normal at level $i$ *iff* $\Gamma \cup \Omega^{\check{\neg}}_i$ *has* **LLL**-*models.* $\Gamma$ *is* normal up to level $i$ *iff* $\Gamma \cup \Omega^{\check{\neg}}_{(i)}$ *has* **LLL**-*models.*

The following is immediate in view of Definition 3.15:

**Fact 3.10** *If* $\Gamma$ *is normal up to level* $i$*, then each of the following holds:*

1. $\Gamma$ *is normal at level* $j$*, for every* $j \leq i$
2. $\Gamma \cup \Omega^{\check{\neg}}_{(j)}$ *is normal at level* $j+1$*, for every* $j < i$
3. $Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(j)})$ *is normal at level* $j+1$*, for every* $j < i$

In the remainder, I use $\mathbf{ULL_i}$ to refer to the upper limit logic of $\mathbf{AL_i}$, i.e. $\mathbf{ULL_i}$ is the monotonic logic that trivializes all abnormalities of level $i$. Likewise, $\mathbf{ULL_{(i)}}$ denotes the upper limit logic of $\mathbf{AL_{(i)}}$ and trivializes all abnormalities up to level $i$. Formally:

**Definition 3.16** $\Gamma \vdash_{\mathbf{ULL_i}} A$ *iff* $\Gamma \cup \Omega^{\check{\neg}}_i \vdash_{\mathbf{LLL}} A$

**Definition 3.17** $\Gamma \vdash_{\mathbf{ULL_{(i)}}} A$ *iff* $\Gamma \cup \Omega^{\check{\neg}}_{(i)} \vdash_{\mathbf{LLL}} A$

**Theorem 3.13** *If* $\Gamma$ *is normal up to level* $i$*, then* $Cn_{\mathbf{ULL_{(i)}}}(\Gamma) \subseteq Cn_{\mathbf{SAL}}(\Gamma)$*.*

*Proof.* Suppose $\Gamma$ is normal up to level $i$. We show by an induction that for all $j \leq i$, $Cn_{\mathbf{SAL_j}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(j)})$ — the rest follows immediately.

($j = 1$) By Fact 3.10.1 and the supposition, $\Gamma$ is normal at level 1. By Fact 3.1.1 and Theorem 2.17 respectively, $Cn_{\mathbf{SAL_1}}(\Gamma) = Cn_{\mathbf{AL_1}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_1)$.

($j \Rightarrow j+1$): By the induction hypothesis, $Cn_{\mathbf{SAL_j}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(j)})$. By Fact 3.10.3 and the supposition, $Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(j)})$ is normal at level $j+1$. But then

[21] Earlier versions of some proofs from this Section appeared in [151], which is co-authored by Christian Straßer.

by Theorem 2.17, $Cn_{\mathbf{SAL_{j+1}}}(\Gamma) = Cn_{\mathbf{AL_{j+1}}}(Cn_{\mathbf{SAL_j}}(\Gamma)) = Cn_{\mathbf{AL_{j+1}}}(Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(j)})) = Cn_{\mathbf{ULL_{j+1}}}(Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(j)})) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(j)}) \cup \Omega^{\check{\neg}}_{j+1}) = Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(j)} \cup \Omega^{\check{\neg}}_{j+1}) = Cn_{\mathbf{LLL}}(\Gamma \cup \Omega^{\check{\neg}}_{(j+1)})$. ■

Lemma 3.20 states that in order for $A$ to be an **SAL**-consequence, there has to be a finite $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. This lemma is will be used to prove that **SAL** is weaker than **ULL** – see Theorem 3.14 below.

**Lemma 3.20** *Each of the following holds:*

1. *for every $i \in I$: if $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$, then there is a $\Delta \subset \Omega_{(i)}$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.*
2. *If $A \in Cn_{\mathbf{SAL}}(\Gamma)$, then there is a $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.*

*Proof. Ad 1.* Let each index $\mathbf{x_i}$ refers to the strategy of the logic $\mathbf{AL_i}$ in the superposition. ($i = 1$) Immediate in view of Theorem 2.6 (for $\mathbf{x_1} = \mathbf{r}$), resp. Theorem 2.7 (for $\mathbf{x_1} = \mathbf{m}$).

($i \Rightarrow i + 1$) Suppose $A \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$. If $\mathbf{x_{i+1}} = \mathbf{r}$, then by Theorem 2.6.1, $Cn_{\mathbf{SAL_i}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega_{i+1}$, whence also $\Delta \subset \Omega$. By Theorem 3.1, $A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$. But then by the induction hypothesis, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta) \check{\vee} Dab(\Theta)$ for a $\Theta \subset \Omega_{(i)}$, whence $\Delta \cup \Theta \subset \Omega_{(i+1)}$.

The reasoning for $\mathbf{x_{i+1}} = \mathbf{m}$ is completely analogous – replace Theorem 2.6.1 by Theorem 2.7.1.

*Ad 2.* Suppose $A \in Cn_{\mathbf{SAL}}(\Gamma)$. By Definition 3.1, there is an $i \in I$ such that $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$. The rest is immediate in view of item 1. ■

**Theorem 3.14** $Cn_{\mathbf{SAL}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$.

*Proof.* Suppose $A \in Cn_{\mathbf{SAL}}(\Gamma)$. By Lemma 3.20, there is a $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By **CL**-properties, $\Gamma \cup \Omega^{\check{\neg}} \vdash_{\mathbf{LLL}} A$. ■

**Theorem 3.15** *If $\Gamma$ is normal, then $Cn_{\mathbf{PAL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.*

*Proof.* Suppose $\Gamma$ is normal. In view of Theorem 3.14, it suffices to prove that $Cn_{\mathbf{ULL}}(\Gamma) \subseteq Cn_{\mathbf{SAL}}(\Gamma)$. So suppose that $\Gamma \cup \Omega^{\check{\neg}} \vdash_{\mathbf{LLL}} A$. By the compactness of **LLL**, there is an $i \in I$ such that $\Gamma \cup \Omega^{\check{\neg}}_{(i)} \vdash_{\mathbf{LLL}}$, whence $A \in Cn_{\mathbf{ULL_{(i)}}}(\Gamma)$. Note that for every $i \in I$, $\Gamma$ is normal up to level $i$. By Theorem 3.13, $A \in Cn_{\mathbf{SAL}}(\Gamma)$. ■

# Chapter 4

# Hierarchic Adaptive Logics

*This Chapter is based on the paper "Hierarchic Adaptive Logics" (Logic Journal of the IGPL 2011, doi: 10.1093/jigpal/jzr025). I am very greatful to Christian Straßer, Diderik Batens, and two anonymous referees for their fruitful comments on that paper. I also thank Peter Verdée for his useful comments on an earlier draft of this chapter.*

In the previous chapter, we saw how it is possible to superpose flat adaptive logics in order to characterize prioritized defeasible reasoning. Notwithstanding the intuitive benefits of this approach, we encountered some meta-theoretic problems, especially with respect to superpositions of ALs with the Minimal Abnormality Strategy.

In the current chapter, I will concentrate on a different and more recent way in which flat adaptive logics have been combined, in order to obtain a prioritized consequence relation. The resulting systems are called *hierarchic adaptive logics*. In Section 4.1, I will explain the general idea behind them, and illustrate it by means of some **K**-based prioritized logics. After that, I will describe their generic semantics (Section 4.2), their proof theory (Section 4.3), and give some meta-theoretic results for these systems (Section 4.4).

## 4.1 General Characteristics of HAL

### 4.1.1 The Syntactic Consequence Relation

The idea behind hierarchic adaptive logics is due to Diderik Batens, who applied it to adaptive logics for inductive generalization in his [24]. After some inquiry, it turned out that we may easily generalize this idea and thereby obtain a generic format for prioritized ALs which has a number of interesting features. Relying on a suggestion by Batens, I was also able to develop a very simple generic proof theory for these systems. Moreover, for a specific subclass of them, an alternative proof theory can be used, from which it follows that all logics in this subclass have a very low degree of complexity – see Section 4.3.3.

To understand Batens' original idea, consider again a sequence of sets of abnormalities, $\langle\Omega_i\rangle_{i\in I}$. As explained in Chapter 1, the overall aim of prioritized ALs is to interpret premise sets "as normally as possible", but in such a way that abnormalities from $\Omega_1$ are considered as the worst abnormalities, next those of $\Omega_2$, etc. One way to do so, for a given $\Gamma$, is the following. We first interpret $\Gamma$ as normally as possible with respect to $\Omega_1$. Put differently, we consider $A \in \Omega_1$ to be false, unless $\Gamma$ entails a minimal disjunction of abnormalities *of rank 1*, in which $A$ is disjunct. Next, we interpret the same premise set $\Gamma$ as normally as possible with respect to $\Omega_{(2)} = \Omega_1 \cup \Omega_2$. So, under this interpretation of $\Gamma$, $B \in \Omega_2$ behaves abnormally iff it is a disjunct in a minimal disjunction of abnormalities of rank 1 or rank 2. After that, we interpret $\Gamma$ as normally as possible in view of $\Omega_{(3)}$, etc.

This means that we apply different flat adaptive logics to the same premise set. More specifically, we apply the logics $\mathbf{AL}^{\mathbf{x}}_{\mathbf{(i)}}$ $(i \in I)$ that are characterized by the triple $\langle\mathbf{LLL}, \Omega_{(i)}, \mathbf{x}\rangle$. This gives us the sets

$$Cn_{\mathbf{AL}^{\mathbf{x}}_{\mathbf{(1)}}}(\Gamma), Cn_{\mathbf{AL}^{\mathbf{x}}_{\mathbf{(2)}}}(\Gamma), Cn_{\mathbf{AL}^{\mathbf{x}}_{\mathbf{(3)}}}(\Gamma), \ldots$$

As will be shown below, $\bigcup_{i\in I} Cn_{\mathbf{AL}^{\mathbf{x}}_{\mathbf{(i)}}}(\Gamma)$ is always **LLL**-satisfiable whenever $\Gamma$ is **LLL**-satisfiable (see Theorem 4.4). This union of sets can be considered as a specific interpretation of $\Gamma$, which can is motivated in terms of the priority of each set $\Omega_i$ $(i \in I)$.

For instance, let $\Gamma_{\mathsf{h}} = \{\Diamond p, \Diamond\Diamond q, \Diamond\Diamond r, \neg p \vee \neg r\}$. Note that either the assumption $p$ or the assumption $r$ has to be withdrawn, in view of the premise $\neg p \vee \neg r$. Since $p$ is more plausible, we expect that a prioritized adaptive logic allows us to (finally) derive $p$. Also, since $q$ is not problematic at all, we expect $q$ to be (finally) derivable by a prioritized AL.

Consider now the flat adaptive logics $\mathbf{K}^{\mathbf{r}}_{\mathbf{(1)}} = \langle\mathbf{K}, \Omega^{\mathbf{K}}_1, \mathbf{r}\rangle$ and $\mathbf{K}^{\mathbf{r}}_{\mathbf{(2)}} = \langle\mathbf{K}, \Omega^{\mathbf{K}}_1 \cup \Omega^{\mathbf{K}}_2, \mathbf{r}\rangle$ (these were introduced on page 44). It can easily be verified that $p \in Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{(1)}}}(\Gamma_{\mathsf{h}})$ and $q \in Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{(2)}}}(\Gamma_{\mathsf{h}})$.

More generally, interpreting $\Gamma$ by means of different logics $\mathbf{AL}^{\mathbf{x}}_{\mathbf{(i)}}$ yields a strong set of consequences, that is justified in terms of the priorities on the corresponding sets of abnormalities. However, this set is in most cases not **LLL**-closed. For instance, although $p, q \in Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{(1)}}}(\Gamma_{\mathsf{h}}) \cup Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{(2)}}}(\Gamma_{\mathsf{h}})$, it can be shown that $p \wedge q \notin Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{(1)}}}(\Gamma_{\mathsf{h}}) \cup Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{(2)}}}(\Gamma_{\mathsf{h}})$. Hence $Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{(1)}}}(\Gamma_{\mathsf{h}}) \cup Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{(2)}}}(\Gamma_{\mathsf{h}})$ does not have the kind of properties we would usually expect from a consequence set – especially if **LLL** is taken to be the standard of deduction in the application of the prioritized adaptive logic.

The solution is straightforward: close the set $\bigcup_{i\in I} Cn_{\mathbf{AL}^{\mathbf{x}}_{\mathbf{(i)}}}(\Gamma)$ under **LLL**.[1] So we obtain the set $Cn_{\mathbf{LLL}}(\bigcup_{i\in I} Cn_{\mathbf{AL}^{\mathbf{x}}_{\mathbf{(i)}}}(\Gamma))$. As will be shown below, the corresponding syntactic consequence relation is sound with respect to a very straightforward semantics, can be fully characterized by a proof theory, and satisfies a number of important metatheoretic requirements. Moreover, while writing this

[1] In view of the results from this chapter, it also seems feasible to close them under a flat AL that has **LLL** as its lower limit logic, but such a combination has not been investigated yet.

thesis, I noticed that we can easily generalize the above construction, allowing the logics $\mathbf{AL_{(i)}}$ to use different strategies, without loss of any results.

This brings us to the following definition. Let **LLL** be a Tarski-logic and let $\langle \Omega_i \rangle_{i \in I}$ be a sequence of sets of abnormalities. Recall that for every $i \in I$, $\Omega_{(i)} =_{\mathsf{df}} \Omega_1 \cup \ldots \cup \Omega_i$. Let every logic $\mathbf{AL_{(i)}}$ $(i \in I)$ be a flat adaptive logic characterized by (i) **LLL**, (ii) $\Omega_{(i)}$ and (iii) a strategy $\mathbf{x_i} \in \{\mathbf{r}, \mathbf{m}\}$. Then the hierarchic combination of the logics $\langle \mathbf{AL_{(i)}} \rangle_{i \in I}$ is defined as follows:

**Definition 4.1** $Cn_{\mathbf{HAL}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{LLL}}(\bigcup_{i \in I} Cn_{\mathbf{AL_{(i)}}}(\Gamma))$

In some cases, it will be useful to restrict the focus to the logics $\mathbf{HAL^r}$ and $\mathbf{HAL^m}$, which are hierarchic combinations of flat ALs that all use the same strategy. Hence, where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$: $Cn_{\mathbf{HAL^x}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{LLL}}(\bigcup_{i \in I} Cn_{\mathbf{AL^x_{(i)}}}(\Gamma))$. Note that, as for sequential superpositions of ALs, we may characterize every logic $\mathbf{HAL^x}$ by a triple, where the second element is a sequence of sets of abnormalities: $\langle \mathbf{LLL}, \langle \Omega_{(i)} \rangle_{i \in I}, \mathbf{x} \rangle$.

In the remainder, we will consider some hierarchic variants of the **K**-based prioritized logics from the preceding Chapter. Their syntactic consequence relation is defined as follows:

$$Cn_{\mathbf{HK^r}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{K}}(\bigcup_{i \in \mathbb{N}} Cn_{\mathbf{K^r_{(i)}}}(\Gamma))$$
$$Cn_{\mathbf{HK^m}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{K}}(\bigcup_{i \in \mathbb{N}} Cn_{\mathbf{K^m_{(i)}}}(\Gamma))$$
$$Cn_{\mathbf{HK2^r}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{K}}(Cn_{\mathbf{K^r_{(1)}}}(\Gamma) \cup Cn_{\mathbf{K^r_{(2)}}}(\Gamma))$$
$$Cn_{\mathbf{HK2^m}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{K}}(Cn_{\mathbf{K^m_{(1)}}}(\Gamma) \cup Cn_{\mathbf{K^m_{(2)}}}(\Gamma))$$

### 4.1.2 Some Generic Properties

As for **SAL**, I will start with a few facts that are easily derivable from the metatheory of flat ALs and the definition of **HAL**. The first one is immediate in view of Definition 4.1 and the fact that **LLL** is a Tarski-logic:

**Theorem 4.1** *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $Cn_{\mathbf{LLL}}(Cn_{\mathbf{HAL}}(\Gamma)) = Cn_{\mathbf{HAL}}(\Gamma)$. *(**LLL**-Closure)*

We can also easily see that **HAL** is **LLL**-invariant:

**Theorem 4.2** *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $Cn_{\mathbf{HAL}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{HAL}}(\Gamma)$. *(**LLL**-Invariance)*

*Proof.* By Theorem 2.19, for every $i \in I$, $Cn_{\mathbf{AL^{x_i}_{(i)}}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma)$. Hence by Definition 4.1, $Cn_{\mathbf{HAL}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{HAL}}(\Gamma)$. ■

The following theorem is also immediate in view of Definition 4.1, the reflexivity of every logic $\mathbf{AL_{(i)}}$ and the reflexivity of **LLL**:

**Theorem 4.3** *Each of the following holds:*

1. $\Gamma \subseteq Cn_{\mathbf{HAL}}(\Gamma)$
2. $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{HAL}}(\Gamma)$

We can also generalize the fact that Minimal Abnormality is stronger than Reliability (see Theorem 2.16) to hierarchic ALs. This is a corollary of the following property:

**Theorem 4.4** $Cn_{\mathbf{HAL^r}}(\Gamma) \subseteq Cn_{\mathbf{HAL}}(\Gamma)$.

*Proof.* Suppose $A \in Cn_{\mathbf{HAL^r}}(\Gamma)$. By Definition 4.1 and the compactness of **LLL**, there are $B_1, \ldots, B_n$ such that (i) $\{B_1, \ldots, B_n\} \vdash_{\mathbf{LLL}} A$ and (ii) for every $i \in \{1, \ldots, n\}$: $B_i \in Cn_{\mathbf{AL^r_{(j_i)}}}(\Gamma)$ for a $j_i \in I$. From (ii), by Theorem 2.16: for every $i \in \{1, \ldots, n\}$: $B_i \in Cn_{\mathbf{AL^{x_{j_i}}_{(j_i)}}}(\Gamma)$ for a $j_i \in I$. By (i), the monotonicity of **LLL** and Definition 4.1, $A \in Cn_{\mathbf{HAL}}(\Gamma)$. ■

**Corollary 4.1** $Cn_{\mathbf{HAL^r}}(\Gamma) \subseteq Cn_{\mathbf{HAL^m}}(\Gamma)$.

Finally, just as for **SAL**, there is a simple criterion to decide whether two premise sets are equivalent in view of a hierarchic AL:

**Theorem 4.5** *If* $\Gamma, \Gamma' \subseteq \mathcal{W}_s$ *and* $\Gamma$ *and* $\Gamma'$ *are* **LLL***-equivalent, then they are* **HAL***-equivalent.*

*Proof.* Suppose the antecedent holds. Then by Theorem 2.22, $Cn_{\mathbf{AL_i}}(\Gamma) = Cn_{\mathbf{AL_i}}(\Gamma')$ for every $i \in I$. The rest is immediate in view of Definition 4.1. ■

## 4.2 A Semantics for HAL

### 4.2.1 The Intersection of the Selections

The selection of the set of **HAL**-models is based on the selections by each of the logics $\mathbf{AL_{(i)}}$. More precisely, the set of **HAL**-models is the set of those $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ that are selected by each flat adaptive logic of the combination:

**Definition 4.2** $\mathcal{M}_{\mathbf{HAL}}(\Gamma) =_{\mathsf{df}} \bigcap_{i \in I} \mathcal{M}_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma)$

As before, we define the semantic consequence relation of **HAL** as follows: $\Gamma \models_{\mathbf{HAL}} A$ iff $A$ is verified by every $M \in \mathcal{M}_{\mathbf{HAL}}(\Gamma)$.

**Example 4.1** *Consider* $\Gamma_{\mathsf{h}} = \{\Diamond p, \Diamond\Diamond q, \Diamond\Diamond r, \neg p \vee \neg r\}$*, which was already mentioned in the first section of this chapter.*

*First of all, note that* $\Gamma_{\mathsf{h}}$ *has no* $Dab_{(1)}$*-consequences. It follows that* $\Gamma_{\mathsf{h}}$ *is normal up to level 1, and hence the set of unreliable formulas for* $\mathbf{K^r_{(1)}}$ *is* $U^{(1)}(\Gamma_{\mathsf{h}}) = \emptyset$*. As a result,*

$$\mathcal{M}_{\mathbf{K^r_{(1)}}}(\Gamma_{\mathsf{h}}) = \{M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{h}}) \mid Ab(M) \cap \Omega^{\mathbf{K}}_{(1)} = \emptyset\}$$

$\Gamma_{\mathsf{h}}$ *has only one minimal Dab-consequence, viz.* $!^1 p \check{\vee} !^2 r$*. Hence, the set of unreliable formulas for* $\mathbf{K^r_{(2)}}$ *is* $U^{(1)}(\Gamma_{\mathsf{h}}) = \{!^1 p, !^2 r\}$*. So we have:*

$$\mathcal{M}_{\mathbf{K^r_{(2)}}}(\Gamma_{\mathsf{h}}) = \{M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{h}}) \mid Ab(M) \cap \Omega^{\mathbf{K}}_{(2)} \subseteq \{!^1 p, !^2 r\}\}$$

*Finally, the set of* $\mathbf{HK2^r}$*-models of* $\Gamma_{\mathsf{h}}$ *is the intersection of* $\mathcal{M}_{\mathbf{K}^{\mathbf{r}}_{(1)}}(\Gamma_{\mathsf{h}})$ *and* $\mathcal{M}_{\mathbf{K}^{\mathbf{r}}_{(2)}}(\Gamma_{\mathsf{h}})$*, and hence:*

$$\mathcal{M}_{\mathbf{HK2^r}}(\Gamma_{\mathsf{h}}) = \{M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{h}}) \mid Ab(M) \cap \Omega^{\mathbf{K}}_{(2)} \subseteq \{!^2 r\}\}$$

*Hence every* $\mathbf{HK2^r}$*-model of* $\Gamma_{\mathsf{h}}$ *falsifies the abnormalities* $!^1 p$ *and* $!^2 q$*, which means it verifies* $p$ *and* $q$*. Also, since every* $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{h}})$ *either verifies* $!^1 p$ *or* $!^2 r$*, it follows that for every* $M \in \mathcal{M}_{\mathbf{HK2^r}}(\Gamma_{\mathsf{h}})$*,* $M \Vdash !^2 r$*.*

As the following theorem states, the syntactic consequence relation of hierarchic ALs is sound with respect to their semantic consequence relation:

**Theorem 4.6** *If* $A \in Cn_{\mathbf{HAL}}(\Gamma)$*, then* $\Gamma \models_{\mathbf{HAL}} A$*.*

*Proof.*[2] Suppose $A \in Cn_{\mathbf{HAL}}(\Gamma)$ and consider an $M \in \mathcal{M}^{\mathbf{HAL}}_{\Gamma}$. By Definition 4.2, $M \in \mathcal{M}^{\mathbf{AL}^{\mathbf{x_i}}_{(\mathbf{i})}}_{\Gamma}$ for every $i \in I$. By Theorem 2.8, $M \Vdash Cn_{\mathbf{AL}^{\mathbf{x_i}}_{(\mathbf{i})}}(\Gamma)$ for every $i \in I$. As $M$ is an **LLL**-model and by the soundness of **LLL**, $M \Vdash Cn_{\mathbf{LLL}}(\bigcup_{i \in I} Cn_{\mathbf{AL}^{\mathbf{x_i}}_{(\mathbf{i})}}(\Gamma))$. By Definition 4.1, $M \Vdash Cn_{\mathbf{HAL}}(\Gamma)$, hence $M \Vdash A$. ■

Before we turn to the Completeness theorem for **HAL**, let me give some additional properties of the **HAL**-semantics. Note that by Definition 4.2, Definition 2.1 (for $\mathbf{x_i} = \mathbf{r}$) and Definition 2.2 (for $\mathbf{x_i} = \mathbf{m}$), the following fact holds:

**Fact 4.1** *For every* $i \in I$*:* $\mathcal{M}_{\mathbf{HAL}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{x_i}}_{(\mathbf{i})}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{LLL}}(\Gamma)$*.*

Let **HAL** be obtained from the hierarchic combination of the logics $\langle \mathbf{AL}^{\mathbf{x_i}}_{(\mathbf{i})} \rangle_{i \in I}$, and let $\mathbf{HAL^m}$ be defined as before. Then we have:

**Lemma 4.1** $\mathcal{M}_{\mathbf{HAL^m}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{HAL}}(\Gamma)$

*Proof.* Consider an $M \in \mathcal{M}_{\mathbf{HAL^m}}(\Gamma)$. By Definition 4.2, $M$ is an $\mathbf{AL}^{\mathbf{m}}_{(\mathbf{i})}$-model for every $i \in I$. Hence by Theorem 2.3, $M$ is an $\mathbf{AL}^{\mathbf{x_i}}_{(\mathbf{i})}$-model for every $i \in I$. By Definition 4.2, $M \in \mathcal{M}_{\mathbf{HAL}}(\Gamma)$. ■

This means that for the more restricted logics $\mathbf{HAL^r}$ and $\mathbf{HAL^m}$, we can generalize Theorem 2.3 to the hierarchic case:

**Fact 4.2** $\mathcal{M}_{\mathbf{HAL^m}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{HAL^r}}(\Gamma)$*.*

### 4.2.2 Completeness for $\mathbf{HAL^r}$

Just as for the restricted class of logics $\mathbf{SAL^r}$, we can prove that $Cn_{\mathbf{HAL^r}}(\Gamma)$ is complete with respect to the set of all $\mathbf{HAL^r}$-models. The proof of completeness relies essentially on Lemma 2.2 from Section 2.6 (see page 34).

**Theorem 4.7** *If* $\Gamma \models_{\mathbf{HAL^r}} A$*, then* $A \in Cn_{\mathbf{HAL^r}}(\Gamma)$*. (Completeness for* $\mathbf{HAL^r}$*)*

[2] Recall that I use $M \Vdash \Delta$ as a shortcut for “$M \Vdash A$ for every $A \in \Delta$”.

*Proof.* Suppose $A \notin Cn_{\mathbf{HAL^r}}(\Gamma)$. By Definition 4.1, $\bigcup_{i\in I} Cn_{\mathbf{AL^r_{(i)}}}(\Gamma) \nvdash_{\mathbf{LLL}} A$. This implies by the completeness of **LLL** that there is an **LLL**-model $M$ such that $M \Vdash \bigcup_{i\in I} Cn_{\mathbf{AL^r_{(i)}}}(\Gamma)$ and $M \nVdash A$. By Lemma 2.2, $M$ is an $\mathbf{AL^r_{(i)}}$-model of $\Gamma$, for every $i \in I$, hence by Definition 4.2, $M$ is an $\mathbf{HAL^r}$-model of $\Gamma$. As a result, $\Gamma \nvDash_{\mathbf{HAL^r}} A$. ■

### 4.2.3 Restricted Completeness for HAL

Where $\Phi(\Gamma)$ is finite, a similar completeness result can be established for the Minimal Abnormality Strategy. As for the completeness-proof of $\mathbf{HAL^r}$, the proof below relies essentially on a lemma that was discussed in Section 2.6.

**Theorem 4.8** *If $\Phi(\Gamma)$ is finite and $\Gamma \vDash_{\mathbf{HAL}} A$, then $A \in Cn_{\mathbf{HAL}}(\Gamma)$. (Restricted Completeness for* **HAL***)*

*Proof.* Suppose that $\Phi(\Gamma)$ is finite and $A \notin Cn_{\mathbf{HAL}}(\Gamma)$. By Definition 4.1, $\bigcup_{i\in I} Cn_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma) \nvdash_{\mathbf{LLL}} A$. This implies, by the completeness of **LLL**, that (†) there is a **LLL**-model $M$ such that $M \Vdash \bigcup_{i\in I} Cn_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma)$ and $M \Vdash \check{\neg} A$.

By the supposition and Lemma 3.6, for every $i \in I$, $\Phi^{(i)}(\Gamma)$ is finite. By Lemma 2.3 for every $i \in I$, $\mathcal{M}_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma) = \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma))$. By (†), $M \in \mathcal{M}_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma)$ for every $i \in I$. Hence by Definition 4.2, $M \in \mathcal{M}_{\mathbf{HAL}}(\Gamma)$. ■

Notwithstanding the importance of this result, especially for many concrete applications, it is possible to construct a premise set for which completeness fails. An example for the class of logics $\mathbf{HAL^m}$ is presented by Diderik Batens in his [25, Chapter 6]; another one can be found in Section C.1 of Appendix C.

### 4.2.4 An Alternative Semantics for Reliability

There is an alternative semantics for $\mathbf{HAL^r}$ as well, one that is more similar to the semantics of $\mathbf{AL^r}$. To see how it works, suppose that $B \in \Omega^{\mathbf{K}}_{(i)}$ and $B \notin U^{(i)}(\Gamma)$. It follows that $\check{\neg} B \in Cn_{\mathbf{AL^r_{(i)}}}(\Gamma)$, whence also $\check{\neg} B \in Cn_{\mathbf{HAL^r}}(\Gamma)$. In other words, whether or not $B \in U^{(j)}(\Gamma)$ for a $j > i$, does not matter for the hierarchic logic. As long as $B \notin U^{(i)}(\Gamma)$, it will be considered reliable.

Relying on this insight, we can obtain an alternative generic semantics for every logic $\mathbf{HAL^r}$. For the sake of convenience, let $\Omega_{(0)} = \emptyset$. For every $\Delta \subset \Omega$, let $i_\Delta$ be such that $\Delta \subset \Omega_{(i)}$ and $\Delta \not\subset \Omega_{(i-1)}$. Define the preferred fragment of $\Delta$ as follows:

$$\mathsf{pf}(\Delta) =_{\mathsf{df}} \Delta - \Omega_{(i_\Delta - 1)}$$

For example, where $\Delta = \{\Diamond^3 p \wedge \neg p, \Diamond^2 q \wedge \neg q, \Diamond^3 r \wedge \neg r, \Diamond^2 s \wedge \neg s\}$, $\mathsf{pf}(\Delta) = \{\Diamond^3 p \wedge \neg p, \Diamond^3 r \wedge \neg r\}$.

Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-consequences of $\Gamma$, define the set of unreliable formulas for the logic $\mathbf{HAL^r}$:

$$U^\star(\Gamma) =_{\mathsf{df}} \mathsf{pf}(\Delta_1) \cup \mathsf{pf}(\Delta_2) \cup \ldots$$

By this definition, $B \in U^\star(\Gamma)$ iff $B \in \Omega_{(i)} - \Omega_{(i-1)}$ for an $i \in I$, and $B$ occurs in a minimal Dab$_{(i)}$-consequence of $\Gamma$. Here we can observe again that the logic imposes a "hierarchy" on the whole set $\Omega$. If the logic is forced by the premises to choose between the abnormality $A \in \Omega_{(i)}$ and the abnormality $B \in \Omega_{(i+j)} - \Omega_{(i)}$, it will consider $B$ as unreliable.

For example, consider $\Gamma' = \{\Diamond p, \Diamond\Diamond q, \neg p \vee \neg q\}$. Note that either $\Diamond p \wedge \neg p$ or $\Diamond\Diamond q \wedge \neg q$ has to be true in view of these premises and the logic **K**. Informally, this means that either a very plausible belief ($p$) has to be given up, or a slightly less plausible belief ($q$). The hierarchic logic $\mathbf{HK^r}$ then chooses for the least harmful of these two options: $\Gamma \vdash_{\mathbf{HK^r}} \Diamond\Diamond q \wedge \neg q$ and also $\Gamma \vdash_{\mathbf{HK^r}} p$.

However, this does not mean that if $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} B$, $A \in \Omega_{(i)}$ and $B \in \Omega_{(i+j)} - \Omega_{(i)}$, then $A$ is necessarily reliable. For example, where the logic is $\mathbf{HK^r}$, and $\Gamma'' = \{\Diamond p, \Diamond^2 q, \Diamond^3 r, \neg p \vee \neg q, \neg q \vee \neg r\}$, $(\Diamond^2 q \wedge \neg q) \vee (\Diamond^3 r \wedge \neg r)$ is a minimal Dab-consequence of $\Gamma''$. Nevertheless, $q$ is still considered unreliable, in view of the minimal Dab-consequence $(\Diamond p \wedge \neg p) \vee (\Diamond^2 q \wedge \neg q)$. In other words, only $p$ will be freed from suspicion by the hierarchic logic.

The set of all $\mathbf{HAL^r}$-models of $\Gamma$ is the set of all **LLL**-models $M$ for which $Ab(M) \subseteq U^\star(\Gamma)$. This is established by the following theorem:

**Theorem 4.9** *$M \in \mathcal{M}_{\mathbf{HAL^r}}(\Gamma)$ iff ($M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $Ab(M) \subseteq U^\star(\Gamma)$).*

*Proof.* ($\Rightarrow$) Suppose $M \in \mathcal{M}_{\mathbf{HAL^r}}(\Gamma)$. By Fact 4.1, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. So assume that $Ab(M) \not\subseteq U^\star(\Gamma)$. Hence $M \Vdash B$ for a $B \in \Omega - U^\star(\Gamma)$. Note that there is an $i \in I$ such that $B \in \Omega_{(i)} - \Omega_{(i-1)}$.

Assume moreover that $B \in U^{(i)}(\Gamma)$. This means that there is a minimal Dab$_{(i)}$-consequence $Dab(\Delta)$ of $\Gamma$, such that $B \in \Delta$. But then, since $B \in \Omega_{(i)} - \Omega_{(i-1)}$, also $B \in \mathsf{pf}(\Delta)$, whence $B \in U^\star(\Gamma)$ — a contradiction.

Hence $B \notin U^{(i)}(\Gamma)$. This implies that $Ab(M) \cap \Omega_{(i)} \not\subseteq U^{(i)}(\Gamma)$. By Definition 2.1, $M \notin \mathcal{M}_{\mathbf{AL^r_{(i)}}}(\Gamma)$, hence by Definition 4.6, $M \notin \mathcal{M}_{\mathbf{HAL^r}}(\Gamma)$, which contradicts the supposition.

($\Leftarrow$) Suppose $M \notin \mathcal{M}_{\mathbf{HAL^r}}(\Gamma)$. By Definition 4.6, $M \notin \mathcal{M}_{\mathbf{AL^r_{(i)}}}(\Gamma)$ for an $i \in I$. If $M \notin \mathcal{M}_{\mathbf{LLL}}(\Gamma)$, the theorem follows immediately. If $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$, then by Definition 2.1, $M \Vdash B$ for a $B \in \Omega_{(i)} - U^{(i)}(\Gamma)$.

Assume now that $B \in U^\star(\Gamma)$. It follows that there is a minimal Dab-consequence $Dab(\Delta)$ of $\Gamma$ such that $B \in \mathsf{pf}(\Delta)$. Let $j \leq i$ be such that $B \in \Omega_{(j)} - \Omega_{(j-1)}$. It follows that $\Delta \subseteq \Omega_{(j)}$, whence $Dab(\Delta)$ is a minimal $Dab_{(j)}$-consequence of $\Gamma$. Since $j \leq i$, $Dab(\Delta)$ is a minimal $Dab_{(i)}$-consequence of $\Gamma$. But then $B \in U^{(i)}(\Gamma)$ — a contradiction.

It follows that $B \notin U^\star(\Gamma)$. As a result, $Ab(M) \not\subseteq U^\star(\Gamma)$. ■

## 4.3 A Proof Theory for HAL

As for **SAL**, it is possible to define a generic proof theory for all logics $Cn_{\mathbf{HAL}}(\Gamma)$. Again, this proof theory is very similar to the existing proof theory for flat adaptive logics. It draws on older proposals from [25] and [149], but is on the one hand simpler, and on the other hand more generic. In Section 4.3.2, the adequacy of this proof theory with respect to Definition 4.1 is proven. After this general approach, a more straightforward and intuitively appealing proof theory for the class of logics $\mathbf{HAL^r}$ is presented in Section 4.3.3.

### 4.3.1 $I$-marking and *-marking

As for superpositions of ALs, the proof theory of **HAL** uses the inference rules **PREM**, **RU** and **RC** of $\mathbf{AL^r} = \langle \mathbf{LLL}, \bigcup_{i\in I}\Omega_i, \mathbf{r}\rangle$. Again, the only difference lies in the marking definitions. To understand these, consider the following $\mathbf{HK^r}$-proof from $\Gamma_{\mathsf{h}}$:

| | | | |
|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\emptyset$ |
| 2 | $\Diamond\Diamond q$ | PREM | $\emptyset$ |
| 3 | $\Diamond\Diamond r$ | PREM | $\emptyset$ |
| 4 | $\neg p \vee \neg r$ | PREM | $\emptyset$ |
| 5 | $p \,\check{\vee} (\Diamond p \wedge \neg p)$ | 1; RU | $\emptyset$ |
| 6 | $p$ | 5; RC | $\{\Diamond p \wedge \neg p\}$ |
| 7 | $(\Diamond p \wedge \neg p) \,\check{\vee} (\Diamond\Diamond r \wedge \neg r)$ | 1,3,4; RU | $\emptyset$ |

At line 6, we have derived $p$ on the condition $\Diamond p \wedge \neg p$. However, this condition occurs in a minimal Dab-formula on line 7. So we may ask ourselves: should line 6 be marked?

Recall the definition of $\mathbf{K^r_{(1)}}$. The set of abnormalities for this logic is $\Omega^{\mathbf{K}}_{(1)} = \Omega^{\mathbf{K}}_{1} = \{\Diamond A \wedge \neg A \mid A \in \mathcal{W}^l_s\}$. This indicates that the formula on line 7 is not a Dab-formula for the logic $\mathbf{K^r_1}$. More generally, we can prove that $\Diamond p \wedge \neg p$ does not occur in any Dab-consequence $Dab(\Delta)$ of $\Gamma$, with $\Delta \subset \Omega_{(1)}$.[3] Hence $\Gamma \vdash_{\mathbf{K^r_{(1)}}} p$, and therefore also $\Gamma \vdash_{\mathbf{HK^r}} p$. So we need to make sure that line 6 is not marked in view of the Dab-formula on line 7.

The marking definition for hierarchic ALs require a classification of Dab-formulas in view of the respective flat adaptive logics $\mathbf{AL_1}, \mathbf{AL_2}, \ldots$. As in the previous chapter, a Dab$_{(i)}$-formula is the classical disjunction of members of $\Omega_{(i)}$. $Dab(\Delta)$ is a Dab$_{(i)}$-formula at stage $s$ iff it has been derived on the empty condition at stage $s$. $Dab(\Delta)$ is a *minimal* Dab$_{(i)}$-formula at stage $s$ iff it is a Dab$_{(i)}$-formula at stage $s$, and there is no Dab$_{(i)}$-formula $Dab(\Delta')$ at stage $s$ such that $\Delta' \subset \Delta$. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab$_{(i)}$-formulas at stage $s$, $\Sigma^{(i)}_s(\Gamma) =_{\mathsf{df}} \{\Delta_1, \Delta_2, \ldots\}$, $U^{(i)}_s(\Gamma) =_{\mathsf{df}} \bigcup \Sigma^{(i)}_s(\Gamma)$ and $\Phi^{(i)}_s(\Gamma)$ is the set of minimal choice sets of $\Sigma^{(i)}_s(\Gamma)$.

Recall that $\Omega_{(1)} \subseteq \Omega_{(2)} \subseteq \ldots$. Hence, if $Dab(\Delta)$ is a Dab$_{(i)}$-formula at stage $s$, then it is also a $Dab_{(j)}$-formula for any $j \in I$ such that $j > i$. It follows that $\Sigma^{(1)}_s(\Gamma) \subseteq \Sigma^{(2)}_s(\Gamma) \subseteq \ldots$, whence also $U^{(1)}_s(\Gamma) \subseteq U^{(2)}_s(\Gamma) \subseteq \ldots$.

Let us return to our example. In view of the above definitions, $U^{(1)}_7(\Gamma_{\mathsf{h}}) = \emptyset$, and $U^{(2)}_7(\Gamma_{\mathsf{h}}) = \{\Diamond p \wedge \neg p, \Diamond\Diamond r \wedge \neg r\}$. The next thing we have to do is to assure that the marking happens in view of the right logic and hence, for the example above, in view of the right $U^{(i)}_s(\Gamma_{\mathsf{h}})$. If we would take $U^{(2)}(\Gamma_{\mathsf{h}})$ into account for

[3] Roughly speaking, the proof goes as follows. Let $\Theta$ be the set of all sentential letters minus $r$. Let $\Theta^{\neg\Diamond\neg} = \{\neg\Diamond\neg A \mid A \in \Theta\}$. Note that $\Lambda = \Gamma \cup \Theta \cup \Theta^{\neg\Diamond\neg} \cup \{\neg\Diamond r\}$ is $\mathbf{K}$-satisfiable. Every $\mathbf{K}$-model that validates all the members of $\Lambda$, falsifies every member of $\Omega_{(1)}$. Hence every Dab-consequence $Dab(\Delta)$ with $\Delta \subset \Omega_{(1)}$ is false in this $\mathbf{K}$-model of $\Gamma$.

the marking of line 6, then this line would become marked. However, since we only need a condition $\Delta \subset \Omega_1$ to derive $p$ on line 6, there is no need to consider disjunctions that contain higher-ranked abnormalities. Put differently, as soon as line 6 is "safe" with respect to $U^{(1)}_7(\Gamma_{\mathsf{h}})$, we have sufficient reasons to consider the formula on line 6 as derived. The following definitions take care of this:

**Definition 4.3 ($i$-unmarking for $\mathbf{x_i = r}$)** *A line with condition $\Delta$ is $i$- unmarked at stage $s$ iff $\Delta \subseteq \Omega_{(i)} - U^{(i)}_s(\Gamma)$.*

**Definition 4.4 ($i$-unmarking for $\mathbf{x_i = m}$)** *A line with formula $A$ and condition $\Delta$ is $i$-unmarked at stage $s$ iff (i) there is a $\varphi \in \Phi^{(i)}_s(\Gamma)$ such that $\Delta \subseteq \Omega_{(i)} - \varphi$, and (ii) for every $\varphi \in \Phi^{(i)}_s(\Gamma)$, $A$ is derived at stage $s$ on a condition $\Theta \subseteq \Omega_{(i)} - \varphi$.*

According to the above marking definitions, line 6 is 1-unmarked at stage 7. Moreover, it will remain 1-marked in any extension of the proof. As before, we say that a line is $I$-unmarked iff it is $i$-unmarked for an $i \in I$; alternatively, line $l$ is $I$-marked iff there is no $i \in I$ such that $l$ is $i$-unmarked.

We are not home yet. Consider the following continuation of the proof from $\Gamma_{\mathsf{h}}$:

| | | | |
|---|---|---|---|
| 8 | $q \vee (\Diamond\Diamond q \wedge \neg q)$ | 2; RU | $\emptyset$ |
| 9 | $q$ | 8; RC | $\{\Diamond\Diamond q \wedge \neg q\}$ |
| 10 | $p \wedge q$ | 6,9; RU | $\{\Diamond p \wedge \neg p, \Diamond\Diamond q \wedge \neg q\}$ |

Note that the condition of line 10 is not a subset of $\Omega_{(1)}$, though it is a subset of $\Omega_{(2)}$. Hence this line cannot be 1-unmarked. Moreover, since its condition has a non-empty intersection with $U^{(2)}_{10}(\Gamma)$, it can also not be 2-unmarked. However, as pointed out in Section 4.1.1, $p \in Cn_{\mathbf{K^m_1}}(\Gamma)$ and $q \in Cn_{\mathbf{K^m_2}}(\Gamma)$. Since $\mathbf{HK^r}$ is closed under $\mathbf{K}$, this means that there should be a way to derive $p \wedge q$ in a proof from $\Gamma_{\mathsf{h}}$ on an unmarked line. This is made possible by the following additional marking definition:[4]

**Definition 4.5 ($*$-unmarking)** *: a line $l$ is $*$-unmarked at stage $s$ iff, where "$l_1, \ldots, l_n$; RU" is its justification, lines $l_1$ and $\ldots$ and $l_n$ are $I$-unmarked or $*$-unmarked at stage $s$.*

In view of Definition 4.5, line 10 is $*$-unmarked. We say that a line is *unmarked* in an **HAL**-proof iff it is $I$-unmarked or $*$-unmarked in this proof. Note that whether or not a line $l$ is $*$-unmarked depends (in part) on whether the lines from which line $l$ was derived are themselves marked. Those lines can be either $I$-unmarked or $*$-unmarked. The general idea is that if $A_1, \ldots, A_n$ are derived on an unmarked line, then every **LLL**-consequence of $\{A_1, \ldots, A_n\}$ can be derived

[4] Batens [25] also introduces an additional rule RU$*$, which allows one to derive **LLL**-consequences from formulas on preceding lines, on a condition $*$. His definition of $*$-marking is then applied to those lines that have a condition $*$. The same proposal is pursued in [149]. It was only afterwards that I found out that such an additional rule is not needed.

on a $*$-unmarked line in an extension of the proof, by applications of the rule **RU**.

To illustrate the above definitions, let me recapitulate the proof from $\Gamma_{\mathsf{h}}$, but now making explicit which lines are unmarked in view of which definition. To avoid clutter, I omit the superscript that refers to the stage of the marking. The symbol ✓ indicates that a line is *not* unmarked in view of any of the above definitions. Where $i_1, \ldots, i_n \in I$, $-_{i_1,\ldots,i_n}$ indicates that a line is $i_1$-unmarked, ..., and $i_n$-unmarked. Finally, *-unmarking is indicated by the subscript *.

| | | | | |
|---|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 2 | $\Diamond\Diamond q$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 3 | $\Diamond\Diamond r$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 4 | $\neg p \vee \neg r$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 5 | $p \check{\vee} (\Diamond p \wedge \neg p)$ | 1; RU | $\emptyset$ | $-_{1,2,*}$ |
| 6 | $p$ | 5; RC | $\{\Diamond p \wedge \neg p\}$ | $-_{1}$ |
| 7 | $(\Diamond p \wedge \neg p) \check{\vee} (\Diamond\Diamond r \wedge \neg r)$ | 1,3,4; RU | $\emptyset$ | $-_{1,2,*}$ |
| 8 | $q \vee (\Diamond\Diamond q \wedge \neg q)$ | 2; RU | $\emptyset$ | $-_{1,2,*}$ |
| 9 | $q$ | 8; RC | $\{\Diamond\Diamond q \wedge \neg q\}$ | $-_{2}$ |
| 10 | $p \wedge q$ | 6,9; RU | $\{\Diamond p \wedge \neg p, \Diamond\Diamond q \wedge \neg q\}$ | $-_{*}$ |

The following extension of the proof shows that we cannot finally derive $r$ from $\Gamma_{\mathsf{h}}$:

| | | | | |
|---|---|---|---|---|
| 11 | $r \check{\vee} (\Diamond\Diamond r \wedge \neg r)$ | 3;RU | $\emptyset$ | $-_{1,2,*}$ |
| 12 | $r$ | 11;RC | $\{\Diamond\Diamond r \wedge \neg r\}$ | ✓ |

The interpretation of a marked (unmarked) line at stage $s$ remains identical, whether it is marked in view of Definition 4.3, 4.4 or 4.5: the second element of an unmarked line is considered as derived at that stage, the second element of a marked line is considered as not derived at that stage.

As for superpositions of ALs, final derivability is given by Definitions 3.8 and 3.9. So we obtain the derivability relation $\vdash_{\mathbf{HAL}}$. In Section 4.3.2, I prove that this derivability relation is sound and complete with respect to the syntactic consequence relation of **HAL**, as given by Definition 4.1.

### 4.3.2 The Adequacy of Final HAL-Derivability

The following is immediate in view of Lemma 2.1 and the fact that the inference rules of **HAL** are the same as those of **AL**:

**Fact 4.3** *There is a* **HAL**-*proof from* $\Gamma$ *that contains a line at which* $A$ *is derived on a condition* $\Delta \subset \Omega$ *iff* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.

Just as for the proof theory of **SAL**, it will be useful to refer to the ($\Gamma$-specific) stage $\mathsf{g}$ in the remainder – see Section 3.4.3 where this notion is introduced. Recall that for every formula $A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$, $A$ is derived on the condition $\Delta$ at stage $\mathsf{g}$. The following facts will be used in the remainder:

**Fact 4.4** *Each of the following holds:*

1. *A line is marked at stage* $\mathsf{g}$ *iff it is marked in every further extension of stage* $\mathsf{g}$.
2. *For every* $i \in I$, $\Sigma^{(i)}_{\mathsf{g}}(\Gamma) = \Sigma^{(i)}(\Gamma)$, *whence also* $U^{(i)}_{\mathsf{g}}(\Gamma) = U^{(i)}(\Gamma)$ *and* $\Phi^{(i)}_{\mathsf{g}}(\Gamma) = \Phi^{(i)}(\Gamma)$.
3. *Where* $i \in I$: *if* $A \in Cn_{\mathbf{AL^m_{(i)}}}(\Gamma)$, *then for every* $\varphi \in \Phi^{(i)}(\Gamma)$, $A$ *is derived on a condition* $\Delta \subseteq \Omega_{(i)} - \varphi$ *at stage* $\mathsf{g}$.

**Lemma 4.2** *Where* $\Gamma \subseteq \mathcal{W}_s$ *and* $i \in I$: *if* $A$ *is derived on an* $i$*-unmarked line at stage* $\mathsf{g}$, *then* $A \in Cn_{\mathbf{AL_i}}(\Gamma)$.

*Proof.* Suppose the antecedent holds for an $i \in I$. ($\mathbf{x_i} = \mathbf{r}$) It follows that $A$ is derived on a condition $\Delta \subseteq \Omega_{(i)} - U^{(i)}_{\mathsf{g}}(\Gamma)$ at stage $\mathsf{g}$. By Facts 4.3 and 4.4.2, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega_{(i)} - U^{(i)}(\Gamma)$. By Theorem 2.6, $A \in Cn_{\mathbf{AL^r_{(i)}}}(\Gamma)$.

($\mathbf{x_i} = \mathbf{m}$) It follows that for every $\varphi \in \Phi^{(i)}_{\mathsf{g}}(\Gamma)$, $A$ is derived on a condition $\Delta$ at stage $\mathsf{g}$ such that $\Delta \cap \varphi = \emptyset$. By Fact 4.3, for every such $\Delta$, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By Fact 4.4.2 and Theorem 2.7, $A \in Cn_{\mathbf{AL^m_{(i)}}}(\Gamma)$. ■

In the remainder, let $\Lambda^{\mathbf{H}}_{\mathsf{g}}(\Gamma)$ be the set of all formulas $A$ that are derived on a unmarked line at stage $\mathsf{g}$.

**Lemma 4.3** *Where* $\Gamma \subseteq \mathcal{W}_s$: *if* $A \in \Lambda^{\mathbf{H}}_{\mathsf{g}}(\Gamma)$, *then* $A \in Cn_{\mathbf{HAL}}(\Gamma)$.

*Proof.* Suppose $A \in \Lambda^{\mathbf{H}}_{\mathsf{g}}(\Gamma)$. Let $l$ be the line at which $A$ is derived, at stage $\mathsf{g}$. By an induction on the length of the proof up to line $l$, I show that (†) there are $B_1, \ldots, B_n$ such that $\{B_1, \ldots, B_n\} \vdash_{\mathbf{LLL}} A$, and each $B_i$ ($i \leq n$) is derived on an $i$-unmarked line at stage $\mathsf{g}$, for an $i \in I$.

(Base Case) Suppose the proof up to line $l$ contains only one line, i.e. line $l$. It follows that the justification of line $l$ cannot refer to any preceding lines in the proof, whence line $l$ cannot be $*$-unmarked. Hence line $l$ is $i$-unmarked for an $i \in I$.

(Induction Step) Suppose the proof up to line $l$ contains $k+1$ lines. Suppose moreover that line $l$ is not $i$-unmarked for any $i \in I$, whence it is $*$-unmarked. It follows that there are lines $l_1, \ldots, l_m$ with formulae $C_1, \ldots, C_m$ such that for every $i \leq m$: (i) line $l_i$ is unmarked at stage $\mathsf{g}$, (ii) the proof up to line $l$ contains at most $k$ lines and (iii) $\{C_1, \ldots, C_m\} \vdash_{\mathbf{LLL}} A$. By the induction hypothesis, (i) and (ii), for every $j \leq m$, there are $D^j_1, \ldots, D^j_n$ such that $\{D^j_1, \ldots, D^j_n\} \vdash_{\mathbf{LLL}} C_j$, and each $D^j_l$ ($l \leq j_n$) is derived on an $i$-unmarked line at stage $\mathsf{g}$, for an $i \in I$. The rest is immediate in view of (iii) and the monotonicity and transitivity of **LLL**.

By (†) and Lemma 4.2, for every $i \leq n$: $B_i \in Cn_{\mathbf{AL_i}}(\Gamma)$ for an $i \in I$. By Definition 4.1, $A \in Cn_{\mathbf{HAL}}(\Gamma)$. ■

**Theorem 4.10** *Where* $\Gamma \subseteq \mathcal{W}_s$: *if* $\Gamma \vdash_{\mathbf{HAL}} A$, *then* $A \in Cn_{\mathbf{HAL}}(\Gamma)$.

*Proof.* Suppose $A$ is finally derived in an **HAL**-proof $\textsc{p}$ from $\Gamma$. Let $l$ be the line on which $A$ is finally derived in $\textsc{p}$. We may further extend $\textsc{p}$ up to stage $\mathsf{g}$. Assume that line $l$ is marked in the extension. By Fact 4.4.1, line $l$ is marked in every further extension of the proof, which contradicts the fact that $A$ is finally derived in P. Hence line $l$ is unmarked at stage $\mathsf{g}$. By Lemma 4.3, $A \in Cn_{\mathbf{HAL}}(\Gamma)$. ■

**Theorem 4.11** *Where $\Gamma \subseteq \mathcal{W}_s$: if $A \in Cn_{\mathbf{HAL}}(\Gamma)$, then $\Gamma \vdash_{\mathbf{HAL}} A$.*

*Proof.* Suppose $A \in Cn_{\mathbf{HAL}}(\Gamma)$. It follows that there is a finite $\Lambda$, such that $\Lambda \subseteq \bigcup_{i \in I} Cn_{\mathbf{AL_i}}(\Gamma)$ and $\Lambda \vdash_{\mathbf{LLL}} A$. Let $\Theta = \Lambda \cap Cn_{\mathbf{LLL}}(\Gamma)$ and let $\Theta' = \Lambda - \Theta = \{B_1, \ldots, B_n\}$. By the compactness and monotonicity of **LLL**, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathbf{LLL}} \Theta$. Note that $\Gamma_0 \cup \Theta' \vdash_{\mathbf{LLL}} \Lambda$, whence by the transitivity of **LLL**, (†) $\Gamma_0 \cup \Theta' \vdash_{\mathbf{LLL}} A$. Also, note that (‡) for every $i \leq n$, there is a $j_i \in I$ such that $B_i \in Cn_{\mathbf{AL_{j_i}}}(\Gamma)$. For every $i \leq n$, let $\Gamma_i$ be a finite subset of $\Gamma$ such that the following holds:

(1) where $\mathbf{x_{j_i}} = \mathbf{r}$: $\Gamma_i \vdash_{\mathbf{LLL}} B_i \check{\vee} Dab(\Delta_i)$ for a $\Delta_i \subseteq \Omega_{(j_i)} - U^{(j_i)}(\Gamma)$.
(2) where $\mathbf{x_{j_i}} = \mathbf{m}$: $\Gamma_i \vdash_{\mathbf{LLL}} B_i \check{\vee} Dab(\Delta_i)$ for a $\Delta_i \subseteq \Omega_{(j_i)} - \varphi$, for a $\varphi \in \Phi^{(j_i)}(\Gamma)$.

Note that for every $B_i$, there is such a $\Gamma_i$, in view of (‡), Theorems 2.6, resp. 2.7, and the compactness of **LLL**.

We start a **HAL**-proof from $\Gamma$ as follows: (i) introduce every member of $\Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_n$ on the empty condition, by the rule PREM, (ii) for every $B_i \in \Theta'$, derive $B_i$ by the rule RC from the formulas in $\Gamma_i$, on a line $l_i$ with condition $\Delta_i$, and (iii) derive $A$ by the rule RU on a line $l$, from all the formulas $C \in \Gamma_0 \cup \Theta'$. Note that (iii) is possible in view of (†). Let $s$ be the last stage of this (finite) proof.

Note that for each $B_i \in \Theta$, $\Delta_i \neq \emptyset$ – otherwise, $B_i$ would be an **LLL**-consequence of $\Gamma$. Hence the only Dab-formulas that have been derived at stage $s$, are members of $\Gamma_0 \cup \ldots \cup \Gamma_n \cap \Omega$, which is a subset of $\Gamma$. We can derive that for every $k \in I$, $U^{(k)}_s(\Gamma) \subseteq U^{(k)}(\Gamma)$, and $\Sigma^{(k)}_s(\Gamma) \subseteq \varphi$ for every $\varphi \in \Phi^{(k)}(\Gamma)$.

By (1), where $\mathbf{x_{j_i}} = \mathbf{r}$: $\Delta_i \subseteq \Omega_{(i_j)} - U^{(i_j)}_s(\Gamma)$. By (2), where $\mathbf{x_{j_i}} = \mathbf{m}$: $\Delta_i \subseteq \Omega_{(i_j)} - \varphi$ for every $\varphi \in \Phi^{(i_j)}_s(\Gamma)$. It follows that all the lines $l_i$ are $j_i$-unmarked at stage $s$. Since also all the formulas $C \in \Gamma_0$ are derived on a 1-unmarked line (their condition is the empty set), it follows that line $l$ is $*$-unmarked at stage $s$.

Suppose line $l$ is not $*$-unmarked in an extension of this proof. We may then further extend the proof up to stage $\mathsf{g}$. Note that if $\mathbf{x_{j_i}} = \mathbf{m}$, then by Fact 4.4.3, for every $\varphi \in \Phi^{(j_i)}(\Gamma)$, $B_i$ is derived on a condition $\Delta \subseteq \Omega_{(j_i)} - \varphi$ at stage $\mathsf{g}$. By (2) and Fact 4.4.2, $\Delta_i \subseteq \Omega_{(j_i)} - \varphi$ for a $\varphi \in \Phi^{(j_i)}_{\mathsf{g}}(\Gamma)$. Also, if $\mathbf{x_{j_i}} = \mathbf{r}$, then in view of (1) and Fact 4.4.2, $\Delta_i \subseteq \Omega_{(j_i)} - U^{(j_i)}_{\mathsf{g}}(\Gamma)$. We can infer that all the lines $l_i$ are $j_i$-unmarked at stage $\mathsf{g}$. Finally, all the lines on which the members of $\Gamma_0$ were derived are marked at stage $\mathsf{g}$, since their condition is the empty set. As a result, line $l$ is $*$-unmarked at stage $\mathsf{g}$. By Definitions 3.8 and 3.9, $\Gamma \vdash_{\mathbf{HAL}} A$. ■

The following corollaries are immediate in view of Theorems 4.6, 4.7 and 4.8:

**Corollary 4.2** *Where $\Gamma \subseteq \mathcal{W}_s$: $\Gamma \vdash_{\mathbf{HAL^r}} A$ iff $\Gamma \models_{\mathbf{HAL^r}} A$.*

**Corollary 4.3** *Where $\Gamma \subseteq \mathcal{W}_s$ and $\Phi(\Gamma)$ is finite: $\Gamma \vdash_{\mathbf{HAL}} A$ iff $\Gamma \models_{\mathbf{HAL}} A$.*

### 4.3.3 An Alternative Proof Theory for Reliability

It is fairly easy to obtain an alternative proof theory for $\mathbf{HAL^r}$, which mimics the alternative semantics spelled out in Section 4.2.4. The inference rules for this proof theory are exactly the same as for the flat logic $\mathbf{AL^r}$ – see Section 2.3. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-formulas at stage $s$, we define a stage-dependent variant of $U^\star(\Gamma)$ as follows:

$$U^\star_s(\Gamma) =_{\mathsf{df}} \mathsf{pf}(\Delta_1) \cup \mathsf{pf}(\Delta_2) \cup \ldots$$

**Definition 4.6** $\mathbf{HAL^r}\star$*-Marking: a line $l$ is marked at stage $s$ iff, where $\Delta$ is its condition, $\Delta \cap U^\star_s(\Gamma) \neq \emptyset$.*

Final derivability for this proof theory can be given by Definitions 3.8 and 3.9 – this yields a proof theory that is sound and complete with respect to $\mathbf{HAL^r}$. However, in this particular case, it is possible to do without the notion of infinite extensions in the definitions for final derivability. This has an important consequence for the upper bound complexity of $\mathbf{HAL^r}$, which is identical to that of flat adaptive logics that use the Reliability Strategy.[5]

**Definition 4.7** *$A$ is* finally derived *from $\Gamma$ on line $l$ of a finite stage $s$ of a $\mathbf{HAL^r}\star$-proof iff (i) $A$ is the second element of line $l$, (ii) line $l$ is unmarked at stage $s$, and (iii) for every finite extension of the proof at stage $s$, in which line $l$ is marked, there is a further finite extension in which line $l$ is unmarked again.*

**Definition 4.8** *$\Gamma \vdash_{\mathbf{HAL^r}\star} A$ iff $A$ is finally derived on a line of a $\mathbf{HAL^r}\star$-proof from $\Gamma$.*

In the remainder of this section, it is proven that $\Gamma \vdash_{\mathbf{HAL^r}\star} A$ iff $A \in Cn_{\mathbf{HAL^r}}(\Gamma)$.

**Lemma 4.4** *For every $i \in I$: if $A \in Cn_{\mathbf{AL^r_{(i)}}}(\Gamma)$, then $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega$ and $\Delta \cap U^\star(\Gamma) = \emptyset$.*

*Proof.* Suppose $A \in Cn_{\mathbf{AL^r_{(i)}}}(\Gamma)$, with $i \in I$. By Theorem 2.6.1, (†) $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega_{(i)}$ and $\Delta \cap U^{(i)}(\Gamma) = \emptyset$. Since $\Omega_{(i)} \subset \Omega$, $\Delta \subset \Omega$. Assume that $\Delta \cap U^\star(\Gamma) \neq \emptyset$. By the definition of $U^\star(\Gamma)$, there is a minimal Dab-consequence $Dab(\Theta)$, with $\Theta \subset \Omega_{(j)}$ and $\Theta \not\subset \Omega_{(j-1)}$, such that $\mathsf{pf}(\Theta) \cap \Delta \neq \emptyset$, whence $(\Theta - \Omega_{(j-1)}) \cap \Delta \neq \emptyset$. As a result, $\Theta \cap \Delta \neq \emptyset$ and $j \leq i$. This implies that $Dab(\Theta)$ is a minimal Dab$_{(i)}$-consequence of $\Gamma$. But then $\Delta \cap U^{(i)}(\Gamma) \neq \emptyset$, which contradicts (†). ■

**Lemma 4.5** *Where $\Gamma \subseteq \mathcal{W}_s$, $i \in I$, and $\Delta \subset \Omega_{(i)} - \Omega_{(i-1)}$ is non-empty and finite: if $\Delta \cap U^\star(\Gamma) = \emptyset$, then $\check{\neg} Dab(\Delta) \in Cn_{\mathbf{HAL^r}}(\Gamma)$.*

[5] As explained in Chapter 2, the upper bound complexity of $\mathbf{AL^r}$ can be reduced to $\Sigma^0_3$ — see page 33.

*Proof.* Suppose (†) $\Delta \cap U^\star(\Gamma) = \emptyset$ for a finite $\Delta \subset \Omega_{(i)} - \Omega_{(i-1)}$. Assume that (‡) $\Delta \cap U^{(i)}(\Gamma) \neq \emptyset$. This implies that there is a minimal $\text{Dab}_{(i)}$-consequence $Dab(\Theta)$ of $\Gamma$, such that $\Delta \cap \Theta \neq \emptyset$. Since $\Delta \subset \Omega_{(i)} - \Omega_{(i-1)}$, $\Theta \not\subset \Omega_{(i-1)}$. By the definition of $U^\star(\Gamma)$, $\Theta - \Omega_{(i-1)} \subseteq U^\star(\Gamma)$. As a result, $\Delta \cap U^\star(\Gamma) \neq \emptyset$, which contradicts (†). Hence (‡) is false: $\Delta \cap U^{(i)}(\Gamma) = \emptyset$, whence $\check{\neg} Dab(\Delta) \in Cn_{\mathbf{AL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma)$. By Definition 4.1 and the reflexivity of **LLL**, $\check{\neg} Dab(\Delta) \in Cn_{\mathbf{HAL}^{\mathbf{r}}}(\Gamma)$. ■

**Theorem 4.12** *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $A \in Cn_{\mathbf{HAL}^{\mathbf{r}}}(\Gamma)$ *iff* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *for a finite* $\Delta \subset \Omega$ *such that* $\Delta \cap U^\star(\Gamma) = \emptyset$.

*Proof.* ($\Rightarrow$) Suppose $A \in Cn_{\mathbf{HAL}^{\mathbf{r}}}(\Gamma)$. By the compactness of **LLL**, there are $B_1, \ldots, B_n$ such that $B_1 \in Cn_{\mathbf{AL}^{\mathbf{r}}_{\mathbf{i_1}}}(\Gamma), \ldots, B_n \in Cn_{\mathbf{AL}^{\mathbf{r}}_{\mathbf{i_n}}}(\Gamma)$ and $\{B_1, \ldots, B_n\} \vdash_{\mathbf{LLL}} A$. By Lemma 4.4: for every $j \in \{1, \ldots, n\}$: $\Gamma \vdash_{\mathbf{LLL}} B_j \check{\vee} Dab(\Delta_j)$ for a $\Delta_j \subset \Omega$ for which $\Delta \cap U^\star(\Gamma) = \emptyset$. By **CL**-properties, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_1 \cup \ldots \cup \Delta_n)$, where $(\Delta_1 \cup \ldots \cup \Delta_n) \subset \Omega$, and $(\Delta_1 \cup \ldots \cup \Delta_n) \cap U^\star(\Gamma) = \emptyset$.

($\Leftarrow$) Suppose that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta$ and $\Delta \cap U^\star(\Gamma) = \emptyset$. For every $i \in I$, define $\Delta_i = \Delta \cap (\Omega_{(i)} - \Omega_{(i-1)})$. Since $\Delta$ is finite, $\Delta = \Delta_1 \cup \ldots \cup \Delta_n$ for an $n \in I$. By Lemma 4.5, $\check{\neg} Dab(\Delta_i) \in Cn_{\mathbf{HAL}^{\mathbf{r}}}(\Gamma)$ for every $i \in \{1, \ldots, n\}$. By **CL**-properties and Definition 4.1, $A \in Cn_{\mathbf{HAL}^{\mathbf{r}}}(\Gamma)$. ■

**Lemma 4.6** *There is a* $\mathbf{HAL}^{\mathbf{r}}\star$*-proof from* $\Gamma$ *that contains a line on which* $A$ *is derived on the condition* $\Delta \subset \Omega$ *iff* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.

*Proof.* Immediate in view of Lemma 2.1 and the fact that the inference rules **AL** are identical to those of $\mathbf{HAL}^{\mathbf{r}}\star$. ■

**Lemma 4.7** *If* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *for a finite* $\Delta \subset \Omega$ *such that* $\Delta \cap U^\star(\Gamma) = \emptyset$*, then there is a finite* $\mathbf{HAL}^{\mathbf{r}}\star$*-proof from* $\Gamma$ *such that* $A$ *is derived in it on an unmarked line with condition* $\Delta$.

*Proof.* Suppose the antecedent holds. Due to the compactness of **LLL**, there is a $\Gamma' = \{A_1, \ldots, A_n\} \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_\psi)$. Let the adaptive proof P be constructed as follows. At line 1 we introduce the premise $A_1$ by PREM, ..., and at line $n$ we introduce the premise $A_n$ by PREM. At line $n+1$ we derive $A$ by RC on the condition $\Delta$. Let $s$ be the stage consisting of lines 1 up to $n+1$. Since $\Gamma' \subseteq \Gamma \subseteq \mathcal{W}$, all Dab-formulas $B_1, \ldots, B_m$ that have been derived at stage $s$ (if any) are members of $\Omega$. Hence $U^\star_s(\Gamma') = \{B_1, \ldots, B_m\} \subseteq U^\star(\Gamma)$. Since $\Delta \cap U^\star(\Gamma) = \emptyset$, also $\Delta \cap U^\star_s(\Gamma') = \emptyset$. Thus, line $n+1$ is unmarked. ■

**Theorem 4.13** *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $\Gamma \vdash_{\mathbf{HAL}^{\mathbf{r}}\star} A$ *iff* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *for a finite* $\Delta \subset \Omega$ *such that* $\Delta \cap U^\star(\Gamma) = \emptyset$.

*Proof.* ($\Rightarrow$) Suppose $\Gamma \vdash_{\mathbf{HAL}^{\mathbf{r}}\star} A$. Hence there is a finite $\mathbf{HAL}^{\mathbf{r}}\star$-proof from $\Gamma$ such that $A$ is finally derived in it on a line $l$. Let $\Delta$ be the condition of line $l$. Note that by Lemma 4.6, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.

Assume that $\Delta \cap U^\star(\Gamma) \neq \emptyset$. It follows that there is a minimal Dab-consequence $Dab(\Theta)$ of $\Gamma$ such that $\mathsf{pf}(\Theta) \cap \Delta \neq \emptyset$. By the compactness of **LLL**, there are $C_1, \ldots, C_m \in \Gamma$ such that $\{C_1, \ldots, C_m\} \vdash_{\mathbf{LLL}} Dab(\Theta)$. So we may extend the proof by introducing every premise $C_i$ $(i \leq m)$ on the condition $\emptyset$, and derive $Dab(\Theta)$ from the latter. Let $t$ be the last stage of this (finite) proof. Since $Dab(\Theta)$ is a minimal Dab-consequence of $\Gamma$, we can derive that at every later stage $t'$, $\mathsf{pf}(\Theta) \subseteq U^\star_{t'}(\Gamma)$. But then line $l$ is marked in every (finite or infinite) extension of the proof at stage $t$ — a contradiction.

($\Leftarrow$) Suppose that (†) $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a finite $\Delta \subset \Omega$ such that $\Delta \cap U^\star(\Gamma) = \emptyset$. By Lemma 4.7, there is a finite $\mathbf{HAL^r}\star$-proof from $\Gamma$ in which $A$ is derived on the condition $\Delta$, on an unmarked line $n+1$.

Suppose that line $n+1$ is marked in a finite extension of this proof. Where $t$ is the stage of this extension, let $\Sigma_t(\Gamma) = \{\Theta_1, \ldots, \Theta_m\}$. For every $i \leq m$, let $\Theta'_i \subseteq \Theta_i$ be a minimal Dab-consequence of $\Gamma$.

By the compactness of **LLL**, for every $i \leq m$, there are $C^i_1, \ldots, C^i_{k_i} \in \Gamma$ such that $\{C^i_1, \ldots, C^i_{k_i}\} \vdash_{\mathbf{LLL}} Dab(\Theta'_i)$. So we may further extend the proof by introducing every premise $C^i_{i_l}$ $(i \leq m$ and $l \leq i_k)$ on the condition $\emptyset$, and derive all formulas $Dab(\Theta'_i)$ from these premises. Where $t'$ is the stage of this (finite) further extension, we can derive that $U^\star_{t'}(\Gamma) \subseteq U^\star(\Gamma)$.[6] It follows that $U^\star_{t'}(\Gamma) \cap \Delta = \emptyset$, whence line $n+1$ is unmarked at stage $t'$. By Definition 2.6, $\Gamma \vdash_{\mathbf{HAL^r}\star} A$. ■

By Theorems 4.12 and 4.13, we obtain:

**Theorem 4.14** *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $\Gamma \vdash_{\mathbf{HAL^r}\star} A$ *iff* $A \in Cn_{\mathbf{HAL^r}}(\Gamma)$*.*

## 4.4 Metatheory of HAL

This section contains some interesting meta-theoretic properties of **HAL**. I will sometimes restrict the focus to $\mathbf{HAL^r}$ and $\mathbf{HAL^m}$ to simplify the discussion – the reader can easily see when and how the results generalize to the whole class of hierarchic logics. As in the previous chapter, I first discuss the semantic property of Strong Reassurance, and afterwards consider properties of the syntactic consequence relation of **HAL**, as given by Definition 4.1.

### 4.4.1 Strong Reassurance

Recall the discussion of the different alternatives for the regular Strong Reassurance property in the penultimate section of the previous chapter (see pages 81-82). As shown below, and in contrast to superposition-logics, all logics **HAL** satisfy property SR3 from that section: if an **LLL**-model $M$ of $\Gamma$ is disselected by the logic **HAL**, then there is a selected model $M'$ and an $i \in I$ such that $Ab(M') \cap \Omega_{(i)} \subset Ab(M) \cap \Omega_{(i)}$. It suffices to prove that $\mathbf{HAL^m}$ has this property – as the proof of Theorem 4.16 shows, we can easily generalize this result to all logics **HAL** in view of Lemma 4.1. In the remainder of this section, let $Ab_{(i)}(M)$ denote the set $\{B \in \Omega_{(i)} \mid M \Vdash B\}$.

**Theorem 4.15** *If* $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{HAL^m}}(\Gamma)$*, then there is an* $i \in I$ *and an* $M' \in \mathcal{M}_{\mathbf{HAL^m}}(\Gamma)$ *such that* $Ab_{(i)}(M') \subset Ab_{(i)}(M)$*.*

[6] Note that the only minimal Dab-formulas at stage $t'$ are $Dab(\Theta'_1), \ldots, Dab(\Theta'_m)$, and abnormalities $B \in \Gamma$ that were introduced by the premise rule.

*Proof.* Suppose $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{HAL^m}}(\Gamma)$. Let $i \in I$ be smallest such that $M \notin \mathcal{M}_{\mathbf{AL^m_{(i)}}}(\Gamma)$, whence (1) for all $j < i$, $M \in \mathcal{M}_{\mathbf{AL^m_{(j)}}}(\Gamma)$. By the Strong Reassurance of $\mathbf{AL^m_{(i)}}$ (see Theorem 2.12), there is an $\mathbf{AL^m_{(i)}}$-model $M'$ such that $Ab_{(i)}(M') \subset Ab_{(i)}(M)$. This implies that for all $j < i$, $Ab_{(j)}(M') \subseteq Ab_{(j)}(M)$, whence by (1), it follows that (2) $M' \in \mathcal{M}_{\mathbf{AL^m_{(j)}}}(\Gamma)$ for all $j \in I$, $j \leq i$. Let $\varphi_{(i)} = Ab_{(i)}(M')$, whence (3) $\varphi_{(i)} \subset Ab_{(i)}(M)$. In the remainder of this proof, I show that there is an $M'' \in \mathcal{M}_{\mathbf{HAL^m}}(\Gamma)$ such that $Ab_{(i)}(M'') = \varphi_{(i)}$, and hence by (3), $Ab_{(i)}(M'') \subset Ab_{(i)}(M)$.

Note that by (2) and Theorem 2.1, the following holds:

(4) $\varphi_{(i)} \cap \Omega_{(j)} \in \Phi^{(j)}(\Gamma)$ for all $j \leq i$.

Define the sets $\varphi_{(i+1)}, \varphi_{(i+2)}, \ldots$ recursively, as follows: for all $j \in I, j > i$, $\varphi_{(j)}$ is an arbitrary element in $\Phi^{[j]}_{\varphi_{(j-1)}}$.[7] In view of the construction, we can derive that (5) for every $j \in I, j > i$, $\varphi_{(j)} \subseteq \Omega_{[j]}$.[8] Let $\varphi^{\oplus} = \varphi_{(i)} \cup \varphi_{(i+1)} \cup \ldots$. It follows from (5) that

(6) $\varphi^{\oplus} \cap \Omega_{(i)} = \varphi_{(i)}$.

In view of Lemma 3.2, we can moreover derive that

(7) for all $j \geq i$, $\varphi^{\oplus} \cap \Omega_{(j)} \in \Phi^{(j)}(\Gamma)$.

From (4) and (7), we can derive:

(8) $\varphi^{\oplus} \in \Phi^{(k)}(\Gamma)$ for every $k \in I$.

I will now prove that $\varphi^{\oplus}$ is a minimal choice set of $\Sigma(\Gamma)$. First of all, assume that $\varphi^{\oplus}$ is not a choice set of $\Sigma(\Gamma)$. Hence there is a minimal Dab-consequence of $\Gamma$, say $Dab(\Theta)$, such that $\varphi^{\oplus} \cap \Theta = \emptyset$. Since $\Theta$ is finite, there is a $k \in I$: $\Theta \subset \Omega_{(k)}$, whence $(\varphi^{\oplus} \cap \Omega_{(k)}) \cap \Theta = \emptyset$. Since $Dab(\Theta)$ is a minimal Dab$_k$-consequence of $\Gamma$, $\varphi^{\oplus} \cap \Omega_{(k)} \notin \Phi^{(k)}(\Gamma)$. This contradicts (8). Hence $\varphi^{\oplus}$ is a choice set of $\Sigma(\Gamma)$.

Assume now that $\varphi^{\oplus}$ is not a minimal choice set of $\Sigma(\Gamma)$. Since $\varphi^{\oplus}$ is a choice set of $\Sigma(\Gamma)$, this implies that there is a $\psi \in \Phi(\Gamma) = \psi \subset \varphi^{\oplus}$. Hence there is a $A \in \varphi^{\oplus} - \psi$, where $A \in \Omega_{(k)}$ for a $k \in I$. However, since by (8), $\varphi^{\oplus} \cap \Omega_{(k)} \in \Phi^{(k)}(\Gamma)$, we have by Lemma 3.1 that there is a minimal Dab$_k$-consequence $Dab(\Theta)$ of $\Gamma$, for which $\Theta \cap \varphi^{\oplus} = A$, hence $\Theta \cap \psi = \emptyset$. This contradicts the fact that $\psi$ is a choice set of $\Sigma(\Gamma)$. As a result, $\varphi^{\oplus} \in \Phi(\Gamma)$.

By Theorem 2.2, there is an $M'' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$, such that $Ab(M'') = \varphi^{\oplus}$. By Theorem 2.1 and (8), this model is an $\mathbf{AL^m_{(k)}}$-model of $\Gamma$ for every $k \in I$. Hence by Definition 4.2, it is an $\mathbf{HAL^m}$-model of $\Gamma$. Furthermore, by (6), $Ab_{(i)}(M'') = \varphi_{(i)}$. ■

**Theorem 4.16** *If $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{HAL}}(\Gamma)$, then there is an $i \in I$ and an $M' \in \mathcal{M}_{\mathbf{HAL}}(\Gamma)$ such that $Ab_{(i)}(M') \subset Ab_{(i)}(M)$.*

*Proof.* Suppose $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{HAL}}(\Gamma)$. By Lemma 4.1, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{HAL^m}}(\Gamma)$. By Theorem 4.15, there is an $i \in I$ and an $M' \in \mathcal{M}_{\mathbf{HAL^m}}(\Gamma)$ such that $Ab_{(i)}(M') \subset Ab_{(i)}(M)$. By Lemma 4.1, $M' \in \mathcal{M}_{\mathbf{HAL}}(\Gamma)$. ■

[7] This set was defined on page 48.

[8] Recall that, where $j \in I - \{1\}$, $\Omega_{[j]} =_{\mathsf{df}} \Omega_j - \Omega_{(j-1)} = \Omega_j - (\Omega_1 \cup \ldots \cup \Omega_{j-1})$.

**Corollary 4.4** *If* $\Gamma$ *has* **LLL***-models, it has* **HAL***-models. (Semantic Reassurance)*

### 4.4.2 Cumulative Indifference and Idempotence

As in the previous chapter, I will only discuss the cumulative indifference property for the restricted classes of hierarchic logics, where each flat logic in the combination uses the same strategy: $\mathbf{HAL^r}$ and $\mathbf{HAL^m}$. Also, I treat Cumulative Transitivity and Cautious Monotonicity separately.

**Cumulative Transitivity.** First of all, $\mathbf{HAL^r}$ does not in general have the Fixed Point property:

**Proposition 4.1** *There is a hierarchic adaptive logic* $\mathbf{HAL^r}$ *and a* $\Gamma \subseteq \mathcal{W}_s$ *such that* $Cn_{\mathbf{HAL^r}}(Cn_{\mathbf{HAL^r}}(\Gamma)) \neq Cn_{\mathbf{HAL^r}}(\Gamma)$.

*Proof.* Consider the hierarchic logic $\mathbf{HK2^r}$ from Section 4.1. Let

$$\Gamma_{\mathsf{t}} = \{\Diamond p, \Diamond\Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg r \vee \neg q\}.$$

Note that $\Gamma_{\mathsf{t}}$ has no minimal Dab$_1$-consequences; the minimal Dab$_2$-consequence of $\Gamma_{\mathsf{t}}$ are:

$$(\Diamond p \wedge \neg p) \vee (\Diamond\Diamond q \wedge \neg q)$$
$$(\Diamond\Diamond r \wedge \neg r) \vee (\Diamond\Diamond q \wedge \neg q)$$

Hence $U^{(1)}(\Gamma_{\mathsf{t}}) = \emptyset$ and $U^{(2)}(\Gamma_{\mathsf{t}}) = \{\Diamond p \wedge \neg p, \Diamond\Diamond q \wedge \neg q, \Diamond\Diamond r \wedge \neg r\}$. It is easy to see that $\Gamma_{\mathsf{t}} \nvdash_{\mathbf{HK2^r}} r$: consider a $\mathbf{K}$-model $M$ of $\Gamma_{\mathsf{t}}$ such that (i) $M \Vdash \check{\neg} A$ for every $A \in \Omega_1^{\mathbf{K}} \cup (\Omega_2^{\mathbf{K}} - \{!^2 q, !^2 r\})$ and (ii) $M \Vdash !^2 q, !^2 r$. In view of the above Dab-consequences, this model is a $\mathbf{K^r_{(1)}}$-model of $\Gamma_{\mathsf{t}}$ and a $\mathbf{K^r_{(2)}}$-model of $\Gamma_{\mathsf{t}}$, whence it is an $\mathbf{HK2^r}$-model of $\Gamma_{\mathsf{t}}$. In view of (ii), $M \nVdash r$.

The only minimal Dab-consequence of $Cn_{\mathbf{HK2^r}}(\Gamma_{\mathsf{t}})$ is $(\Diamond\Diamond q \wedge \neg q)$. That is, since $\neg(\Diamond p \wedge \neg p)$ is $\mathbf{K^r_{(1)}}$-derivable from $\Gamma_{\mathsf{t}}$, it is $\mathbf{HK2^r}$-derivable from $\Gamma_{\mathsf{t}}$, and hence $\Diamond\Diamond q \wedge \neg q \in Cn_{\mathbf{HK2^r}}(\Gamma_{\mathsf{t}})$. This means that $\Diamond\Diamond r \wedge \neg r$ is no longer considered unreliable when the consequence relation is iterated, such that $r \in Cn_{\mathbf{HK2^r}}(Cn_{\mathbf{HK2^r}}(\Gamma_{\mathsf{t}}))$. ■

By the reflexivity of $\mathbf{HK2^r}$ and Lemma 2.7, we can immediately infer that $\mathbf{HK2^r}$ is also not cumulatively transitive. Also, logics $\mathbf{HAL^m}$ are not in general idempotent or cumulatively transitive – I refer to Section C.2 in Appendix C for a counterexample. In the same chapter, it will be shown that given certain (weak) restrictions on the premise sets, $\mathbf{HAL^m}$ is cumulatively transitive and hence also idempotent.

**Cautious Monotonicity** Logics in $\mathbf{HAL^x}$-format are not in general cautiously monotonic. For $\mathbf{HAL^r}$, this can be seen in view of a very simple example. Let $\Gamma_{\mathsf{m}} = \{\Diamond p, \Diamond\Diamond q, \neg p \vee \neg q\}$. Note that $p \in Cn_{\mathbf{HK2^r}}(\Gamma_{\mathsf{m}})$, whence also $!^2 q \in Cn_{\mathbf{HK2^r}}(\Gamma_{\mathsf{m}})$. This implies that e.g. $!^2 q \vee !^2 r \in Cn_{\mathbf{HK2^r}}(\Gamma_{\mathsf{m}})$. Also, $!^2 q$ is the only unreliable abnormality in view of $\mathbf{HK2^r}$, whence e.g. $\neg!^2r$ is an $\mathbf{HK2^r}$-consequence of $\Gamma_{\mathsf{m}}$.

Note that $!^2q\,\check{\vee}!^2r$ is a minimal Dab-consequence of $\Gamma_{\mathsf{m}} \cup \{!^2q\vee!^2r\}$. Hence $!^2r$ is an unreliable formula for the logic $\mathbf{K^r_{(2)}}$: $!^2r \in U^{(2)}(\Gamma \cup \{!^2q\vee!^2r\})$. As a result, $\neg!^2r \notin Cn_{\mathbf{HK2^r}}(\Gamma_{\mathsf{m}} \cup \{!^2q\vee!^2r\})$.

For logics $\mathbf{HAL^m}$, a counterexample of the Cautious Monotonicity property is presented in Section C.4 of Appendix C. However, it can be shown that the semantic consequence relation of $\mathbf{HAL^m}$ is cumulatively indifferent.[9] Hence, in view Theorem 4.8, it seems plausible (but was not yet proven) that given certain restrictions on the premise sets – e.g. that $\Phi(\Gamma)$ is finite –, $\mathbf{HAL^m}$ is cautiously monotonic.[10]

### 4.4.3 Normal Premise Sets

Suppose that $\Gamma \vdash_{\mathbf{HAL^r}} A$. By Theorem 4.12, there is a $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \,\check{\vee}\, Dab(\Delta)$ and $\Delta \cap U^\star(\Gamma) = \emptyset$. It immediately follows that $\Gamma \vdash_{\mathbf{ULL}} A$. Together with Corollary 4.1, this implies:

**Theorem 4.17** $Cn_{\mathbf{HAL}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$

Also, **HAL** is just as well-behaved for premise sets up to level $i$ as **SAL**:

**Theorem 4.18** *If $\Gamma$ is normal up to level $i$, then $Cn_{\mathbf{ULL_{(i)}}}(\Gamma) \subseteq Cn_{\mathbf{HAL}}(\Gamma)$.*

*Proof.* Immediate in view of the fact that $Cn_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma) \subseteq Cn_{\mathbf{HAL^m}}(\Gamma)$ (see Definition 4.1), Definition 3.15, and Theorem 2.17. ■

In view of Theorem 4.18, Theorem 4.17 and the monotonicity and compactness of **LLL**, the proof of the following can be safely left to the reader:

**Theorem 4.19** *If $\Gamma$ is normal, then $Cn_{\mathbf{HAL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.*

### 4.4.4 Relations Between Logics

In Chapter 1, I wrote that prioritized ALs are often stronger than their flat counterparts. This point requires some qualification. Recall that $\mathbf{AL^x}$ was defined by (i) **LLL**, (ii) $\Omega = \bigcup_{i\in I}\Omega_i$ and (iii) the strategy $x \in \{\mathbf{r}, \mathbf{m}\}$. We have already seen numerous examples where $\mathbf{HK^r}$ ($\mathbf{HK^m}$) is stronger than $\mathbf{K^r}$ ($\mathbf{HK^r}$), but does this hold for all premise sets $\Gamma$, and all logics $\mathbf{HAL^x}$? Where $\mathbf{x} = \mathbf{r}$, the hierarchic logic is always at least as strong as its flat nephew, as is stated by the following theorem and the subsequent proposition:

**Theorem 4.20** $Cn_{\mathbf{AL^r}}(\Gamma) \subseteq Cn_{\mathbf{HAL^r}}(\Gamma)$.

[9] This is an immediate corollary of the equivalence of this semantics to the $\mathbf{AL^m_{\sqsubset}}$-semantics (as shown in Chapter 6), and the fact that the latter is cumulatively indifferent (see Chapter 5).

[10] The difficulty lies in the proof of the following lemma: whenever $\Gamma' \subseteq Cn_{\mathbf{HAL}}(\Gamma)$ and $\Phi(\Gamma)$ is finite, then also $\Phi(\Gamma \cup \Gamma')$ is finite.

*Proof.* Suppose $A \in Cn_{\mathbf{AL^r}}(\Gamma)$. By Theorem 2.6, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, for a (finite) $\Delta \subset \Omega$ such that $\Delta \cap U(\Gamma) = \emptyset$. Consider the smallest $i \in I$ for which $\Delta \subset \Omega_{(i)}$. Since $U^{(i)}(\Gamma) \subseteq U(\Gamma)$, $\Delta \cap U^{(i)}(\Gamma) = \emptyset$. Hence, by Theorem 2.6, $A \in Cn_{\mathbf{AL^r_{(i)}}}(\Gamma)$. By Definition 4.1 and the reflexivity of **LLL**, $A \in Cn_{\mathbf{HAL^r}}(\Gamma)$. ■

Hence, it is the Minimal Abnormality Strategy that enforced the use of the adverb "often". First of all, when $\Phi(\Gamma)$ is finite, the counterpart of Theorem 4.20 can easily be proven:

**Theorem 4.21** *Where $\Phi(\Gamma)$ is finite: $Cn_{\mathbf{AL^m}}(\Gamma) \subseteq Cn_{\mathbf{HAL^m}}(\Gamma)$.*

*Proof.* Suppose $\Phi(\Gamma)$ is finite and $A \in Cn_{\mathbf{AL^m}}(\Gamma)$. Let $\Phi(\Gamma) = \{\varphi_1, \ldots, \varphi_n\}$. By Theorem 2.7: ($\star$) for every $\varphi_j$ with $j \in \{1, \ldots, n\}$, there is a $\Delta_j \subset \Omega$ such that $\Delta_j \cap \varphi_j = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_j)$. Consider the smallest $i \in I$ for which $(\Delta_1 \cup \ldots \cup \Delta_n) \subseteq \Omega_{(i)}$. By Lemma 3.4, for every $\varphi \in \Phi^{(i)}(\Gamma)$, there is a $\varphi_j$ with $j \in \{1, \ldots, n\}$ such that $\varphi_j \cap \Omega_{(i)} = \varphi$. By ($\star$): for every $\varphi \in \Phi^{(i)}(\Gamma)$, there is a $\Delta_j \subset \Omega_{(i)}$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_j)$ and $\Delta_j \cap \varphi = \Delta_j \cap (\varphi_j \cap \Omega_{(i)}) = \Delta_j \cap \varphi_j = \emptyset$. By Theorem 2.7, $\Gamma \vdash_{\mathbf{AL^m_{(i)}}} A$. By Definition 4.1 and the reflexivity of **LLL**, $A \in Cn_{\mathbf{HAL^m}}(\Gamma)$. ■

There are $\Gamma$ for which $Cn_{\mathbf{K^m}}(\Gamma) \subset Cn_{\mathbf{HK^m}}(\Gamma)$ – again, see the example $\Gamma_{\mathsf{h}}$ in Section 4.1.1. More generally, a formula $A$ that is derivable from the premise set by the logic $\mathbf{AL^x}$, is usually derivable by one of the logics $\mathbf{AL^x_{(i)}}$ from the combined logic $\mathbf{HAL^x}$. However, this does not hold in general:

**Proposition 4.1** *There are $\Gamma \subseteq \mathcal{W}_s$ for which $Cn_{\mathbf{K^m}}(\Gamma) \nsubseteq Cn_{\mathbf{HK^m}}(\Gamma)$.*

An example of this can be found in Section C.6 of Appendix C. The crucial feature of this example is that infinitely many conditions, belonging to infinitely many different $\Omega_i^{\mathbf{K}}$'s are necessary to derive the formula $A$. As a result, no logic $\mathbf{K^m_{(i)}}$ can yield $A$, whereas the flat adaptive logic that takes as its set of abnormalities $\bigcup_{i \in \mathbb{N}} \Omega_i^{\mathbf{K}}$ does yield $A$.

Note that this difficulty can only arise for hierarchic logics that are built up from an infinite number of flat adaptive logics – where $I = \{1, \ldots, n\}$, obviously $Cn_{\mathbf{AL^x}}(\Gamma) = Cn_{\mathbf{AL^x_{(n)}}}(\Gamma)$, whence $Cn_{\mathbf{AL^x}}(\Gamma) \subseteq Cn_{\mathbf{HAL}}(\Gamma)$ by the reflexivity of **LLL**. Hence if there are finitely many flat adaptive logics in the combination, the hierarchic adaptive logics are always at least as strong as their flat nephews.

For those who might still see Proposition 4.1 as a drawback of hierarchic adaptive logics, note that we can simply change Definition 4.1, adding $Cn_{\mathbf{AL^x}}(\Gamma)$ inside the brackets, to ensure that $\mathbf{HAL^x}$ is always at least as strong as $\mathbf{AL^x}$. It seems highly likely – but was not proven so far – that both the proof theory and semantics of **HAL** can easily be generalized in order to incorporate the resulting class of logics.

# Chapter 5

# Lexicographic Adaptive Logics

*This Chapter is based on the paper "Extending the Standard Format of Adaptive Logics to the Prioritized Case" (Logique & Analyse, to appear), which was co-authored by Christian Straßer. I thank Peter Verdée for his critical remarks on an earlier draft.*

In this chapter, we develop a third format for prioritized ALs, i.e. that of *lexicographic adaptive logics*. As far as we know, logics in this format cannot be reduced to (a combination of) ALs in standard format. This format is however very close to the standard format in numerous respects. It also makes use of the characterization by a triple, but now replacing the set of abnormalities $\Omega$ by a sequence of sets of abnormalities $\langle\Omega_1, \Omega_2, \ldots\rangle$, where the different subscripts of the sets refer to their priority ranking. Both proof theory and semantics of the new format have the same overall structure as the standard format. The difference is that the strategy is adjusted to the prioritized setting.

Since we see no way to reduce the new format to the standard format, we will have to transfer a lot of meta-theoretic results from the latter to the former. However, in view of the strong similarity between both, much of the work can easily be achieved through an adaptation of the meta-proofs from [21, 25]. As a result, the format of lexicographic ALs inherits most of the nice properties of the standard format – to be sure, it inherits *all* those properties which we were able to check so far.

This chapter is structured as follows. In Section 5.1, we present the basic idea behind lexicographic ALs, and how this is implemented at the semantic level. The semantics are illustrated by means of a concrete prioritized adaptive logic similar to the **K**-based logics from previous chapters. Next, we present a generic proof theory for logics in this new format (Section 5.2). In the rather lengthy Section 5.3, the core metatheoretic properties for the format are established.

**Notational Conventions – a Brief Reminder** Before we start, let us insert a brief reminder of some notational conventions from the two preceding chapters.

Recall that $I$ is used as a metavariable for initial subsequences of $\mathbb{N} = \{1, 2, \ldots\}$. Where the sequence of sets of abnormalities $\langle \Omega_i \rangle_{i \in I}$ is given, $\Omega =_{\mathsf{df}} \bigcup_{i \in I} \Omega_i$ and $\Omega_{(i)} = \Omega_1 \cup \ldots \cup \Omega_i$.

Where **LLL** and $\langle \Omega_i \rangle_{i \in I}$ are given, $\mathbf{AL^x}$ refers to the flat AL defined by (i) **LLL**, (ii) $\Omega$ and (iii) a strategy $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$. We use $Dab(\Delta)$ as a metavariable for disjunctions of members of $\Omega$, and $\Sigma(\Gamma), U(\Gamma), \Phi(\Gamma), \Sigma_s(\Gamma), U_s(\Gamma), \Phi_s(\Gamma)$ to denote the sets of (sets of) abnormalities that are used in the semantics, resp. proof theory of $\mathbf{AL^r}$, resp. $\mathbf{AL^m}$ – we refer to Chapter 2 for the precise definitions. Finally, $Ab(M) =_{\mathsf{df}} \{A \in \Omega \mid M \Vdash A\}$.

As in previous chapters, we will use **K**-based logics to illustrate the format of prioritized adaptive logics studied in this chapter. Recall that **K** is Kripke's normal modal logic, extended with a number of axioms to deal with the checked connectives – see page 28. The language of **K** is $\check{\mathcal{L}}_m$ and the associated set of formulas $\check{\mathcal{W}}_m$. Where $i \in \mathbb{N}$, let $\lozenge^i A$ abbreviate $A$, preceded by $i$ diamonds. Let $!^i A$ abbreviate $\lozenge^i A \wedge \neg A$. Finally, $\Omega_i^{\mathbf{K}} =_{\mathsf{df}} \{!^i A \mid A \in \mathcal{W}_c^l\}$, where $\mathcal{W}_c^l$ denotes the set of all sentential letters and their negation.

## 5.1 The Lexicographic Turn

### 5.1.1 General Characteristics of $\mathbf{AL}_{\sqsubset}$

Before we explain the idea behind a lexicographic selection of models, let us start with a general characterization of lexicographic ALs. As they are very similar to ALs in standard format, we can be rather brief here. Every logic $\mathbf{AL}_{\sqsubset} : \wp(\check{\mathcal{W}}_s) \to \wp(\check{\mathcal{W}}_s)$ is characterized by a triple:

1. A lower limit logic **LLL**
2. A sequence of sets of abnormalities: $\langle \Omega_i \rangle_{i \in I}$
3. A strategy: $\sqsubset$-Minimal Abnormality or $\sqsubset$-Reliability

Let us briefly discuss the elements of the above triple. First of all, just like **AL**, every logic $\mathbf{AL}_{\sqsubset}$ is built on top of a logic **LLL**, which is obtained from **LLL** as described in Section 2.1. The upper limit logic of $\mathbf{AL}_{\sqsubset}$ is identical to the upper limit logic of **AL**, which was denoted by **ULL**.

Recall that intuitively, every logic **AL** avoids abnormalities "as much as possible". Likewise, every logic $\mathbf{AL}_{\sqsubset}$ avoids abnormalities "as much as possible, *in view of their rank*". How this is done precisely will be explained below, and depends on the strategy of $\mathbf{AL}_{\sqsubset}$. As for **AL**, the two strategies give rise to two subclasses of prioritized ALs: $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ and $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$.

Since the $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-semantics is technically less involving than the $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$-semantics, we will start with the former in Section 5.1.2. In Section 5.1.3, we will present an example of a logic in the new format: $\mathbf{K}^{\mathbf{m}}_{\sqsubset}$. After that, an alternative way to characterize the $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-models of a premise set is discussed. Finally, in Section 5.1.5, we show how a Reliability-variant is obtained from this alternative characterization.

### 5.1.2 The $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-semantics

In Chapter 2, we explained that flat ALs select a subset of the **LLL**-models of a premise set in view of their abnormal part. For $\mathbf{AL}^{\mathbf{m}}$, a model $M$ is selected iff its abnormal part $Ab(M)$ is minimal with respect to set-inclusion. The prioritized logic $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ also selects **LLL**-models in view of their abnormal part, but takes into account the rank of abnormalities. In view of the prioritization $Ab(M)$ is not flat but is structured and may be represented by the tuple $\langle Ab(M) \cap \Omega_1, Ab(M) \cap \Omega_2, \ldots \rangle$. Just like the flat abnormal parts were partially ordered in the standard format by $\subset$, the structured abnormal parts of lexicographic ALs may be partially ordered by the lexicographic order $\sqsubset_{\mathsf{lex}}$:[1]

**Definition 5.1** *Where $\Delta, \Delta' \subseteq \Omega$: $\langle \Delta \cap \Omega_i \rangle_{i \in I} \sqsubset_{\mathsf{lex}} \langle \Delta' \cap \Omega_i \rangle_{i \in I}$ iff (1) there is an $i \in I$ such that for all $j < i$, $\Delta \cap \Omega_j = \Delta' \cap \Omega_j$, and (2) $\Delta \cap \Omega_i \subset \Delta' \cap \Omega_i$. We write $\Delta \sqsubset \Delta'$ iff $\langle \Delta \cap \Omega_i \rangle_{i \in I} \sqsubset_{\mathsf{lex}} \langle \Delta' \cap \Omega_i \rangle_{i \in I}$.*

Just as for flat ALs, the **LLL**-models were selected whose abnormal part was $\subset$-minimal, we now select the **LLL**-models whose abnormal part is $\sqsubset$-minimal:

**Definition 5.2** *$M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and there is no $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$.*

As we did with $\subset$-minimally abnormal models, we can speak of $\sqsubset$-minimally abnormal models. Lemma 5.1 below states that the $\subset$-order on $\wp(\Omega)$ is included in the $\sqsubset$-order on $\wp(\Omega)$.

**Lemma 5.1** *Where $\Delta, \Delta' \subseteq \Omega$: if $\Delta \subset \Delta'$, then $\Delta \sqsubset \Delta'$.*

*Proof.* Suppose $\Delta \subset \Delta'$. Then for all $i \in I$, $\Delta \cap \Omega_i \subseteq \Delta' \cap \Omega_i$ and there is an $i \in I$ such that $\Delta \cap \Omega_i \subset \Delta' \cap \Omega_i$. Take the smallest $i \in I$ for which $\Delta \cap \Omega_i \subset \Delta' \cap \Omega_i$, whence for all $j < i$, $\Delta \cap \Omega_i = \Delta' \cap \Omega_i$. By Definition 5.1, $\Delta \sqsubset \Delta'$. ■

By Lemma 5.1, we immediately obtain:

**Theorem 5.1** *Every $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-model of $\Gamma$ is an $\mathbf{AL}^{\mathbf{m}}$-model of $\Gamma$.*

### 5.1.3 An Example: $\mathbf{K}^{\mathbf{m}}_{\sqsubset}$

The lexicographic adaptive logic $\mathbf{K}^{\mathbf{m}}_{\sqsubset}$ is characterized by the following triple:

1. The modal logic **K**
2. The sequence $\langle \Omega^{\mathbf{K}}_i \rangle_{i \in \mathbb{N}}$
3. The Strategy: $\sqsubset$-Minimal Abnormality

[1] Lexicographic orders are a well-known ordering type and are mentioned in any representative mathematical dictionary or encyclopedia (see e.g. [82, p. 1170]). Lexicographic orders have already previously proven to be useful for the formal explication of reasoning on the basis of prioritized information. Lehmann employed them to deal with priorities among defaults [89], Nebel [114] in order to deal with prioritized theory bases and Hansen [69] applied Nebel's preference order to the context of prioritized imperatives.

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ |
|---|---|---|---|---|---|---|---|---|
| | $p, q,$ $r, s$ | $p, q,$ $\neg r, s$ | $\neg p, q,$ $r, s$ | $\neg p, q,$ $r, \neg s$ | $\neg p, q,$ $\neg r, s$ | $\neg p, q,$ $\neg r, \neg s$ | $\neg p, \neg q,$ $r, s$ | $\neg p, \neg q,$ $\neg r, s$ |
| $!^1 p$ | | | + | + | + | + | + | + |
| $!^1 \neg q$ | + | + | + | + | + | + | | |
| $!^1 r$ | | + | | | + | + | | + |
| $!^2 s$ | | | | + | | + | | |
| $!^2 \neg s$ | + | + | + | | + | | + | + |

Table 5.1: A representation of the **K**-models of $\Gamma_1$. The first row shows the non-modal propositions each model validates, the second row the abnormalities of rank 1 and the third row the abnormalities of rank 2.

To compare the format for prioritized logics with flat adaptive logics, it will be convenient to refer to the logics $\mathbf{K^m}$ and $\mathbf{K^r}$, defined by (i) **K**, (ii) $\Omega^{\mathbf{K}} = \bigcup_{i \in \mathbb{N}} \Omega_i^{\mathbf{K}}$ and (iii) Minimal Abnormality, respectively Reliability.

The logic $\mathbf{K^m_\sqsubset}$ allows for the defeasible inference from $\lozenge^i A$ (where $i \in \mathbb{N}$) to $A$. This is done by defining "$A$ is plausible (to degree $i$), but false" as an abnormality (of rank $i$).[2] Consider the prioritized belief base $\Psi_{\mathsf{ex}} = \langle \{p \supset q, q \vee s, p \supset s\}, \{p, \neg q \wedge r\}, \{s, \neg s\} \rangle$. The translation gives us $\Psi^{\lozenge}_{\mathsf{ex}} = \{p \supset q, q \vee s, p \supset s, \lozenge p, \lozenge(\neg q \wedge r), \lozenge\lozenge s, \lozenge\lozenge\neg s\}$. To facilitate the reading, let henceforth $\Gamma_1 = \Psi^{\lozenge}_{\mathsf{ex}}$. Let us take a look at the **K**-models of $\Gamma_1$. Note that every such model validates the modal formulas $\lozenge p, \lozenge\neg q, \lozenge r, \lozenge\lozenge s$ and $\lozenge\lozenge\neg s$. Table 5.1 represents these models in terms of (1) the non-modal literals they validate and (2) their abnormal part. For reasons of simplicity, we restrict the scope to those propositional letters that occur in $\Gamma_1$.[3]

Figure 5.1 shows the partial order imposed on the models from Table 5.1 by the two logics $\mathbf{K^m}$ and $\mathbf{K^m_\sqsubset}$. $M_1, M_4, M_7$ are $\subset$-minimally abnormal. From these, $M_4$ is not $\sqsubset$-minimally abnormal: $Ab(M_1) \cap \Omega_1^{\mathbf{K}} \subset Ab(M_4) \cap \Omega_1^{\mathbf{K}}$, whence $Ab(M_1) \sqsubset Ab(M_4)$. $M_1$ and $M_7$ are incommensurable in view of $\Omega_1$, whence $Ab(M_1) \not\sqsubset Ab(M_7)$ and $Ab(M_7) \not\sqsubset Ab(M_1)$. Recall that the set of $\mathbf{AL^m_\sqsubset}$-models is always a subset of the $\mathbf{AL^m}$-models, whence in this particular case, $M_1$ and $M_7$ are the only $\sqsubset$-minimally abnormal models. As a result, $s$ and $p \vee \neg q$ are semantic $\mathbf{K^m_\sqsubset}$-consequences of $\Gamma_1$. Note that in view of $M_4$, these are not semantic $\mathbf{K^m}$-consequences of $\Gamma_1$.

We can explain this outcome as follows. In view of $\Gamma_1$, both $p$ and $\neg q$ are plausible, but one of them has to be false (although we do not know which one). So if we want to privilege our most plausible beliefs, all we can do is assume that one of both holds: $p \vee \neg q$. So all the selected models either verify $p$ or they verify $\neg q$. Since $\Gamma_1 \cup \{p \vee \neg q\} \vdash_{\mathbf{K}} s$, these models also verify $s$. The logic $\mathbf{K^m}$ cannot

[2] Note that for all $i, j \in \mathbb{N}$ such that $i \neq j$, $\Omega_i^{\mathbf{K}} \cap \Omega_j^{\mathbf{K}} = \emptyset$. This is not required for a logic to fit the format of $\mathbf{AL}_\sqsubset$; all that is required is that each $\Omega_i$ is characterized by a logical form.

[3] It is provable for that (1) for every $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_1)$, $Ab(M) \supseteq Ab(M_i)$ for a "model" $M_i$ in the table and (2), for every "model" $M_i$ in the table, there is a $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_1)$ such that $Ab(M) = Ab(M_i)$. Hence it suffices to look at these limited representations, to decide which abnormalities hold in the minimal abnormal models. This allows one to derive the claims about $Cn_{\mathbf{K^m_\sqsubset}}(\Gamma_1)$ that are made in this section.


(a) (b)

Figure 5.1: : A graphic comparison of the partial orders $\subset$ (5.1(a)) and $\sqsubset$ (5.1(b)) on the abnormal parts of the models $M_1, \ldots, M_8$.

achieve this result, since it considers $M_1$ and $M_4$ as incommensurable.

### 5.1.4 An Alternative Characterization of $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$

In Section 2.2 of Chapter 2, it was pointed out that the set of $\mathbf{AL}^{\mathbf{m}}$-models of $\Gamma$ can be characterized alternatively, in view of the minimal Dab-consequences of $\Gamma$. A similar characterization can be given of $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. We say that $\varphi$ is a $\sqsubset$-minimal choice set of $\Sigma$ iff there is no choice set $\psi$ of $\Sigma$ such that $\psi \sqsubset \varphi$. Let $\Sigma(\Gamma)$ be defined as in Section 2.2.

**Definition 5.3** $\Phi^{\sqsubset}(\Gamma)$ *is the set of* $\sqsubset$*-minimal choice sets of* $\Sigma(\Gamma)$.

Note that the following theorem follows immediately from Lemma 5.1:

**Theorem 5.2** $\Phi^{\sqsubset}(\Gamma) \subseteq \Phi(\Gamma)$.

In Section 5.2.4, it is proven that for every $\Gamma$, $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$ – see Theorem 5.11. We will now show that, just as the set $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}}(\Gamma)$ can be characterized in view of $\Phi(\Gamma)$, the set $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ can be characterized in view of $\Phi^{\sqsubset}(\Gamma)$.

**Lemma 5.2** *Where* $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$*,* $Ab(M)$ *is a choice set of* $\Sigma(\Gamma)$.

*Proof.* Suppose $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. Let $Dab(\Delta)$ be an arbitrary minimal Dab-consequence of $\Gamma$. By the soundness of **LLL**, $\Gamma \models_{\mathbf{LLL}} Dab(\Delta)$. Hence $M \Vdash Dab(\Delta)$, which implies that $M \Vdash A$ for an $A \in \Delta$. Hence $Ab(M) \cap \Delta \neq \emptyset$. ■

**Lemma 5.3** *If* $\Gamma$ *has* **LLL***-models, then for every choice set* $\varphi$ *of* $\Sigma(\Gamma)$*, there is a* **LLL***-model* $M$ *of* $\Gamma$ *such that* $Ab(M) \subseteq \varphi$.

*Proof.* Suppose (†) $\Gamma$ has **LLL**-models. Let $\varphi$ be a choice set of $\Sigma(\Gamma)$. Suppose there is no $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $Ab(M) \subseteq \varphi$. Hence $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$ has no **LLL**-models. By the compactness of **LLL**, there is a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta \subseteq \Omega - \varphi$ such that $\Gamma' \cup \Delta^{\check{\neg}}$ has no **LLL**-models. However, by (†) and the monotonicity of **LLL**, $\Gamma'$ has **LLL**-models, whence $\Delta \neq \emptyset$. By **CL**-properties, $\Gamma' \vdash_{\mathbf{LLL}} Dab(\Delta)$, whence by the monotonicity of **LLL**, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$. Note that there is a minimal non-empty $\Delta' \subseteq \Delta$ such that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta')$, and also $\Delta' \cap \varphi = \emptyset$. Hence, $\varphi$ is not a choice set of $\Sigma(\Gamma)$ — a contradiction. ■

**Theorem 5.3** *$M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ iff ($M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $Ab(M) \in \Phi^{\sqsubset}(\Gamma)$).*

*Proof.* ($\Rightarrow$) Suppose $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By Definition 5.2, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. Suppose (†) $Ab(M) \notin \Phi^{\sqsubset}(\Gamma)$, and let $Ab(M) = \varphi$. By Lemma 5.2, $Ab(M)$ is a choice set of $\Sigma(\Gamma)$, whence by (†), there is a choice set $\psi$ of $\Sigma(\Gamma)$ such that $\psi \sqsubset \varphi$. By Lemma 5.3, there is a **LLL**-model $M'$ of $\Gamma$ such that $Ab(M') \subseteq \psi$.
*Case 1:* $Ab(M') = \psi$. Hence, $Ab(M') \sqsubset \varphi$.
*Case 2:* $Ab(M') \subset \psi$. Hence, $Ab(M') \sqsubset \psi$ in view of Lemma 5.1. By the transitivity of $\sqsubset$, $Ab(M') \sqsubset \varphi$.

Hence in either case, there is a **LLL**-model $M'$ of $\Gamma$ such that $Ab(M') \sqsubset Ab(M)$, which contradicts the fact that $M \in \mathcal{M}_{\mathbf{AL}_{\sqsubset}\mathbf{m}}(\Gamma)$.

($\Leftarrow$) Suppose $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$, but $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. Then there is a $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) : Ab(M') \sqsubset Ab(M)$. By Lemma 5.2, $Ab(M')$ is a choice set of $\Sigma(\Gamma)$, whence in view of Definition 5.3, $Ab(M) \notin \Phi^{\sqsubset}(\Gamma)$. ■

Note that the above theorem nicely parallels Theorem 2.1. The theorem below states that whenever $\Gamma$ has **LLL**-models, then we can also go in the opposite direction: the set $\Phi^{\sqsubset}(\Gamma)$ can be defined in view of $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$.

**Theorem 5.4** *If $\Gamma$ has* **LLL***-models, then $\Phi^{\sqsubset}(\Gamma) = \{Ab(M) \mid M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)\}$.*

*Proof.* Suppose $\Gamma$ has **LLL**-models. That $\{Ab(M) \mid M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)\} \subseteq \Phi^{\sqsubset}(\Gamma)$ is immediate in view of Theorem 5.3. Let $\varphi \in \Phi^{\sqsubset}(\Gamma)$. By Lemma 5.3, there is a $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $Ab(M) = \varphi$. By Theorem 5.3, $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. ■

Below we will see that $\Phi^{\sqsubset}(\Gamma)$ has a proof-theoretic counterpart, $\Phi^{\sqsubset}_s(\Gamma)$, that determines the marking of lines of a proof at stage $s$. Hence Theorems 5.3 and 5.4 function as a bridge between the proof theory and semantics of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$.

### 5.1.5 The $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$-semantics

Recall that $U(\Gamma) = \bigcup \Phi(\Gamma)$ – see page 20 –, where $U(\Gamma)$ is associated with $\mathbf{AL}^{\mathbf{r}}$ and $\Phi(\Gamma)$ with $\mathbf{AL}^{\mathbf{m}}$. In view of Theorem 2.1, this implies that an abnormality is unreliable iff it is verified by a $\subset$-minimally abnormal model: $U(\Gamma) = \{A \in \Omega \mid M \Vdash A$ for a $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}}(\Gamma)\}$.

Let us now take a look at $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$. Just as $U(\Gamma)$, the set of $\sqsubset$-unreliable abnormalities can be characterized in two equivalent ways: (i) syntactically, as the union of all the members of $\Phi^{\sqsubset}(\Gamma)$ and (ii) semantically, as the set of those abnormalities that are verified by a $\sqsubset$-minimally abnormal model. To simplify the meta-theory and to stay as close as possible to the standard format, we will use (i) as the official definition of the set of $\sqsubset$-unreliable abnormalities:

**Definition 5.4** $U^{\sqsubset}(\Gamma) =_{\mathsf{df}} \bigcup \Phi^{\sqsubset}(\Gamma)$

By Theorem 5.3, $U^{\sqsubset}(\Gamma) = \{A \in \Omega \mid M \Vdash A$ for a $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)\}$. We can now define the set of $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$-models of $\Gamma$ as we did for $\mathcal{M}_{\mathbf{AL}^{\mathbf{r}}}(\Gamma)$:

**Definition 5.5** *$M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$ iff ($M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $Ab(M) \subseteq U^{\sqsubset}(\Gamma)$)*

The following theorem follows immediately from the above definition:

**Theorem 5.5** $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$.

*Proof.* Suppose $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By Theorem 5.3, $Ab(M) = \varphi$ for a $\varphi \in \Phi^{\sqsubset}(\Gamma)$. Hence $Ab(M) \subseteq \bigcup \Phi^{\sqsubset}(\Gamma)$, whence by Definitions 5.4 and 5.5, $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$. ■

In view of Theorem 5.2, the fact that $U(\Gamma) = \bigcup \Phi(\Gamma)$ and Definition 5.4, we also have:

**Theorem 5.6** $U^{\sqsubset}(\Gamma) \subseteq U(\Gamma)$

**Theorem 5.7** *Every* $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$*-model of* $\Gamma$ *is a* $\mathbf{AL}^{\mathbf{r}}$*-model of* $\Gamma$.

Let us reconsider the example from Section 5.1.3 from the viewpoint of the $\mathbf{K}^{\mathbf{r}}_{\sqsubset}$-semantics. In view of the above definitions, it is required that we first look at the minimal Dab-consequences of a set $\Gamma$, to find the set of $\sqsubset$-unreliable formulas. The set $\Gamma_1 = \{p \supset q, q \vee s, p \supset s, \Diamond p, \Diamond(\neg q \wedge r), \Diamond\Diamond s, \Diamond\Diamond\neg s\}$ has four minimal *Dab*-consequences: $!^1 p \check{\vee} !^1\neg q$, $!^1 p \check{\vee} !^2\neg s$, $!^1\neg q \check{\vee} !^2\neg s$, and $!^2 s \check{\vee} !^2\neg s$.

Hence $\Sigma(\Gamma_1) = \{\{!^1 p, !^1\neg q\}, \{!^1 p, !^2\neg s\}, \{!^1\neg q, !^2\neg s\}, \{!^2 s, !^2\neg s\}\}$. The set of $\sqsubset$-minimal choice sets of $\Sigma(\Gamma_1)$ is $\Phi^{\sqsubset}(\Gamma_1) = \{\{!^1\neg q, !^2\neg s\}, \{!^1 p, !^2\neg s\}\}$. Remark that these sets correspond to the $\sqsubset$-minimal abnormal models $M_1$ and $M_7$ depicted in Table 5.1. As a result, $U^{\sqsubset}(\Gamma_1) = \{!^1 p, !^1\neg q, !^2\neg s\}$.

This means that all $\sqsubset$-reliable models falsify $!^2 s$, whence in view of $\Gamma_1$, they verify $s$. Hence $s$ is also a semantic $\mathbf{K}^{\mathbf{r}}_{\sqsubset}$-consequence of $\Gamma_1$. Note however that $\mathcal{M}_{\mathbf{K}^{\mathbf{r}}_{\sqsubset}}(\Gamma_1) \neq \mathcal{M}_{\mathbf{K}^{\mathbf{m}}_{\sqsubset}}(\Gamma_1)$: for the model $M_3$ represented in Section 5.1.3, we have that $M_3 \in \mathcal{M}_{\mathbf{K}^{\mathbf{r}}_{\sqsubset}}(\Gamma_1) - \mathcal{M}_{\mathbf{K}^{\mathbf{m}}_{\sqsubset}}(\Gamma_1)$. This implies that $p \vee \neg q$ is not a semantic $\mathbf{K}^{\mathbf{r}}_{\sqsubset}$-consequence of $\Gamma_1$.

## 5.2 The Proof Theory of $\mathbf{AL}_{\sqsubset}$

### 5.2.1 The Generic Proof Theory for $\mathbf{AL}_{\sqsubset}$

As for **SAL** and **HAL**, the inference rules of an $\mathbf{AL}_{\sqsubset}$-proof are identical to those of a **AL**-proof – see page 21 where these are spelled out. As a result, Lemma 2.1 holds also for $\mathbf{AL}_{\sqsubset}$-proofs. So again, apart from the marks, every **AL**-proof is a $\mathbf{AL}_{\sqsubset}$-proof and vice versa. The distinctive feature of an $\mathbf{AL}_{\sqsubset}$-proof lies in its marking definition. Let $\Sigma_s(\Gamma)$ be defined as in Section 2.3.

**Definition 5.6** $\Phi^{\sqsubset}_s(\Gamma)$ *is the set of* $\sqsubset$*-minimal choice sets of* $\Sigma_s(\Gamma)$.

In Section 5.2.4, we prove that for every $\Gamma$ and at every stage $s$ of a $\mathbf{AL}_{\sqsubset}$-proof from $\Gamma$, $\Phi^{\sqsubset}_s(\Gamma) \neq \emptyset$. Of course, it may be the case that $\Phi^{\sqsubset}_s(\Gamma) = \{\emptyset\}$, i.e. whenever $\Sigma_s(\Gamma) = \emptyset$. Marking in view of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ is now done in the same way as for $\mathbf{AL}^{\mathbf{m}}$, replacing $\Phi_s(\Gamma)$ by $\Phi^{\sqsubset}_s(\Gamma)$:

**Definition 5.7** $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$*-Marking: a line $l$ with formula $A$ is marked at stage $s$ iff, where its condition is $\Delta$: (i) no $\varphi \in \Phi^{\sqsubset}_s(\Gamma)$ is such that $\varphi \cap \Delta = \emptyset$, or (ii) for a $\varphi \in \Phi^{\sqsubset}_s(\Gamma)$, there is no line on which $A$ is derived on a condition $\Theta$ for which $\Theta \cap \varphi = \emptyset$.*

The set of $\sqsubset$-unreliable formulas at stage $s$ is defined as the union of the members of $\Phi_s^{\sqsubset}(\Gamma)$:

**Definition 5.8** $U_s^{\sqsubset}(\Gamma) =_{\mathsf{df}} \bigcup \Phi_s^{\sqsubset}(\Gamma)$

**Definition 5.9** $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$*-Marking: a line $l$ with formula $A$ is marked at stage $s$ iff, where its condition is $\Delta$, $\Delta \cap U_s^{\sqsubset}(\Gamma) \neq \emptyset$.*

Derivability at a stage and final derivability are defined as by Definitions 3.8 and 3.9. This gives us the relations $\vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}$ and $\vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}$.

The following is an immediate consequence of Lemma 5.1:

**Fact 5.1** *At every stage $s$ of a proof from $\Gamma$, $\Phi_s^{\sqsubset}(\Gamma) \subseteq \Phi_s(\Gamma)$.*

This fact implies that at every stage $s$ of a proof from $\Gamma$, we can first check which choice sets of $\Sigma_s(\Gamma)$ are $\subset$-minimal, and only afterwards select the set of $\sqsubset$-minimal choice sets from these. Also, from Fact 5.1, the fact that at every stage $s$, $U_s(\Gamma) = \bigcup \Phi_s(\Gamma)$ and Definition 5.8, we can derive:

**Fact 5.2** *At every stage $s$ of a proof from $\Gamma$, $U_s^{\sqsubset}(\Gamma) \subseteq U_s(\Gamma)$.*

Facts 5.1 and 5.2 imply that whenever a line is unmarked in an $\mathbf{AL}^{\mathbf{x}}$-proof (where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$), it is unmarked in an $\mathbf{AL}^{\mathbf{x}}_{\sqsubset}$-proof as well – recall that apart from the marking, these proofs are interchangeable. Hence if something is (finally) derived in an $\mathbf{AL}^{\mathbf{x}}$-proof, then it is finally derived in an $\mathbf{AL}^{\mathbf{x}}_{\sqsubset}$-proof as well. This allows us to safely infer:

**Theorem 5.8** *Where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$: $Cn_{\mathbf{AL}^{\mathbf{x}}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{x}}_{\sqsubset}}(\Gamma)$.*

By Theorem 2.9 and Theorem 2.16 respectively, we immediately have:

**Theorem 5.9** *Each of the following holds:*
*1. $\Gamma \subseteq Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma)$ (Reflexivity)*
*2. $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma)$ (**LLL** is weaker than or identical to $\mathbf{AL}_{\sqsubset}$)*

### 5.2.2 Example of a $\mathbf{K}^{\mathbf{x}}_{\sqsubset}$-proof

**$\sqsubset$-Minimal Abnormality.** To illustrate the new marking definitions, let us take a look at a particular $\mathbf{K}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma_1 = \{p \supset q, q \vee s, p \supset s, \lozenge p, \lozenge(\neg q \wedge r), \lozenge\lozenge s, \lozenge\lozenge\neg s\}$:

| | | | |
|---|---|---|---|
| 1 | $q \vee s$ | PREM | $\emptyset$ |
| 2 | $\lozenge(\neg q \wedge r)$ | PREM | $\emptyset$ |
| 3 | $\lozenge\neg q$ | 2;RU | $\emptyset$ |
| 4 | $\neg q$ | 3;RC | $\{!^1\neg q\}$ |
| 5 | $s$ | 1,4;RU | $\{!^1\neg q\}$ |
| 6 | $\lozenge\lozenge\neg s$ | PREM | $\emptyset$ |
| 7 | $!^1\neg q \check{\vee} !^2\neg s$ | 1,3,6;RU | $\emptyset$ |

Note that $\Sigma_7(\Gamma_1) = \{\{!^1\neg q, !^2\neg s\}\}$. This implies that the set of $\sqsubset$-minimal choice sets at stage 7, $\Phi^{\sqsubset}_7(\Gamma_1)$ only contains one member, i.e. $\{!^2\neg s\}$ – note that $\{!^2\neg s\} \sqsubset \{!^1\neg q\}$. Since the condition of line 5 has an empty intersection with this set, line 5 is unmarked.

Suppose we extend the proof as follows (we repeat from line 5 on):

| | | | |
|---|---|---|---|
| 5 | $s$ | 1,4;RU | $\{!^1\neg q\}$ $\checkmark^{10}$ |
| 6 | $\Diamond\Diamond\neg s$ | PREM | $\emptyset$ |
| 7 | $!^1\neg q \,\check{\vee}\, !^2\neg s$ | 1,3,6;RU | $\emptyset$ |
| 8 | $p \supset q$ | PREM | $\emptyset$ |
| 9 | $\Diamond p$ | PREM | $\emptyset$ |
| 10 | $!^1 p \,\check{\vee}\, !^1\neg q$ | 3,8,9;RU | $\emptyset$ |

$\Sigma_{10}(\Gamma_1) = \{\{!^1\neg q, !^2\neg s\}, \{!^1 p, !^1\neg q\}\}$, whence there are two $\sqsubset$-minimal choice sets at this stage: $\Phi^{\sqsubset}_{10}(\Gamma_1) = \Phi_{10}(\Gamma_1) = \{\{!^1\neg q\}, \{!^1 p, !^2\neg s\}\}$. In view of the first choice set, line 5 is marked. We can however further extend the proof in such a way that line 5 is again unmarked:

| | | | |
|---|---|---|---|
| 5 | $s$ | 1,4;RU | $\{!^1\neg q\}$ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | $!^1 p \,\check{\vee}\, !^1\neg q$ | 3,8,9;RU | $\emptyset$ |
| 11 | $p$ | 9;RC | $\{!^1 p\}$ |
| 12 | $p \supset s$ | PREM | $\emptyset$ |
| 13 | $s$ | 11,12;RU | $\{!^1 p\}$ |

Note that since no new Dab-formula has been derived, $\Phi^{\sqsubset}_{13}(\Gamma_1) = \Phi^{\sqsubset}_{10}(\Gamma_1)$. However, $s$ is now also derived on a condition that has an empty intersection with $\{!^1\neg q\}$. As a result, lines 5 and 13 are unmarked.

**$\sqsubset$-Reliability.** If the marking definition for $\sqsubset$-Reliability is applied, the above proof does not suffice to finally derive $s$. That is, $U^{\sqsubset}_{13}(\Gamma_1) = \bigcup \Phi^{\sqsubset}_{13}(\Gamma_1) = \{!^1 p, !^1\neg q, !^2\neg s\}$. As a result, both line 5 and line 13 are marked.

Nevertheless, $s$ is finally derivable in a $\mathbf{K}^{\mathbf{r}}_{\sqsubset}$-proof from $\Gamma_1$. To show how, let us recapitulate lines 5–15 from the above proof, marking lines according to Definition 5.9:

| | | | |
|---|---|---|---|
| 5 | $s$ | 1,4;RU | $\{!^1\neg q\}$ $\checkmark^{15}$ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 11 | $p$ | 9;RC | $\{!^1 p\}$ $\checkmark^{15}$ |
| 12 | $p \supset s$ | PREM | $\emptyset$ |
| 13 | $s$ | 11,12;RU | $\{!^1 p\}$ $\checkmark^{15}$ |
| 14 | $\Diamond\Diamond s$ | PREM | $\emptyset$ |
| 15 | $s$ | 14;RC | $\{!^2 s\}$ |

Note that this time, lines 5 and 13 are marked. However, we have derived $s$ on a condition that is not $\sqsubset$-unreliable at stage 15. As I explained in Section 5.1,

$!^2 s$ is not contained in any $\sqsubset$-minimal choice set of $\Sigma(\Gamma_1)$. This warrants that $s$ is finally derived in the proof. To explain why, consider the following extension of the proof:

| | | | |
|---|---|---|---|
| 15 | $s$ | 14;RC | $\{!^2 s\}$ ✓[16] |
| 16 | $!^2 s \check{\vee} !^2 \neg s$ | 6,14;RU | $\emptyset$ |

$\Sigma_{16}(\Gamma_1) = \{\{!^1\neg q, !^2\neg s\}, \{!^1 p, !^1\neg q\}, \{!^2 s, !^2\neg s\}\}$, whence $\Phi^{\sqsubset}_{16}(\Gamma_1) = \{\{!^1 p, !^2\neg s\}, \{!^1\neg q, !^2 s\}, \{!^1\neg q, !^2\neg s\}\}$. As a result, $U^{\sqsubset}_{16}(\Gamma_1) = \{!^1 p, !^1\neg q, !^2 s, !^2\neg s\}$. However, it suffices to derive the fourth minimal Dab-consequence of $\Gamma_1$ (see page 115) to undo the marking of line 15:

| | | | |
|---|---|---|---|
| 15 | $s$ | 14;RC | $\{!^2 s\}$ |
| 16 | $!^2 s \check{\vee} !^2 \neg s$ | 6,14;RU | $\emptyset$ |
| 17 | $!^1 p \check{\vee} !^2 \neg s$ | 6,9,12;RU | $\emptyset$ |

At stage 17, all minimal Dab-consequences of $\Gamma_1$ have been derived, whence $U^{\sqsubset}_{17}(\Gamma_1) = U^{\sqsubset}(\Gamma_1) = \{!^1 p, !^1\neg q, !^2\neg s\}$ – see Section 5.1.5. As a result, line 15 is unmarked again and will remain unmarked in every further extension of this proof.

### 5.2.3 The Standard Format as a Border-line Case

At the start of this chapter, it was mentioned that the standard format is a border case of the format of lexicographic ALs. Let us briefly spell out why this holds. Consider the sequence of sets of abnormalities: $S = \langle \Omega_i \rangle_{i \in I}$, where $\Omega_i = \Omega_j$ for every $i, j \in I$. Note that this is the case e.g. whenever $I = \{1\}$, i.e. whenever there is only one set in the sequence. As before, let $\Omega = \bigcup_{i \in I} \Omega_i$. We leave it to the reader to prove that in this case (†) $\Delta \sqsubset \Delta'$ iff $\Delta \subset \Delta'$.

For the sake of clarity, let me use the name $\mathbf{BAL}^{\mathbf{x}}_{\sqsubset}$ for the border case logic defined by (i) **LLL**, (ii) $S$ and (iii) a strategy $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$. By (†) and Definitions 2.2 and 5.2, we immediately have that $\mathcal{M}_{\mathbf{BAL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}}(\Gamma)$. Also, since in this case $\Phi^{\sqsubset}(\Gamma) = \Phi(\Gamma)$, we have by Definition 5.4 that $U^{\sqsubset}(\Gamma) = U(\Gamma)$. This implies by Definitions 2.1 and 5.5 that $\mathcal{M}_{\mathbf{BAL}^{\mathbf{r}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}}(\Gamma)$.

Similar results can be established for the proof theory. In view of Definition 5.3, it easy to see that by (†), for every stage $s$ of a proof from $\Gamma$, $\Phi^{\sqsubset}_s(\Gamma) = \Phi_s(\Gamma)$. From this and Definition 5.8, it follows that $U^{\sqsubset}_s(\Gamma) = U_s(\Gamma)$. Hence, where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$, a line is unmarked in a $\mathbf{AL}^{\mathbf{x}}$-proof iff it is unmarked in a $\mathbf{BAL}^{\mathbf{x}}_{\sqsubset}$-proof. This implies that where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$, $Cn_{\mathbf{BAL}^{\mathbf{x}}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{x}}}(\Gamma)$.

So every AL in standard format is equivalent to a logic in the new format. Remark that the equivalence is not restricted to the respective consequence sets, but to all the crucial concepts in the semantics and proof theory of both logics. This implies that all the meta-theoretic properties of $\mathbf{AL}_{\sqsubset}$ hold for **AL** as well.

### 5.2.4 $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$

The proof of the title of this section is highly similar to the proof for Lemma 3.3 in Chapter 3. However, in the current section, we will first prove a stronger claim, i.e. that for every stage $s$ of an $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma$, $\Phi^{\sqsubset}_s(\Gamma) \neq \emptyset$ — that $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$ follows almost immediately from the latter property.

As in Chapter 3, let $\Omega_{[1]} = \Omega_1$ and for all $i \in I, i > 1$, let $\Omega_{[i]} = \Omega_i - \Omega_{(i-1)}$. Recall that where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal $\mathrm{Dab}_{(i)}$-formulas at stage $s$ of a proof of from $\Gamma$, $\Sigma^{(i)}_s(\Gamma) =_{\mathsf{df}} \{\Delta_1, \Delta_2, \ldots\}$, and that $\Phi^{(i)}_s(\Gamma)$ is the set of ($\subset$-)minimal choice sets of $\Sigma^{(i)}_s(\Gamma)$.

Note that for each $\Delta \in \Sigma^{(i+1)}_s(\Gamma) - \Sigma^{(i)}_s(\Gamma)$, $\Delta \cap \Omega_{[i+1]} \neq \emptyset$. Where $\varphi \in \Phi^{(i)}_s(\Gamma)$, let $\Phi^{[i+1]}_{\varphi,s}(\Gamma)$ be the set of minimal choice sets of $\{\Delta \cap \Omega_{[i+1]} \mid \Delta \in \Sigma^{(i+1)}_s(\Gamma), \varphi \cap \Delta = \emptyset\}$.

**Lemma 5.4** *For all $\varphi \in \Phi^{(i)}_s(\Gamma)$ and all $\varphi' \in \Phi^{[i+1]}_{\varphi,s}(\Gamma)$, $\varphi \cup \varphi' \in \Phi^{(i+1)}_s(\Gamma)$.*

*Proof.* Let $\varphi \in \Phi^{(i)}_s(\Gamma)$ and consider an arbitrary $\varphi' \in \Phi^{[i+1]}_{\varphi,s}(\Gamma)$. We first prove that $\varphi \cup \varphi'$ is a choice set of $\Sigma^{(i+1)}_s(\Gamma)$. Let $\Delta \in \Sigma^{(i+1)}_s(\Gamma)$; we need to show that $\Delta \cap (\varphi \cup \varphi') \neq \emptyset$. If $\Delta \cap \varphi \neq \emptyset$, this holds trivially. So suppose $\Delta \cap \varphi = \emptyset$. Then $\Delta \notin \Sigma^{(i)}_s(\Gamma)$ since $\varphi \in \Phi^{(i)}_s(\Gamma)$. In this case $\Delta \cap \Omega_{[i+1]} \neq \emptyset$. Hence $\varphi' \cap \Delta \neq \emptyset$, since $\varphi' \in \Phi^{[i+1]}_{\varphi,s}$.

We now prove that $\varphi \cup \varphi'$ is also a *minimal* choice set of $\Sigma^{(i+1)}_s(\Gamma)$. By the right-left direction of Lemma 3.1 and the fact that $\varphi \in \Phi^{(i)}_s(\Gamma)$, for every $A \in \varphi$ there is a $\Delta \in \Sigma^{(i)}_s(\Gamma)$ such that $\Delta \cap \varphi = \{A\}$. Moreover, for all these $\Delta$, $\varphi' \cap \Delta = \emptyset$, since $\varphi' \subseteq \Omega_{[i+1]}$. Finally, $\Sigma^{(i)}_s(\Gamma) \subseteq \Sigma^{(i+1)}_s(\Gamma)$, which gives us:

(1) for every $A \in \varphi$ there is a $\Delta \in \Sigma^{(i+1)}_s(\Gamma)$ such that $\Delta \cap (\varphi \cup \varphi') = \{A\}$.

From the right-left direction of Lemma 3.1: for every $A \in \varphi'$, there is a $\Theta \in \Phi^{[i+1]}_{\varphi,s}$ such that $\Theta \cap \varphi' = \{A\}$, where $\Theta = \Delta \cap \Omega_{[i+1]}$ for a $\Delta \in \Sigma^{(i+1)}_s(\Gamma)$. Since $\varphi' \subseteq \Omega_{[i+1]}$, $\Delta \cap \varphi' = \{A\}$. Moreover, in view of the definition of $\Phi^{[i+1]}_{\varphi,s}$, $\Delta \cap \varphi = \emptyset$. Hence we have:

(2) for every $A \in \varphi'$, there is a $\Delta \in \Sigma^{(i+1)}_s(\Gamma)$ such that $\Delta \cap (\varphi \cup \varphi') = \{A\}$.

By (1) and (2): for every $A \in \varphi \cup \varphi'$, there is a $\Delta \in \Sigma^{(i+1)}_s(\Gamma)$ such that $\Delta \cap (\varphi \cup \varphi') = \{A\}$. By the left-right direction of Lemma 3.1, $\varphi \cup \varphi'$ is a minimal choice set of $\Sigma^{(i+1)}_s(\Gamma)$. Hence, $\varphi \cup \varphi' \in \Phi^{(i+1)}_s(\Gamma)$. ■

**Theorem 5.10** *For every stage $s$ of a proof from $\Gamma$, $\Phi^{\sqsubset}_s(\Gamma) \neq \emptyset$.*

*Proof.* Note that at every stage $s$ of a proof, $\Sigma^1_s(\Gamma)$ is a set of finite sets. By Fact 2.2, $\Phi^{(1)}_s(\Gamma) \neq \emptyset$. Let $\varphi_1 \in \Phi^{(1)}_s(\Gamma)$, and for all $i > 1$, let $\varphi_i$ be an arbitrary element in $\Phi^{[i]}_{\varphi_{i-1},s}$. Let $\varphi^{\oplus} = \varphi_1 \cup \varphi_2 \cup \ldots$. Note that for every $j \in I$, $\varphi_j \subseteq \Omega_{[j]}$. As a result, for every $j \in I$, $\varphi^{\oplus} \cap \Omega_{(j)} = \varphi_1 \cup \ldots \cup \varphi_j$, whence by Lemma 5.4:

(†) for every $j \in I$, $\varphi^{\oplus} \cap \Omega_{(j)} \in \Phi^{(j)}_s(\Gamma)$.

We will now prove that $\varphi^{\oplus}$ is a $\sqsubset$-minimal choice set of $\Sigma_s(\Gamma)$. Let $\Delta \in \Sigma_s(\Gamma)$. Then there is a $k \in I$ such that $\Delta \subseteq \Omega_{(k)}$. It follows immediately by (†) that $\varphi^{\oplus} \cap \Delta \neq \emptyset$. Hence, $\varphi^{\oplus}$ is a choice set of $\Sigma_s(\Gamma)$.

Assume now that $\varphi^{\oplus} \notin \Phi_s^{\sqsubset}(\Gamma)$. Hence, there is a choice set of $\Sigma_s(\Gamma)$, say $\psi$, such that for an $n \in I$, $\psi \cap \Omega_m = \varphi^{\oplus} \cap \Omega_m$ for all $m < n$ and $\psi \cap \Omega_n \subset \varphi^{\oplus} \cap \Omega_n$. Note that since $\Sigma_s^{(n)}(\Gamma) \subseteq \Sigma_s(\Gamma)$, $\psi$ is a choice set of $\Sigma_s^{(n)}(\Gamma)$, whence also $\psi \cap \Omega_{(n)}$ is a choice set of $\Sigma_s^{(n)}(\Gamma)$. This implies that $\varphi^{\oplus} \cap \Omega_{(n)}$ is not a minimal choice set of $\Sigma_s^{(n)}(\Gamma)$, which contradicts (†). ■

**Theorem 5.11** *For every* $\Gamma$, $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$.

*Proof.* Let the stage $\mathsf{g}$ be the same as in the two preceding chapters. Note that every minimal Dab-consequence of $\Gamma$ is derived at stage $\mathsf{g}$. It follows that $\Sigma_{\mathsf{g}}(\Gamma) = \Sigma(\Gamma)$. By Definitions 5.3 and 5.6, $\Phi_{\mathsf{g}}^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma)$. By Theorem 5.10, $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$. ■

## 5.3 Metatheory of $\mathbf{AL}_{\sqsubset}$

In this section, we prove that all the meta-theoretic properties discussed in Section 2.5 hold for lexicographic ALs as well. Some of the proofs are obtained by very small variations on proofs from the meta-theory of the standard format – see [25] for their most recent formulation.

### 5.3.1 Soundness and Completeness

**Lemma 5.5** *For every* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *and* $\Delta \cap \varphi = \emptyset$ *for a* $\varphi \in \Phi^{\sqsubset}(\Gamma)$*, then there is a finite* $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$*-proof from* $\Gamma$ *in which* $A$ *is derived on the condition* $\Delta$ *at an unmarked line.*

*Proof.* Suppose the antecedent holds. Due to the compactness of **LLL**, there is a $\Gamma' = \{A_1, \ldots, A_n\} \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. Let the adaptive proof P be constructed as follows. At line 1 we introduce the premise $A_1$ by PREM, …, and at line $n$ we introduce the premise $A_n$ by PREM. At line $n+1$ we derive $A$ by RC on the condition $\Delta$. Let $s$ be the stage consisting of lines 1 up to $n+1$. Since $\Gamma' \subseteq \Gamma \subseteq \mathcal{W}$, all Dab-formulas $B_1, \ldots, B_m$ that have been derived at stage $s$ (if any) are members of $\Omega$. Hence $\Phi_s^{\sqsubset}(\Gamma') = \{\{B_1, \ldots, B_m\}\}$. Due to the monotonicity of **LLL**, also $\Gamma \vdash_{\mathbf{LLL}} B_i$ for all these abnormalities $B_i$. Then $\{B_1, \ldots, B_m\} \subseteq \psi$ for all $\psi \in \Phi^{\sqsubset}(\Gamma)$. Since $\varphi \cap \Delta = \emptyset$ and $\varphi \in \Phi^{\sqsubset}(\Gamma)$, also $\Delta \cap \{B_1, \ldots, B_m\} = \emptyset$. Thus, line $n+1$ is unmarked. ■

**Lemma 5.6** *If* $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$*, then each of the following holds:*

1. *$A$ is derivable on a line $l$ of a finite $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma$, on a condition $\Delta$ such that $\Delta \cap \varphi = \emptyset$ for a $\varphi \in \Phi^{\sqsubset}(\Gamma)$*
2. *For every $\varphi \in \Phi^{\sqsubset}(\Gamma)$, there is a finite $\Delta \subseteq \Omega - \varphi$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.*

*Proof.* Suppose $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. By Definition 2.5, there is a finite $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$- proof P from $\Gamma$, such that (i) $A$ is derived in this proof on an unmarked line $l$ with a condition $\Delta$, (ii) every extension of the proof in which line $l$ is marked can be further extended such that line $l$ is unmarked again. We now extend P to a stage

$s$ such that all minimal Dab-consequences are derived on the empty condition. Note $\Phi^{\sqsubset}_{s}(\Gamma) = \Phi^{\sqsubset}(\Gamma)$ and that at every later stage $s'$, $\Phi^{\sqsubset}_{s'}(\Gamma) = \Phi^{\sqsubset}_{s}(\Gamma)$.
*Ad 1.* Suppose there is no $\varphi \in \Phi^{\sqsubset}(\Gamma)$ such that $\Delta \cap \varphi = \emptyset$. By Definition 5.7, line $l$ is marked at stage $s$ and at every later stage $s'$, which contradicts (ii).
*Ad 2.* Suppose there is a $\varphi \in \Phi^{\sqsubset}(\Gamma)$ for which there is no $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap \varphi = \emptyset$. By Definition 5.7 line $l$ is marked at stage $s$, and we cannot further extend the proof such that line $l$ is unmarked – this again contradicts (ii). ■

**Lemma 5.7** *Where $\Gamma \subseteq \mathcal{W}_s$: if for every $\varphi \in \Phi^{\sqsubset}(\Gamma)$, there is a finite $\Delta \subseteq \Omega - \varphi$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, then $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$.*

*Proof.* Suppose that for every $\varphi \in \Phi^{\sqsubset}(\Gamma)$ there is a finite $\Delta_\varphi \subseteq \Omega - \varphi$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta_\varphi)$. Due to Lemma 5.5, for every such $\Delta_\varphi$ there is a finite $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma$ in which $A$ is derived on the condition $\Delta_\varphi$ at an unmarked line $l$. Let P be any such proof (since $\Phi^{\sqsubset}(\Gamma)$ is non-empty there is at least one). Suppose the proof is extended to a stage $s$ in which line $l$ is marked. We extend the proof further to a stage $s'$ in which (i) all minimal Dab-formulas have been derived on the empty condition, and (ii) for all $\varphi \in \Phi(\Gamma)$, $A$ has been derived on the condition $\Delta_\varphi$. By Definition 5.7, line $l$ is unmarked at stage $s'$. ■

**Theorem 5.12** *If $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$, then $\Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. (Soundness)*

*Proof.* Suppose $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. If $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) = \emptyset$, the theorem follows immediately. Suppose $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \neq \emptyset$. Let $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, whence $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. By Theorem 5.3, $Ab(M) \in \Phi^{\sqsubset}(\Gamma)$. By Lemma 5.6.2, there is a $\Delta \subseteq \Omega$ such that $Ab(M) \cap \Delta = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By the soundness of **LLL**, $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. Since $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $M \Vdash \check{\neg} Dab(\Delta)$, $M \Vdash A$. ■

Where $\varphi \in \Phi(\Gamma)$, let $\mathcal{M}^{\varphi} =_{\mathsf{df}} \{M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) \mid Ab(M) = \varphi\}$. To obtain a completeness proof, we first establish two lemmas about this set of models:

**Lemma 5.8** *Where $\varphi \in \Phi(\Gamma)$: if $M$ is an **LLL**-model of $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$, then $M \in \mathcal{M}^{\varphi}$.*

*Proof.* Suppose (†) $\varphi \in \Phi(\Gamma)$ and $M$ is an **LLL**-model of $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$. Hence (1) $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. Note that $Ab(M) \subseteq \varphi$. By Lemma 5.2, $Ab(M)$ is a choice set of $\Sigma(\Gamma)$, whence by (†), $Ab(M) \not\subset \varphi$. Hence (2) $Ab(M) = \varphi$. By (1) and (2), $M \in \mathcal{M}^{\varphi}$. ■

**Lemma 5.9** *Where $\varphi \in \Phi(\Gamma)$: if all members of $\mathcal{M}^{\varphi}$ verify $A$, then $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega - \varphi$.*

*Proof.* Suppose all members of $\mathcal{M}^{\varphi}$ verify $A$. By Lemma 5.8, all **LLL**-models of $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$ verify $A$. This implies by the completeness of **LLL**: $\Gamma \cup (\Omega - \varphi)^{\check{\neg}} \vdash_{\mathbf{LLL}} A$. By the compactness of **LLL**, $\Gamma' \cup \Delta^{\check{\neg}} \vdash_{\mathbf{LLL}} A$, for a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta \subseteq \Omega - \varphi$. By the Deduction Theorem, $\Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, and by the monotonicity of **LLL**, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. ■

**Theorem 5.13** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$*, then* $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$*. (Completeness)*

*Proof.* Suppose (†) $\Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. Consider a $\varphi \in \Phi^{\sqsubset}(\Gamma)$. By Theorem 5.2, $\varphi \in \Phi(\Gamma)$. By Theorem 5.3, we have that for every $M \in \mathcal{M}^{\varphi}$, $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. In view of (†), it follows that for every $M \in \mathcal{M}^{\varphi}$, $M \Vdash A$. By Lemma 5.9, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega - \varphi$. Since this holds for all $\varphi \in \Phi^{\sqsubset}(\Gamma)$, we obtain by Lemma 5.7 that $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. ■

**Corollary 5.1** *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$ *iff* $\Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$ *(Soundness and Completeness)*

**Lemma 5.10** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *and* $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$*, then each of the following holds:*

1. *There is a finite* $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$*-proof from* $\Gamma$ *in which* $A$ *is derived on the condition* $\Delta$ *at an unmarked line*
2. $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$

*Proof. Ad 1.* The proof proceeds analogous to the proof for Lemma 5.5. We again construct the proof P as above. Note that since $\Gamma \vdash_{\mathbf{LLL}} B_i$ for all the derived abnormalities $B_i$, $U^{\sqsubset}_s(\Gamma') = \{B_1, \ldots, B_m\} \subseteq U^{\sqsubset}(\Gamma)$. Since $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$, also $\Delta \cap U^{\sqsubset}_s(\Gamma') = \emptyset$. Thus, line $n+1$ is unmarked.
*Ad 2.* Suppose that there is a finite $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. By item 1, there is a finite proof from $\Gamma$ such that $A$ is derived on the condition $\Delta$, on an unmarked line $l$. Suppose the proof is extended such that line $l$ becomes marked. In that case, we can further extend the proof, deriving every minimal Dab-consequence of $\Gamma$. Then where $s'$ is the stage of the second extension, $U^{\sqsubset}_{s'}(\Gamma) = U^{\sqsubset}(\Gamma)$, whence line $l$ is unmarked again. ■

**Lemma 5.11** *If* $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$*, then* $A$ *is derivable in a* $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$*-proof* P *from* $\Gamma$ *on line* $l$ *with condition* $\Delta$ *such that* $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$*.*

*Proof.* Suppose that $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. So A is finally derived on line $l$ of a $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$-proof from $\Gamma$. Let $\Delta$ be the condition of line $l$. Suppose that $\Delta \cap U^{\sqsubset}(\Gamma) \neq \emptyset$. In that case, we can extend P to a stage $s$ such that every minimal Dab-consequence of $\Gamma$ is derived in it. We have that $U^{\sqsubset}_s(\Gamma) = U^{\sqsubset}(\Gamma)$ and for all later stages $s'$, $U^{\sqsubset}_{s'}(\Gamma) = U^{\sqsubset}_s(\Gamma)$. As a result, line $l$ is marked at stage $s$ and remains marked in every further extension of the proof, which contradicts the antecedent in view of Definition 2.5. ■

**Theorem 5.14** $\Gamma \models_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$ *iff* $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *for a finite* $\Delta$ *such that* $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$*.*

*Proof.* (⇒) Suppose that $\Gamma \models_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$, whence for every $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$, $M \Vdash A$. By Definition 5.5, for every $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $Ab(M) \subseteq U^{\sqsubset}(\Gamma)$, $M \Vdash A$. Then $\Gamma \cup (\Omega - U^{\sqsubset}(\Gamma))^{\check{\neg}} \models_{\mathbf{LLL}} A$. As **LLL** is compact, $\Gamma' \cup (\Delta)^{\check{\neg}} \models_{\mathbf{LLL}} A$ for a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta \subseteq (\Omega - U^{\sqsubset}(\Gamma))$. Hence $\Gamma' \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. So, as **LLL** is monotonic, $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.

(⇐) Suppose there is a finite $\Delta \subseteq \Omega$ such that $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. Note that by Definition 5.5, for every $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$, $M \Vdash \check{\neg} Dab(\Delta)$. This implies that $M \Vdash A$ and we are done. ■

**Theorem 5.15** *If* $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$*, then* $\Gamma \models_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$*. (Soundness)*

*Proof.* Suppose $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. By Lemma 5.11, $A$ is derivable in a $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$-proof $\text{P}$ from $\Gamma$ on line $l$ with condition $\Delta$ such that $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. By Lemma 2.1 $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By the soundness of **LLL**, $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By Theorem 5.14, $\Gamma \models_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. ■

**Theorem 5.16** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Gamma \models_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$*, then* $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$*. (Completeness)*

*Proof.* Suppose $\Gamma \models_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. By Theorem 5.14, $\Gamma \models_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta$ such that $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. By the completeness of **LLL**, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. By Lemma 5.10.2, $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. ■

### 5.3.2 Strong Reassurance

Just as for all logics **HAL**, it can be shown that all logics $\mathbf{AL}_{\sqsubset}$ satisfy the kind of Strong Reassurance denoted by SR3 (see page 81). This property reads: if a model $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ is not selected by the adaptive logic, then there is a selected model $M'$ and an $i \in I$, such that $Ab(M') \cap \Omega_{(i)} \subset Ab(M) \cap \Omega_{(i)}$. It can easily be verified that SR3 is equivalent to the following:

(SR3′) $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{PAL}}(\Gamma)$ iff there is an $M' \in \mathcal{M}_{\mathbf{PAL}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$.

Hence, the claim that lexicographic ALs satisfy SR3, boils down to the claim that the relation $\sqsubset$ and the abnormal part relation impose a smooth partial order on the set of **LLL**-models of a premise set $\Gamma$.

As we will see below, that $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ satisfies SR3′, almost immediately implies that also $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$ satisfies this property, since every $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-model is also an $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$-model (see Theorem 5.5). So we first prove SR3′ for $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$.[4] Recall that the flat adaptive logics $\mathbf{AL}^{\mathbf{m}}_{\mathbf{i}}$ ($i \in I$) are defined by (i) **LLL**, (ii) $\Omega_i$ and (iii) Minimal Abnormality. The proof of SR3′ for $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ relies on the Strong Reassurance of each of these flat adaptive logics (see Theorem 2.12).

**Theorem 5.17** *If* $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$*, then there is an* $M' \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ *such that* $Ab(M') \sqsubset Ab(M)$*.*

*Proof.*[5] Suppose $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. Let $\mathcal{M}$ be the set of all $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$ — note that $\mathcal{M} \neq \emptyset$ since $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. For each $M' \in \mathcal{M}$, let $i_{M'} \in I$ be such that for all $j < i_{M'}$, $Ab(M) \cap \Omega_j =$

[4] It is possible to prove (Semantic) Reassurance for $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$, relying on Theorems 5.4 and 5.11. That is, Theorem 5.4 implies that whenever $\Gamma$ has **LLL**-models, then for every $\varphi \in \Phi^{\sqsubset}(\Gamma)$, there is an $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ such that $Ab(M) = \varphi$. Moreover, by Theorem 5.11, $\Phi^{\sqsubset}(\Gamma)$ is non-empty. It follows that whenever $\Gamma$ has **LLL**-models, it also has $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-models. However, for the proof of *Strong* Reassurance as given by SR3′, we need a deeper argument.

[5] In fact, Theorem 5.17 follows from (i) the fact that $\mathbf{HAL}^{\mathbf{m}}$ satisfies SR3 and (ii) the equivalence of the $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-semantics and the $\mathbf{HAL}^{\mathbf{m}}$-semantics – (ii) is proven in Chapter 6. However, to keep the current chapter homogeneous, I decided to stick with the current (independent) proof. I am very grateful to Peter Verdée for spotting various unclarities in the next to last version of this proof.

$Ab(M') \cap \Omega_j$, and $Ab(M') \cap \Omega_{i_{M'}} \subset Ab(M) \cap \Omega_{i_{M'}}$. Let $k = \min(\{i_{M'} \mid M' \in \mathcal{M}\})$, and let $M''$ be an arbitrary model in $\mathcal{M}$ be such that $i_{M''} = k$.

If $k = 1$, let $M_k$ be an arbitrary model in $\mathcal{M}_{\mathbf{AL^m_1}}(\Gamma)$ such that $Ab(M_k) \cap \Omega_1 \subseteq Ab(M'') \cap \Omega_1$. Note that $M_k$ exists in view of the Strong Reassurance of $\mathbf{AL^m_1}$ and the fact that $M''$ is an **LLL**-model of $\Gamma$.

If $k > 1$, let $M_i = M$ for every $i < k$, and let $\Delta_i = (\Omega_i - Ab(M_i))^{\check{\neg}}$ for every $i < k$. Moreover, let $M_k$ be an arbitrary model in $\mathcal{M}_{\mathbf{AL^m_k}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k-1})$ such that $Ab(M_k) \cap \Omega_k \subseteq Ab(M'') \cap \Omega_k$. Note that $M_k$ exists in view of the Strong Reassurance of $\mathbf{AL^m_k}$, and the fact that $M''$ is an **LLL**-model of $\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k-1}$.

Define the sets $\Delta_k, \Delta_{k+1}, \ldots$ and the models $M_{k+1}, M_{k+2}, \ldots$ recursively, as follows:

- Let $\Delta_k = (\Omega_k - Ab(M_k))^{\check{\neg}}$.
- Let $M_{k+1}$ be an arbitrary model in $\mathcal{M}_{\mathbf{AL^m_{k+1}}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_k)$.
- Let $\Delta_{k+1} = (\Omega_{k+1} - Ab(M_{k+1}))^{\check{\neg}}$.
- Let $M_{k+2}$ be an arbitrary model in $\mathcal{M}_{\mathbf{AL^m_{k+2}}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k+1})$.
- Let $\Delta_{k+2} = (\Omega_{k+2} - Ab(M_{k+2}))^{\check{\neg}}$.
- ...

It can easily be shown by an induction that each model $M_{k+j}$ (with $k+j \in I$) exists, and hence that the sets $\Delta_{k+j}$ are well-defined. For the base case ($j = 0$), this was already shown in the preceding paragraph. For the induction step, suppose that $M_{k+j}$ exists. By the construction, $M_{k+j} \in \mathcal{M}_{\mathbf{AL^m_{k+j}}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k+j-1})$. It follows that (i) $M_{k+j}$ is an **LLL**-model of $\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k+j-1}$. Moreover, in view of the construction, (ii) $M_{k+j} \Vdash \Delta_{k+j}$. By (i) and (ii), $M_{k+j}$ is an **LLL**-model of $\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k+j}$. This means that $\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k+j}$ has **LLL**-models. Hence by the Reassurance of $\mathbf{AL^m_{k+j+1}}$, the set $\mathcal{M}_{\mathbf{AL^m_{k+j+1}}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k+j})$ is non-empty, which means that the model $M_{k+j+1}$ exists.

**Proposition 5.1** *$\Gamma \cup \bigcup_{i \in I} \Delta_i$ has* **LLL**-*models.*

*Subproof.* Let $\Gamma'$ be an arbitrary finite subset of $\Gamma \cup \bigcup_{i \in I} \Delta_i$. Let $i \in I$ be such that $\Gamma' \subseteq (\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i)$. In view of the construction, $M_i$ is an **LLL**-model of $\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i$, whence by the monotonicity of **LLL**, $M_i$ is also an **LLL**-model of $\Gamma'$.

So every finite subset of $\Gamma \cup \bigcup_{i \in I} \Delta_i$ has **LLL**-models. By the compactness of **LLL**, $\Gamma \cup \bigcup_{i \in I} \Delta_i$ has **LLL**-models. ∎

Let $M_\star$ be an arbitrary model in $\mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \bigcup_{i \in I} \Delta_i)$ – note that $M_\star$ exists in view of Proposition 5.1. In the remainder, we prove that $M_\star$ is an $\mathbf{AL^m_\sqsubset}$-model of $\Gamma$, and $Ab(M_\star) \sqsubset Ab(M)$ – see Proposition 5.3, resp. 5.4. Before doing so, we first need to prove the following:

**Proposition 5.2** *For all $i \in I$, $Ab(M_\star) \cap \Omega_i = Ab(M_i) \cap \Omega_i$.*

*Subproof.* By the construction, we have that (†) for all $i \in I$, $Ab(M_\star) \cap \Omega_i \subseteq Ab(M_i) \cap \Omega_i$. Assume now that there is an $i \in I$ for which $Ab(M_\star) \cap \Omega_i \subset Ab(M_i) \cap \Omega_i$.

*Case 1:* $i < k$. Note that in view of the construction, (‡) for all $j < k$, $Ab(M_j) = Ab(M)$. Let $m \leq i$ be smallest such that $Ab(M_\star) \cap \Omega_m \subset Ab(M_m) \cap \Omega_m$. Note that $m < k$. By (‡), it follows that (i) for all $n < m$, $Ab(M_\star) \cap \Omega_n = Ab(M_n) \cap \Omega_n = Ab(M) \cap \Omega_n$. Also by (‡), (ii) $Ab(M_\star) \cap \Omega_m \subset Ab(M) \cap \Omega_m$. It follows that $Ab(M_\star) \sqsubset Ab(M)$, whence $M_\star \in \mathcal{M}$. But then $m \in \{i_{M'} \mid M' \in \mathcal{M}\}$, which is a contradiction to the fact that $k = \min(\{i_{M'} \mid M' \in \mathcal{M}\})$.

*Case 2:* $i \geq k$.

*Case 2.1:* $i = 1 = k$. By the monotonicity of **LLL**, $M_\star \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. By the assumption, $Ab(M_\star) \cap \Omega_1 \subset Ab(M_1) \cap \Omega_1$. But this contradicts the fact that, when $k = 1$, $M_1 \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma)$.

*Case 2.2:* $i > 1$. By the monotonicity of **LLL**, $M_\star \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\mathbf{i}}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{i-1})$. By the assumption, $Ab(M_\star) \cap \Omega_i \subset Ab(M_i) \cap \Omega_i$. But this contradicts the fact that $M_i \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\mathbf{i}}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{i-1})$. ∎

**Proposition 5.3** $M_\star \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$.

*Subproof.* Assume that there is an $M''' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that Let $l \in I$ be such that (i) for all $m < l$, $Ab(M''') \cap \Omega_m = Ab(M_\star) \cap \Omega_m$ and (ii) $Ab(M''') \cap \Omega_l \subset Ab(M_\star) \cap \Omega_l$. By the transitivity of $\sqsubset$, $Ab(M''') \sqsubset Ab(M)$ and hence $M''' \in \mathcal{M}$.

*Case 1:* $l < k$. In this case, for all $m < l$, $Ab(M_\star) \cap \Omega_m = Ab(M) \cap \Omega_m$, and also $Ab(M_\star) \cap \Omega_l = Ab(M) \cap \Omega_l$. By (i) and (ii), respectively: (i)' for all $m < l$, $Ab(M''') \cap \Omega_m = Ab(M) \cap \Omega_m$ and (ii)' $Ab(M''') \cap \Omega_l \subset Ab(M) \cap \Omega_m$. By this means that $l = i_{M'''} < k$, which is in contradiction with the minimality of $k$ in $\{i_{M'} \mid M' \in \mathcal{M}\}$.

*Case 2:* $l \geq k$.

*Case 2.1:* $l = 1 = k$. By (ii) $Ab(M''') \cap \Omega_1 \subset Ab(M_\star) \cap \Omega_1$. By Proposition 5.2, $Ab(M_\star) \cap \Omega_1 = Ab(M_1) \cap \Omega_1$. Thus $Ab(M''') \cap \Omega_1 \subset Ab(M_1) \cap \Omega_1$. This contradicts the fact that $M_1 \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma)$.

*Case 2.2:* $l > 1$. Note that due to (i) and Proposition 5.2, for all $m < l$, $Ab(M''') \cap \Omega_m = Ab(M_\star) \cap \Omega_m = Ab(M_m) \cap \Omega_m$. It follows that $M''' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{l-1})$. Due to (ii) and Proposition 5.2, $Ab(M''') \cap \Omega_l \subset Ab(M_\star) \cap \Omega_l = Ab(M_l) \cap \Omega_l$. This contradicts the fact that $M_l \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\mathbf{l}}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{l-1})$. ∎

**Proposition 5.4** $Ab(M_\star) \sqsubset Ab(M)$.

*Subproof.* Immediate in view of the fact that (i) for all $i < k$, $Ab(M_\star) \cap \Omega_i = Ab(M) \cap \Omega_i$ (see Proposition 5.2), and (ii) $Ab(M_\star) \cap \Omega_k = Ab(M_k) \cap \Omega_k \subseteq Ab(M'') \cap \Omega_k \subset Ab(M) \cap \Omega_k$. ∎

■

**Theorem 5.18** *If* $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$, *then there is an* $M' \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$ *such that* $Ab(M') \sqsubset Ab(M)$.

*Proof.* Suppose $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$. By Theorem 5.5, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By Theorem 5.17, there is a $M' \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$. By Theorem 5.5, $M' \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$. ■

**Corollary 5.2** *If* $\Gamma$ *has* **LLL**-*models, it has* $\mathbf{AL}_{\sqsubset}$-*models. (Reassurance)*

### 5.3.3 Cumulative Indifference and the Deduction Theorem

In the remainder of Section 5.3, $\Gamma'$ is a metavariable for subsets of $\check{\mathcal{W}}_s$.

**Theorem 5.19** *If $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, then $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$.*

*Proof.* Suppose (†) $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. Consider a $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. By Definition 5.2, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma')$ and hence $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. Suppose that $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By Theorem 5.17, there is a $M' \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$. However, in view of (†), $M' \Vdash A$ for every $A \in \Gamma'$, whence also $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma')$. By Definition 5.2, $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$, which contradicts the supposition.

Consider a $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By (†), $M \Vdash A$ for every $A \in \Gamma'$. By Definition 5.2, $M$ is a **LLL**-model of $\Gamma$. We thus obtain that $M$ is a **LLL**-model of $\Gamma \cup \Gamma'$. Suppose $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. By Theorem 5.17, there is a $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma')$: $Ab(M') \sqsubset Ab(M)$. Hence $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. By Definition 5.2, $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, whence we have obtained a contradiction. ■

**Lemma 5.12** *If $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, then $\Phi^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$.*

*Proof.* Suppose $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. If $\Gamma$ has no **LLL**-models, then $\Gamma$ and $\Gamma \cup \Gamma'$ are **LLL**-trivial, whence $\Phi^{\sqsubset}(\Gamma) = \{\Omega\} = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$.

If (1) $\Gamma$ has **LLL**-models, then in view of the reassurance of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$, there is a $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By Theorem 5.19, $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$, whence also (2) $\Gamma \cup \Gamma'$ has **LLL**-models. By Theorem 5.19, $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. By (1), (2) and Theorem 5.4, this means that $\Phi^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$. ■

**Theorem 5.20** *If $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, then $Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. (Cautious Monotonicity)*

*Proof.* Suppose $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, whence by Lemma 5.12, (†) $\Phi^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$. Suppose $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. By Lemma 5.6.2 and (†), we have that (‡) for every $\varphi \in \Phi^{\sqsubset}(\Gamma \cup \Gamma')$, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta$ such that $\varphi \cap \Delta = \emptyset$. By Lemma 5.6.1 and (‡), there is a finite $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-proof P from $\Gamma$ such that $A$ is derived at an unmarked line $l$ with condition $\Delta$, and $\Delta \cap \varphi = \emptyset$ for a $\varphi \in \Phi^{\sqsubset}(\Gamma \cup \Gamma')$. Note that P is also a proof from $\Gamma \cup \Gamma'$.

Suppose line $l$ is marked in an extension of P. We may extend this extension further such that (a) all minimal Dab-consequences of $\Gamma \cup \Gamma'$ are derived on the empty condition and (b) for every $\varphi \in \Phi^{\sqsubset}(\Gamma \cup \Gamma')$, $A$ is derived on a condition $\Delta$ such that $\Delta \cap \varphi = \emptyset$ – this is possible in view of (‡). Let $s$ be the stage of this second extension of P.

Note that by (a), for every later stage $s'$, $\Phi_{s'}(\Gamma \cup \Gamma') = \Phi(\Gamma \cup \Gamma')$. By (b), at every later stage $s'$, for every $\varphi \in \Phi_{s'}(\Gamma \cup \Gamma')$, $A$ is derived on a condition $\Delta$ such that $\Delta \cap \varphi = \emptyset$. By Definition 5.7, line $l$ is unmarked at every such stage $s'$, whence $A$ is finally derived in the proof. Hence $\Gamma \cup \Gamma' \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. ■

**Theorem 5.21** *Where $\Gamma \subseteq \mathcal{W}_s$: if $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, then $Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. (Cumulative Transitivity)*

*Proof.* Suppose $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, whence by Theorem 5.19, (†) $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. Suppose $\Gamma \cup \Gamma' \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. By the soundness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$, $\Gamma \cup \Gamma' \models_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. By (†), $\Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. By the completeness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$, $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. ■

**Theorem 5.22** *$Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$.*

*Proof.*[6] Suppose $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. By Lemma 5.11, there is a finite $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$-proof P from $\Gamma$, in which $A$ occurs on an unmarked line $l$ with condition $\Delta$, and $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. Let $s$ be the stage of this proof. Since line $l$ is unmarked, we have that (†) $\Delta \cap U^{\sqsubset}_s(\Gamma) = \emptyset$. Since $U^{\sqsubset}(\Gamma) = \bigcup \Phi^{\sqsubset}(\Gamma)$, we can derive that (‡) $\Delta \cap \varphi = \emptyset$ for every $\varphi \in \Phi^{\sqsubset}(\Gamma)$.

Note that P is also an $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma$. By (†) and the fact that $U^{\sqsubset}_s(\Gamma) = \bigcup \Phi^{\sqsubset}_s(\Gamma)$, we can derive that $\Delta \cap \varphi = \emptyset$ for every $\varphi \in \Phi^{\sqsubset}_s(\Gamma)$. Hence line $l$ is also unmarked in P if the strategy is $\sqsubset$-Minimal Abnormality.

Suppose line $l$ is $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-marked in a further extension of the proof. We then extend the proof further to a stage $s'$, such that every minimal Dab-consequence of $\Gamma$ is derived at stage $s'$. Note that $\Phi^{\sqsubset}_{s'}(\Gamma) = \Phi^{\sqsubset}(\Gamma)$. By (‡) and Definition 5.7, line $l$ is unmarked at stage $s'$. ■

**Lemma 5.13** *If $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$, then $U^{\sqsubset}(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma)$.*

*Proof.* Suppose $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$. By Theorem 5.22, $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By Lemma 5.12, $\Phi^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$, whence by Definition 5.4, $U^{\sqsubset}(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma)$. ■

**Theorem 5.23** *If $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$, then $\mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$.*

*Proof.* Suppose $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$. By Lemma 5.13, (†) $U^{\sqsubset}(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma)$. Suppose $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$. By the supposition and the soundness of $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma')$. By (†) and Definition 5.5, $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$.

Suppose $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$. By Definition 5.5, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma')$ and $Ab(M) \subseteq U^{\sqsubset}(\Gamma \cup \Gamma')$. Then by the monotonicity of **LLL** and (†), $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $Ab(M) \subseteq U^{\sqsubset}(\Gamma)$. By Definition 5.5, $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$. ■

**Theorem 5.24** *If $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$, then $Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$. (Cautious Monotonicity)*

*Proof.* Suppose $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$, whence by Lemma 5.13, (†) $U^{\sqsubset}(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma)$. Suppose $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$, whence by Lemma 5.11, $A$ is derivable in an $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$-proof P from $\Gamma$ on line $l$ with condition $\Delta$ such that $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. Note that P is a $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$-proof from $\Gamma \cup \Gamma'$ as well.

Suppose that line $l$ is marked in an extension of P. We may then further extend the extension, such every minimal Dab-consequence of $\Gamma \cup \Gamma'$ is derived in it on the empty condition. Where the stage of the second extension is $s$, we have that $U^{\sqsubset}_s(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma \cup \Gamma')$. By (†), $\Delta \cap U^{\sqsubset}(\Gamma \cup \Gamma') = \emptyset$. As a result, line $l$ is unmarked at stage $s$. ■

[6]Some readers might wonder if this theorem is not an immediate consequence of Theorem 5.5 above. However, since the completeness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ and $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$ is restricted to premise sets $\Gamma \subseteq \mathcal{W}_s$, we need a syntactic argument to prove that $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ is always at least as strong as $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$.

**Theorem 5.25** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$*, then* $Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$*. (Cumulative Transitivity)*

*Proof.* Suppose $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$, whence by Theorem 5.23, (†) $\mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$. Now suppose $\Gamma \cup \Gamma' \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. By the soundness of $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$, $\Gamma \cup \Gamma' \vDash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. By (†), $\Gamma \vDash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. By the completeness of $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$, $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. ■

In view of Theorems 5.20, 5.21, 5.24, and 5.25, we immediately have:

**Corollary 5.3** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Gamma' \subseteq Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma)$*, then* $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma \cup \Gamma')$*. (Cumulative Indifference)*

By Corollary 5.3, Theorem 5.9.1 and Lemma 2.7:

**Theorem 5.26** *Where* $\Gamma \subseteq \mathcal{W}_s$*,* $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{AL}_{\sqsubset}}(Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma))$*. (Fixed Point)*

**Theorem 5.27** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Gamma \cup \{A\} \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} B$*, then* $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A \check{\supset} B$*. (Deduction Theorem for* $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$*)*[7]

*Proof.* Suppose $\Gamma \cup \{A\} \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} B$, whence by the soundness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$: (†) every $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-model of $\Gamma \cup \{A\}$ verifies $B$. Assume that $\Gamma \nvdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A \check{\supset} B$ — we derive a contradiction. By the completeness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$, there is a $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-model $M$ of $\Gamma$ such that $M \Vdash A \check{\wedge} \check{\neg} B$. Note that $M$ is a **LLL**-model of $\Gamma \cup \{A\}$. In view of (†), $M$ is not a $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-model of $\Gamma \cup \{A\}$, whence there is a **LLL**-model $M'$ of $\Gamma \cup \{A\}$ such that $Ab(M') \sqsubset Ab(M)$. However, by the monotonicity of **LLL**, $M'$ is a **LLL**-model of $\Gamma$. By Definition 5.2, $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. ■

### 5.3.4 Relations Between Logics

**Theorem 5.28** $Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$*.*

*Proof.* Suppose $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$. By Lemma 5.6.2, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega$. By **CL**-properties, $\Gamma \cup \Omega^{\check{\neg}} \vdash_{\mathbf{LLL}} A$, whence in view of the definition of **ULL**, $\Gamma \vdash_{\mathbf{ULL}} A$. ■

Let $\mathbf{AL}^{\mathbf{r}}$ and $\mathbf{AL}^{\mathbf{m}}$ be defined as in Section 5.1, i.e. by (i) **LLL**, (ii) $\Omega = \bigcup_{i \in I} \Omega_i$ and (iii) Reliability, resp. Minimal Abnormality. In view of Theorems 5.9, 5.22, 5.28 and 5.8, we have:

**Corollary 5.4** *Each of the following holds:*

1. $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$
2. $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$
3. $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$

**Theorem 5.29** *Where* $\Gamma \subseteq \mathcal{W}_s$*,* $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{AL}_{\sqsubset}}(Cn_{\mathbf{LLL}}(\Gamma))$*. (**LLL**-invariance)*

[7]The Deduction Theorem does not hold for $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$. This follows immediately in view of the fact that it does not hold for $\mathbf{AL}^{\mathbf{r}}$ – see page 31 – and the fact that every logic $\mathbf{AL}^{\mathbf{r}}$ is a logic in the extended format as well – see Section 5.2.3.

*Proof.* By Theorem 5.9.2, (‡) $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma)$. By (‡) and Corollary 5.3, $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma \cup Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma)$. Since **LLL** is reflexive, $\Gamma \subseteq Cn_{\mathbf{LLL}}(\Gamma)$, whence $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma \cup Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{AL}_{\sqsubset}}(Cn_{\mathbf{LLL}}(\Gamma))$ and we are done. ■

**Theorem 5.30** *Where $\Gamma \subseteq \mathcal{W}_s$, $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma))$. (**LLL**-closure)*

*Proof.* That $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma))$ follows from the reflexivity of **LLL**. For the other direction of the inclusion, suppose $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma))$. By the soundness of **LLL**, (†) $A$ is true in every **LLL**-model $M$ of $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma)$. By the soundness of $\mathbf{AL}_{\sqsubset}$, every $M \in \mathcal{M}_{\mathbf{AL}_{\sqsubset}}(\Gamma)$ is a **LLL**-model of $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma)$. But then by (†), every $M \in \mathcal{M}_{\mathbf{AL}_{\sqsubset}}(\Gamma)$ verifies $A$. By the completeness of $\mathbf{AL}_{\sqsubset}$, $A \in Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma)$. ■

### 5.3.5 Normal Premise Sets

By Theorem 2.17 and Corollary 5.4, we can derive:

**Corollary 5.5** *If $\Gamma$ is normal, then $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.*

As for superpositions of ALs and hierarchic ALs, we can easily show the well-behavedness of $\mathbf{AL}_{\sqsubset}$ in the case of premise sets that are normal up to a certain level $i$. The proof relies on two lemmas:

**Lemma 5.14** *If $\Gamma$ is normal up to level $i$, then there is no $\Delta \subset \Omega_{(i)}$ such that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$.*

*Proof.* Suppose the antecedent holds, and let $\Delta$ be a finite subset of $\Omega_{(i)}$. By the supposition, there is an $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Omega_{(i)}^{\check{\neg}})$, whence $M \nVdash Dab(\Delta)$. By the soundness of **LLL**, $\Gamma \nvdash_{\mathbf{LLL}} Dab(\Delta)$. ■

**Lemma 5.15** *If $\Gamma$ is normal up to level $i$, then at every stage $s$ of a proof from $\Gamma$, $\varphi \cap \Omega_{(i)} = \emptyset$ for every $\varphi \in \Phi_s^{\sqsubset}(\Gamma)$.*

*Proof.* Suppose the antecedent holds and let $s$ be a stage of a proof from $\Gamma$. Assume that for a $\varphi \in \Phi_s^{\sqsubset}(\Gamma)$, $\varphi \cap \Omega_{(i)} \neq \emptyset$. Let $\psi = \bigcup\{\Theta - \Omega_{(i)} \mid \Theta \in \Sigma_s^{(i)}(\Gamma)\}$. By the supposition and Lemma 5.14, every $\Theta \in \Sigma_s^{(i)}(\Gamma)$ is such that $\Theta - \Omega_{(i)} \neq \emptyset$, whence $\psi$ is a choice set of $\Sigma_s^{(i)}(\Gamma)$. However, since $\psi \cap \Omega_{(i)} = \emptyset$, it follows that $\psi \sqsubset \varphi$ — a contradiction. ■

**Theorem 5.31** *If $\Gamma$ is normal up to level $i$, then $Cn_{\mathbf{ULL}_{(i)}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$.*

*Proof.* Suppose the antecedent holds and $A \in Cn_{\mathbf{ULL}_{(i)}}(\Gamma)$, whence $\Gamma \cup \Omega_{(i)}^{\check{\neg}} \vdash_{\mathbf{LLL}} A$. By the compactness of **LLL**, there are $B_1, \ldots, B_n \in \Gamma$ and there is a finite $\Delta \subset \Omega_{(i)}$ such that $\{B_1, \ldots, B_n\} \cup \Delta^{\check{\neg}} \vdash_{\mathbf{LLL}} A$. By the Deduction Theorem, $\{B_1, \ldots, B_n\} \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.

Let P be an $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma$, obtained by (i) introducing all the premises $B_i$ ($i \leq n$) and (ii) deriving $A$ on the condition $\Delta$ from these premises, by the rule RC. Let $l$ be the line on which $A$ is derived.

Assume that $l$ is marked. It follows that there is a $\varphi \in \Phi_s^{\sqsubset}(\Gamma)$ such that $\varphi \cap \Delta \neq \emptyset$, whence $\varphi \cap \Omega_{(i)} \neq \emptyset$. By Lemma 5.15, $\Gamma$ is not normal up to level $i$ — a contradiction. By the same reasoning, it follows that in every extension of P, line $l$ remains unmarked. Hence, $A$ is finally derived in P, whence $A \in Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. ■

### 5.3.6 Equivalent Premise Sets

Recall the criteria for the **AL**-equivalence of two premise sets $\Gamma, \Gamma'$ from Section 2.5 in Chapter 2. In the remainder, we will show that each of these criteria apply to all logics $\mathbf{AL}_{\sqsubset}$. This requires that we first establish some additional results, each of which were shown for flat ALs in [33] and recapitulated in Chapter 2.

**Theorem 5.32** *Where $\Gamma, \Gamma' \subseteq \mathcal{W}_s$: if $\Gamma$ and $\Gamma'$ are* **LLL***-equivalent, then they are $\mathbf{AL}_{\sqsubset}$-equivalent.*

*Proof.* Suppose $\Gamma$ and $\Gamma'$ are **LLL**-equivalent, whence $Cn_{\mathbf{LLL}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma')$. By Theorem 5.30, $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{AL}_{\sqsubset}}(Cn_{\mathbf{LLL}}(\Gamma))$ and $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma') = Cn_{\mathbf{AL}_{\sqsubset}}(Cn_{\mathbf{LLL}}(\Gamma'))$. Hence $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma')$. ■

**Theorem 5.33** *Every monotonic logic that is weaker than or identical to $\mathbf{AL}_{\sqsubset}$ is weaker than or identical to* **LLL***. (Maximality of* **LLL***)*

*Proof.* *($\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$)* This follows immediately in view of (i) Lemma 5.6.2 from the current chapter, (ii) the proof of Theorem 10 in [33]— replace $\Phi(\Gamma \cup \Gamma')$ in that proof by $\Phi^{\sqsubset}(\Gamma \cup \Gamma')$ and Theorem 4 in that proof by Lemma 5.6.2 from the current chapter.
*($\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$)* This follows immediately in view of the fact that $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ is stronger than $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$ — see Theorem 5.22 — and item 1. ■

**Fact 5.3** *Where* **L** *is a Tarski-logic weaker than or identical to* **LLL***: if $\Gamma$ and $\Gamma'$ are* **L***-equivalent, then they are* **LLL***-equivalent.*

**Theorem 5.34** *Where $\Gamma, \Gamma' \subseteq \mathcal{W}_s$, $Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma')$ if one of the following holds:*

*(C1)* $\Gamma' \subseteq Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma)$ *and* $\Gamma \subseteq Cn_{\mathbf{AL}_{\sqsubset}}(\Gamma')$
*(C2) Where* **L** *is a Tarski-logic weaker than or identical to $\mathbf{AL}_{\sqsubset}$: $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$*
*(C3) Where* **L** *is a Tarski-logic and for every $\Theta \subseteq \mathcal{W}_s$, $Cn_{\mathbf{AL}_{\sqsubset}}(\Theta) = Cn_{\mathbf{L}}(Cn_{\mathbf{AL}_{\sqsubset}}(\Theta))$: $Cn_{\mathbf{L}}(\Gamma) = Cn_{\mathbf{L}}(\Gamma')$*

*Proof. Ad 1.* Immediate in view of Corollary 5.3 and Lemma 2.8.
*Ad 2. and 3.* It was proven in [33] that (C2) and (C3) are coextensive whenever (i) $\mathbf{AL}_{\sqsubset}$ is reflexive and has the Fixed Point property, and (ii) **L** is monotonic. Hence in view of Theorem 5.9.1 and Theorem 5.26, it suffices to prove item 2.

Suppose **L** is a Tarski-logic weaker than or identical to $\mathbf{AL}_{\sqsubset}$. By Theorem 5.33, **L** is weaker than or identical to **LLL**. Now suppose $\Gamma$ and $\Gamma'$ are **L**-equivalent. By Fact 5.3, $\Gamma$ and $\Gamma'$ are **LLL**-equivalent. By Theorem 5.32, $\Gamma$ and $\Gamma'$ are $\mathbf{AL}_{\sqsubset}$-equivalent. ■

# Chapter 6

# Comparing the Formats

*Sections 6.2-6.5 of this chapter are based on the paper "Three Formats of Prioritized Adaptive Logics. A Comparative Study" (Logic Journal of the IGPL 2012, doi:10.1093/jigpal/JZS004), which was co-authored by Christian Straßer. We thank two anonymous referees for their valuable comments on that paper. Sections 6.6 and 6.7 contain some unpublished results, and Section 6.8 presents a general conclusion of Part I of this thesis. I am indebted to Peter Verdée for his critical remarks on previous drafts of this chapter.*

## 6.1 Introduction

In this chapter, I will compare various formats from the preceding chapters in terms of the logical strength of systems defined in these formats. That is, for any given $I \subseteq \mathbb{N}$, any sequence $\langle \Omega_i \rangle_{i \in I}$ of sets of abnormalities, and any lower limit logic **LLL**, I will compare the logics $\mathbf{AL^x}, \mathbf{SAL^x}, \mathbf{SAL^x_{(I)}}, \mathbf{HAL^x}, \mathbf{AL^x_{\sqsubset}}$ defined from them, both for the case where $\mathbf{x} = \mathbf{r}$ and $\mathbf{x} = \mathbf{m}$.[1] Recall that these logics are defined as follows:

- $\mathbf{AL^x}$ is the flat adaptive logic defined by the triple $\langle \mathbf{LLL}, \bigcup_{i \in I} \Omega_i, \mathbf{x} \rangle$
- Where $i \in I$, $\mathbf{AL^x_i}$ is the flat adaptive logic defined by the triple $\langle \mathbf{LLL}, \Omega_i, \mathbf{x} \rangle$
- Where $i \in I$, $\Omega_{(i)} = \Omega_1 \cup \ldots \cup \Omega_i$
- Where $i \in I$, $\mathbf{AL^x_{(i)}}$ is the flat adaptive logic defined by the triple $\langle \mathbf{LLL}, \Omega_{(i)}, \mathbf{x} \rangle$
- $\mathbf{SAL^x}$ is obtained by the superposition of the logics $\langle \mathbf{AL^x_i} \rangle_{i \in I}$
- $\mathbf{SAL^x_{(I)}}$ is obtained by the superposition of the logics $\langle \mathbf{AL^x_{(i)}} \rangle_{i \in I}$
- $\mathbf{HAL^x}$ is obtained by the hierarchic combination of the logics $\langle \mathbf{AL^x_{(i)}} \rangle_{i \in I}$
- $\mathbf{AL^x_{\sqsubset}}$ is the lexicographic adaptive logic defined by the triple $\langle \mathbf{LLL}, \langle \Omega_i \rangle_{i \in I}, \mathbf{x} \rangle$

[1] I will not make pairwise comparisons of logics that have a different strategy in this chapter. It was shown in the preceding chapters that flat ALs, hierarchic ALs, and lexicographic ALs that have the Minimal Abnormality Strategy are at least as strong as their respective Reliability-variants. I also illustrated that in general, $\mathbf{SAL^r}$ and $\mathbf{SAL^m}$ are incomparable – see Chapter 3, Section 3.1.2. As shown at the end of Section 6.7, every logic $\mathbf{SAL^m_{(I)}}$ is at least as strong as the corresponding logic $\mathbf{SAL^r_{(I)}}$ given certain weak restrictions on the premise set (see Corollary 6.17).

I will first consider the Minimal Abnormality-variants of these formats. For systematic reasons, it is easiest to start with the logics $\mathbf{SAL^m_{(I)}}$, $\mathbf{HAL^m}$ and $\mathbf{AL^m_{\sqsubset}}$. In Sections 6.2-6.4, it will be shown that each of the following holds:

(i) The semantic consequence relations of $\mathbf{HAL^m}$, $\mathbf{SAL^m_{(I)}}$ and $\mathbf{AL^m_{\sqsubset}}$ are equivalent (Section 6.2, Corollary 6.1)

(ii) The syntactic consequence relation of $\mathbf{AL^m_{\sqsubset}}$ is always at least as strong as that of $\mathbf{SAL^m_{(I)}}$ and $\mathbf{HAL^m}$ (Section 6.3, Corollaries 6.5 and 6.6)

(iii) The syntactic consequence relations of $\mathbf{SAL^m_{(I)}}$ and $\mathbf{HAL^m}$ are complete and equivalent to $\mathbf{AL^m_{\sqsubset}}$, given certain (weak) restrictions on the premise sets (Section 6.4, Corollaries 6.11 and 6.12)

To arrive at (ii), it is first shown that every logic in the format of $\mathbf{SAL^m_{(I)}}$ is sound with respect to its semantics (see Section 6.3, Theorem 6.5.2).

In view of (i) and the soundness and completeness of $\mathbf{AL^m_{\sqsubset}}$ for all $\Gamma \subseteq \mathcal{W}_s$, we obtain three different semantic characterizations of the $\mathbf{AL^m_{\sqsubset}}$-consequence relation. (ii) and (iii) are of particular interest in view of the fact that $\mathbf{AL^m_{\sqsubset}}$, $\mathbf{SAL^m_{(I)}}$ and $\mathbf{HAL^m}$ are each characterized by their own format-specific semantics and proof theory. This means that we obtain a great variety of methods to prove that a formula is (not) an $\mathbf{AL^m_{\sqsubset}}$-consequence of a set of premises. As shown in Section 6.5, whenever the restrictions mentioned in (iii) hold, properties such as Idempotence and the Deduction Theorem can easily be transferred from $\mathbf{AL^m_{\sqsubset}}$ to $\mathbf{HAL^m}$ and $\mathbf{SAL^m_{(I)}}$, relying on (i)-(iii).

In Section 6.6, the above three systems are compared to the corresponding logics in the format $\mathbf{SAL^m}$, with the following result:

(iv) whenever $\Phi(\Gamma)$ is finite, then every logic $\mathbf{SAL^m}$ is at least as strong as the corresponding logics $\mathbf{SAL^m_{(I)}}$, $\mathbf{HAL^m}$ and $\mathbf{AL^m_{\sqsubset}}$.

Hence, superpositions of flat ALs are usually stronger than the corresponding hierarchic logics and lexicographic ALs. By means of a simple example, I will also show that the converse of (iv) fails. The main results concerning the logical strength of the Minimal Abnormality-variants are summarized in Corollary 6.14. Examples that illustrate why (iii) and (iv) are restricted to specific classes of premise sets, are presented in Appendix C.

In Section 6.7, I will consider the Reliability-variants of the various formats. There I will show each of the following:

(v) every logic $\mathbf{SAL^r_{(I)}}$ is at least as strong as the corresponding logic $\mathbf{HAL^r}$

(vi) every logic $\mathbf{SAL^r}$ is at least as strong as the corresponding logic $\mathbf{SAL^r_{(I)}}$

(vii) every logic $\mathbf{AL^r_{\sqsubset}}$ is at least as strong as the corresponding logic $\mathbf{SAL^r_{(I)}}$

Again, simple examples will be given to illustrate that the converses of (v)-(vii) often fail. It will also be shown that $\mathbf{AL^r_{\sqsubset}}$ and $\mathbf{SAL^r}$ are in general incomparable. Corollaries 6.15 and 6.16 summarize the main results of this and preceding chapters, concerning the logical strength of the Reliability-variants of flat and prioritized ALs.

On the basis of the results from this and the preceding chapters, I will formulate some general conclusions about the use and disuse of the formats for prioritized adaptive logics, presented in this thesis (Section 6.8). There I will also return to the restrictions of the current study, and discuss various topics for further research.

**Some More Notational Conventions** To close the gap between, on the one hand, combinations in terms of flat adaptive logics, and, on the other hand, lexicographic ALs, it will be convenient to define lexicographic ALs that only consider abnormalities up to a certain rank $i \in I$. Define $\sqsubset_{(i)}$, the lexicographic order up to level $i$, as follows:

**Definition 6.1** *Where* $\Delta, \Delta' \subseteq \Omega$*:* $\Delta \sqsubset_{(i)} \Delta'$ *iff* $\langle \Delta \cap \Omega_j \rangle_{j \leq i} \sqsubset_{\mathsf{lex}} \langle \Delta' \cap \Omega_j \rangle_{j \leq i}$.

In view of the above definition and Definition 5.1, we have:

**Fact 6.1** *Where* $i \in I$ *and* $\Delta, \Delta' \subseteq \Omega$*, each of the following holds:*

1. $\Delta \sqsubset_{(i)} \Delta'$ *iff* $\Delta \cap \Omega_{(i)} \sqsubset_{(i)} \Delta' \cap \Omega_{(i)}$
2. *If* $\Delta \sqsubset_{(i)} \Delta'$*, then* $\Delta \sqsubset \Delta'$
3. *If* $\Delta \sqsubset_{(i)} \Delta'$*, then* $\Delta \sqsubset_{(j)} \Delta'$ *for every* $j \in I, i \leq j$

Where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$ and $i \in I$, the logic $\mathbf{AL}^{\mathbf{x}}_{\sqsubset_{(\mathbf{i})}}$ is a lexicographic AL, defined by $\langle \mathbf{LLL}, \langle \Omega_j \rangle_{j \leq i}, \mathbf{x} \rangle$. Note that $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and there is no $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $Ab(M') \sqsubset_{(i)} Ab(M)$.

In a similar vein as for $\mathbf{AL}^{\mathbf{x}}_{\sqsubset}$, we can characterize the $\mathbf{AL}^{\mathbf{x}}_{\sqsubset_{(\mathbf{i})}}$-semantics syntactically, in terms of a specific set of (sets of) abnormalities – see also Section 5.1.4 in Chapter 5. Let $\Phi^{\sqsubset_{(i)}}(\Gamma)$ denote the set of $\sqsubset_{(i)}$-minimal choice sets of $\Sigma^{(i)}(\Gamma)$, and let $U^{\sqsubset_{(i)}}(\Gamma) = \bigcup \Phi^{\sqsubset_{(i)}}(\Gamma)$. In view of the preceding chapter, we can infer that $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $Ab(M) \cap \Omega_{(i)} \in \Phi^{\sqsubset_{(i)}}(\Gamma)$, and that $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $Ab(M) \cap \Omega_{(i)} \subseteq U^{\sqsubset_{(i)}}(\Gamma)$.

Note also that, where $\Delta, \Delta' \subseteq \Omega$, $\Delta \sqsubset_{(1)} \Delta'$ iff $\Delta \cap \Omega_{(1)} \subset \Delta' \cap \Omega_{(1)}$. From this, we can infer:

**Fact 6.2** *Each of the following holds:*

1. $\Phi^{(1)}(\Gamma) = \Phi^{\sqsubset_{(1)}}(\Gamma)$
2. $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{(\mathbf{1})}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{1})}}}(\Gamma)$
3. $Cn_{\mathbf{AL}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{m}}_{(\mathbf{1})}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{1})}}}(\Gamma)$
4. $U^1(\Gamma) = U^{(1)}(\Gamma) = U^{\sqsubset_{(1)}}(\Gamma)$
5. $\mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{(\mathbf{1})}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset_{(\mathbf{1})}}}(\Gamma)$
6. $Cn_{\mathbf{AL}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{r}}_{(\mathbf{1})}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset_{(\mathbf{1})}}}(\Gamma)$

**Generic Properties** Recall that the subset-relation is central in the definition of the $\mathbf{AL^m}$-semantics and $\mathbf{AL^m}$–proof theory. For $\mathbf{AL^m_\sqsubset}$, this relation is replaced by the lexicographic order $\sqsubset$. To shorten some proofs in the remainder, it will be useful to state one definition and two theorems generically, using $\prec$ as a metavariable for $\subset$ and $\sqsubset$.

Let $\Phi^\subset(\Gamma) =_{\mathsf{df}} \Phi(\Gamma)$ and let $\mathbf{AL^m_\subset} =_{\mathsf{df}} \mathbf{AL^m}$. Where $\prec\ \in \{\subset, \sqsubset\}$, we have:

**Definition 6.2** *$M \in \mathcal{M}_{\mathbf{AL^m_\prec}}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and there is no $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $Ab(M') \prec Ab(M)$.*

The following theorems are corollaries of theorems from Chapters 2 and 5:

**Theorem 6.1** *If $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL^m_\prec}}(\Gamma)$, then there is a $M' \in \mathcal{M}_{\mathbf{AL^m_\prec}}(\Gamma)$ such that $Ab(M') \prec Ab(M)$.*

**Theorem 6.2** *Each of the following holds:*

1. *$M \in \mathcal{M}_{\mathbf{AL^m_\prec}}(\Gamma)$ iff ($M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ and $Ab(M) \in \Phi^\prec(\Gamma)$)*
2. *If $\Gamma$ has $\mathbf{LLL}$-models, then $\Phi^\prec(\Gamma) = \{Ab(M) \mid M \in \mathcal{M}_{\mathbf{AL^m_\prec}}(\Gamma)\}$*

## 6.2 Three Equivalent Semantic Characterizations

**Superpositions and Lexicographic ALs** Recall that the $\mathbf{SAL^m_{(I)}}$-semantics is defined in terms of the sequential superpositions of minimally abnormal selections, in view of the sets $\langle\Omega_{(i)}\rangle_{i\in I}$ – see Definition 3.4 on page 50. Below, it is shown that this sequential selection leads to the same result as the lexicographic selection of the $\mathbf{AL^m_\sqsubset}$-semantics – see Theorem 6.3.

We start with a number of facts about the $\mathbf{SAL^m_{(I)}}$-semantics. By Facts 3.2 and 6.2.2, we have:

**Fact 6.3** *$\mathcal{M}_{\mathbf{SAL^m_1}}(\Gamma) = \mathcal{M}_{\mathbf{AL^m_1}}(\Gamma) = \mathcal{M}_{\mathbf{AL^m_{(1)}}}(\Gamma) = \mathcal{M}_{\mathbf{AL^m_{\sqsubset_{(1)}}}}(\Gamma)$.*

Also, in view of Definition 3.4, the sequence $\mathcal{M}_{\mathbf{SAL^m_1}}(\Gamma), \mathcal{M}_{\mathbf{SAL^m_2}}(\Gamma), \ldots$ decreases monotonically:

**Fact 6.4** *Each of the following holds for every $i \in I$:*

1. *$\mathcal{M}_{\mathbf{SAL^m_i}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{SAL^m_{i-1}}}(\Gamma)$.*
2. *$\mathcal{M}_{\mathbf{SAL^m_i}}(\Gamma) = \bigcap_{j\leq i} \mathcal{M}_{\mathbf{SAL^m_j}}(\Gamma)$*

**Lemma 6.1** *Let $i \in I$. If $M \in \mathcal{M}_{\mathbf{SAL^m_i}}(\Gamma)$ and $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ is such that $Ab(M') \cap \Omega_{(i)} \subseteq Ab(M) \cap \Omega_{(i)}$, then $M' \in \mathcal{M}_{\mathbf{SAL^m_i}}(\Gamma)$.*

*Proof.* Assume that the antecedent holds, but $M' \notin \mathcal{M}_{\mathbf{SAL^m_i}}(\Gamma)$. Let $j \leq i$ be the smallest $j \in I$ such that $M' \notin \mathcal{M}_{\mathbf{SAL^m_j}}(\Gamma)$. By Definition 3.4, there is an $M'' \in \mathcal{M}_{\mathbf{SAL^m_{j-1}}}(\Gamma)$ such that $Ab(M'') \cap \Omega_j \subset Ab(M') \cap \Omega_j$. In view of the supposition, $Ab(M') \cap \Omega_j \subseteq Ab(M) \cap \Omega_j$, whence (†) $Ab(M'') \cap \Omega_j \subset Ab(M) \cap \Omega_j$. By the supposition and Definition 3.4, $M \in \mathcal{M}_{\mathbf{SAL^m_{j-1}}}(\Gamma)$. By (†) and Definition 3.4, $M \notin \mathcal{M}_{\mathbf{SAL^m_j}}(\Gamma)$, whence also $M \notin \mathcal{M}_{\mathbf{SAL^m_i}}(\Gamma)$ — a contradiction. ■

**Theorem 6.3** $\mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$.

*Proof.* ($\mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$) Assume that $M \in \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By Definition 3.4, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. Hence, by Definition 5.1 and Definition 5.2, there is an $i \in I$ and an $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that (1) for every $j < i$: $Ab(M') \cap \Omega_j = Ab(M) \cap \Omega_j$, and (2) $Ab(M') \cap \Omega_i \subset Ab(M) \cap \Omega_i$. It follows that (1') for every $j < i$: $Ab(M') \cap \Omega_{(j)} = Ab(M) \cap \Omega_{(j)}$ and (2') $Ab(M') \cap \Omega_{(i)} \subset Ab(M) \cap \Omega_{(i)}$. By Definition 3.4, $M \in \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i-1})}}(\Gamma)$. But then also $M' \in \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i-1})}}(\Gamma)$, which implies that $M \notin \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma)$. By Definition 3.4, $M \notin \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$ — a contradiction.

($\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$) Assume that $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$. Take the smallest $i \in I$ for which $M \notin \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma)$. By Definition 3.4, there is an $M' \in \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i-1})}}(\Gamma)$ such that $Ab(M') \cap \Omega_{(i)} \subset Ab(M) \cap \Omega_{(i)}$. Note that $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. Also, it follows that there is a $k \leq i$ such that (1) for every $j \in I, j < k$: $Ab(M') \cap \Omega_j = Ab(M) \cap \Omega_j$, and (2) $Ab(M') \cap \Omega_k \subset Ab(M) \cap \Omega_k$. But then $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ by Definitions 5.1 and 5.2. ■

**Hierarchic and Lexicographic ALs** As explained in Chapter 4, the $\mathbf{HAL}^{\mathbf{m}}$-semantics is obtained by taking the intersection of the sets $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma)$ with $i \in I$. Just as for $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$, we can easily prove that this semantics is equivalent to that of the logic $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$, characterized by the triple $\langle \mathbf{LLL}, \langle \Omega_i \rangle_{i \in I}, \mathbf{m} \rangle$:

**Theorem 6.4** $\mathcal{M}_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$

*Proof.* ($\mathcal{M}_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$) Suppose $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By Definition 5.1 and Definition 5.2, there is an $i \in I$ and an $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that (1) for every $j < i$: $Ab(M') \cap \Omega_j = Ab(M) \cap \Omega_j$, and (2) $Ab(M') \cap \Omega_i \subset Ab(M) \cap \Omega_i$. It follows that (2') $Ab(M') \cap \Omega_{(i)} \subset Ab(M) \cap \Omega_{(i)}$. Thus $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma)$, whence by Definition 4.2, $M \notin \mathcal{M}_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma)$.

($\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma)$) Suppose $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma)$. Hence there is an $i \in I$: $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma)$, whence by Definition 5.2, there is an $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$: $Ab(M') \cap \Omega_{(i)} \subset Ab(M) \cap \Omega_{(i)}$. It follows that there is a $k \leq i$ such that (1) for every $j < k$: $Ab(M') \cap \Omega_j = Ab(M) \cap \Omega_j$, and (2) $Ab(M') \cap \Omega_k \subset Ab(M) \cap \Omega_k$. Thus $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. ■

**Some Corollaries** The following corollary is one of the central results presented in this chapter. It shows that the semantics of hierarchic adaptive logics and logics in the $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-format define the same consequence relation. Moreover, there is a class of superposed adaptive logics for which the semantics is equivalent to that of $\mathbf{HAL}^{\mathbf{m}}$ and $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ as well. This class of superposed adaptive logics is characterized by the same sequences of flat adaptive logics as hierarchical adaptive logics, namely sequences of the form $\langle \mathbf{AL}^{\mathbf{m}}_{(\mathbf{1})}, \mathbf{AL}^{\mathbf{m}}_{(\mathbf{2})}, \ldots \rangle$. The fact that three different paths of devising selection semantics for prioritized logics lead to the same semantic consequence relation demonstrates the centrality, robustness and usefulness of the latter.

**Corollary 6.1** $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma) = \mathcal{M}_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma)$. *Hence,* $\Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$ *iff* $\Gamma \models_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}} A$ *iff* $\Gamma \models_{\mathbf{HAL}^{\mathbf{m}}} A$.

In the remainder of this section, let $\mathbf{PAL} \in \{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}, \mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}, \mathbf{HAL}^{\mathbf{m}}\}$. Since $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ is sound and complete with respect to $\models_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}$ (see Corollary 5.1), the corollary equips us with alternative semantic selection procedures for $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$:

**Corollary 6.2** *Each of the following holds:*

1. *If* $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$, *then* $\Gamma \models_{\mathbf{PAL}} A$.
2. *Where* $\Gamma \subseteq \mathcal{W}_s$: *if* $\Gamma \models_{\mathbf{PAL}} A$, *then* $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A$.

By Theorems 5.17 and 6.3, it follows that $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ satisfies a specific kind of Strong Reassurance:[2]

**Corollary 6.3** $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ *satisfies* SR3*: if* $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$, *then there is an* $M' \in \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$ *and an* $i \in I$ *such that* $Ab(M') \cap \Omega_{(i)} \subset Ab(M) \cap \Omega_{(i)}$. *If* $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$. *Alternatively, if* $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) - \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$, *then there is an* $M' \in \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$ *such that* $Ab(M') \sqsubset Ab(M)$.

## 6.3 Soundness of $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$

In view of the soundness of $\mathbf{HAL}^{\mathbf{m}}$ (see Theorem 4.6), the completeness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ and the preceding section, it can easily be inferred that for every $\Gamma \subseteq \mathcal{W}_s$, $Cn_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ – see Corollary 6.6 below. To obtain a similar result for $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$, I first prove that these logics are also sound with respect to their semantics. Recall that **SAL**-logics are not in general sound and complete; nor are logics of the more restricted class $\mathbf{SAL}^{\mathbf{m}}$ – see Chapter 3, Section 3.3.3.

**Soundness of $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$** For the proof of Theorem 6.5 below, we need to establish a specific property of the $\mathbf{AL}^{\mathbf{m}}_{\prec}$-semantics. It states that if every member of a set $\Gamma'$ is true in all $\mathbf{AL}^{\mathbf{m}}_{\prec}$-models of $\Gamma$, then $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma \cup \Gamma') = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma)$:

**Lemma 6.2** *If* $\Gamma' \subseteq \{A \mid \Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\prec}} A\}$, *then* $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma \cup \Gamma') = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma)$.[3]

*Proof.* Suppose (†) $\Gamma' \subseteq \{A \mid \Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\prec}} A\}$.

($\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma \cup \Gamma') \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma)$) Consider an $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma \cup \Gamma')$. By Definition 6.2, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma')$ and hence $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. Assume that $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma)$. By Theorem 6.1, there is an $M' \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma)$ such that $Ab(M') \prec Ab(M)$. However, in view of (†), $M' \Vdash A$ for every $A \in \Gamma'$, whence also $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma')$. By Definition 6.2, $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\prec}}(\Gamma \cup \Gamma')$ — a contradiction.

[2]See page 81 where SR3 was introduced and compared to other variants of Strong Reassurance. Recall that SR3 was already proven for $\mathbf{HAL}^{\mathbf{m}}$ and $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ independently – see Theorems 4.15, resp. 5.17.

[3]This lemma generalizes Lemma 5.19 from Chapter 5, Section 5.3.3 – there it was shown that the consequent of the lemma holds for $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ whenever $\Gamma' \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$.

($\mathcal{M}_{\mathbf{AL^m_{\precsim}}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL^m_{\precsim}}}(\Gamma \cup \Gamma')$) Consider an $M \in \mathcal{M}_{\mathbf{AL^m_{\precsim}}}(\Gamma)$. By (†), $M \Vdash A$ for every $A \in \Gamma'$. By Definition 6.2, $M$ is an **LLL**-model of $\Gamma$. We thus obtain that $M$ is a **LLL**-model of $\Gamma \cup \Gamma'$. Assume that $M \notin \mathcal{M}_{\mathbf{AL^m_{\precsim}}}(\Gamma \cup \Gamma')$. By Theorem 6.1, there is an $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma')$: $Ab(M') \prec Ab(M)$. Hence $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. By Definition 6.2, $M \notin \mathcal{M}_{\mathbf{AL^m_{\precsim}}}(\Gamma)$ — a contradiction. ■

To understand the following theorem, recall that the logic $\mathbf{SAL^m_{(i)}}$ ($i \in I$) is obtained by the sequential superposition of the logics $\langle \mathbf{AL^m_{(j)}} \rangle_{j \leq i}$. Item 2 of the theorem below states that every logic $\mathbf{SAL^m_{(i)}}$ is sound with respect to its semantics. As shown below, the soundness of $\mathbf{SAL^m_{(I)}}$ is an immediate corollary of this.

**Theorem 6.5** *Each of the following holds for every $i \in I$:*

1. *For every $M \in \mathcal{M}_{\mathbf{SAL^m_{(i)}}}(\Gamma)$, $Ab(M) \cap \Omega_{(i)} \in \Phi^{(i)}(Cn_{\mathbf{SAL^m_{(i-1)}}}(\Gamma))$*
2. *If $A \in Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma)$, then $\Gamma \models_{\mathbf{SAL^m_{(i)}}} A$.*

*Proof.* ($i = 1$) *Ad 1.* Suppose $M \in \mathcal{M}_{\mathbf{SAL^m_{(1)}}}(\Gamma)$. By Fact 3.2, $M \in \mathcal{M}_{\mathbf{AL^m_{(1)}}}(\Gamma)$. Hence by Theorem 2.1.1, $Ab(M) \cap \Omega_{(1)} \in \Phi^{(1)}(\Gamma)$. The rest is immediate in view of Fact 2.1.

*Ad 2.* Suppose $A \in Cn_{\mathbf{SAL^m_{(1)}}}(\Gamma) = Cn_{\mathbf{AL^m_{(1)}}}(\Gamma)$. By the soundness of $\mathbf{AL^m_{(1)}}$, $A$ is true in every $M \in \mathcal{M}_{\mathbf{AL^m_{(1)}}}(\Gamma)$. By Fact 3.2, $A$ is true in every $M \in \mathcal{M}_{\mathbf{SAL^m_1}}(\Gamma)$.

($i \Rightarrow i+1$) *Ad 1.* By item 2 of the induction hypothesis, Fact 6.4 and Theorem 6.3 respectively, $Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma) \subseteq \{A \mid \Gamma \models_{\mathbf{SAL^m_{(i)}}} A\} \subseteq \{A \mid \Gamma \models_{\mathbf{SAL^m_{(i+1)}}} A\} = \{A \mid \Gamma \models_{\mathbf{AL^m_{\sqsubset(i+1)}}} A\}$. By Lemma 6.2,

$$\mathcal{M}_{\mathbf{AL^m_{\sqsubset(i+1)}}}(\Gamma) = \mathcal{M}_{\mathbf{AL^m_{\sqsubset(i+1)}}}(\Gamma \cup Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma))$$

whence by the reflexivity of $\mathbf{SAL^m_{(i)}}$, $\mathcal{M}_{\mathbf{AL^m_{\sqsubset(i+1)}}}(\Gamma) = \mathcal{M}_{\mathbf{AL^m_{\sqsubset(i+1)}}}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma))$. By Theorem 6.3,

$$\mathcal{M}_{\mathbf{SAL^m_{(i+1)}}}(\Gamma) = \mathcal{M}_{\mathbf{SAL^m_{(i+1)}}}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma)) \tag{6.1}$$

Suppose (†) $M \in \mathcal{M}_{\mathbf{SAL^m_{(i+1)}}}(\Gamma)$. By (6.1), $M \in \mathcal{M}_{\mathbf{SAL^m_{(i+1)}}}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma))$. By Definition 3.4, $M$ is an **LLL**-model of $Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma)$, whence by Fact 5.2, $Ab(M) \cap \Omega_{(i+1)}$ is a choice set of $\Sigma^{(i+1)}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma))$.

Suppose $Ab(M) \cap \Omega_{(i+1)} \notin \Phi^{(i+1)}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma))$. Hence, there is a choice set $\psi$ of $\Sigma^{(i+1)}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma))$, such that $\psi \subset Ab(M) \cap \Omega_{(i+1)}$. By Fact 5.3, there is an **LLL**-model $M'$ of $Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma)$ such that $Ab(M') \cap \Omega_{(i+1)} \subseteq \psi$, whence $Ab(M') \cap \Omega_{(i+1)} \subset Ab(M) \cap \Omega_{(i+1)}$. Note that since $M \in \mathcal{M}_{\mathbf{SAL^m_{(i)}}}(\Gamma)$, by Lemma 6.1, $M' \in \mathcal{M}_{\mathbf{SAL^m_{(i)}}}(\Gamma)$. But then by Definition 3.4, $M \notin \mathcal{M}_{\mathbf{SAL^m_{(i+1)}}}(\Gamma)$, which contradicts (†).

*Ad 2.* Suppose $A \in Cn_{\mathbf{SAL^m_{(i+1)}}}(\Gamma)$, whence by Definition 3.1, $A \in Cn_{\mathbf{AL^m_{(i+1)}}}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma))$. By Theorem 2.7.1, for every $\varphi \in \Phi^{(i+1)}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma))$, there is a $\Delta \subset \Omega_{(i+1)}$ such that (†) $Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap \varphi = \emptyset$. By Theorem 3.1.1, (‡) $A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma)$ for every such $\Delta$.

Let $M \in \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i+1})}}(\Gamma)$. By item 1, there is a $\varphi \in \Phi^{(i+1)}(Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma))$ such that $Ab(M) \cap \Omega_{(i+1)} = \varphi$. By Definition 3.4, $M \in \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma)$. By the induction hypotheses, $M \Vdash B$ for every $B \in Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma)$, whence by (‡), $M \Vdash A \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega_{(i+1)}$ such that $\varphi \cap \Delta = \emptyset$. But then, since $Ab(M) \cap \Omega_{(i+1)} = \varphi$, $M \Vdash \check{\neg} Dab(\Delta)$, whence $M \Vdash A$. ■

Note that if $A \in Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$, then by Definition 3.1, there is an $i \in I$ such that $A \in Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma)$. Also, by Definition 3.4, if $\Gamma \models_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}} A$ for an $i \in I$, then $\Gamma \models_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}} A$. Hence, in view of Theorem 6.5.2, we immediately obtain:

**Corollary 6.4** *If $A \in Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$, then $\Gamma \models_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}} A$.*

**Relations between the three consequence relations** By Corollary 5.1, Corollary 6.1 and Theorem 6.3, we obtain:

**Corollary 6.5** *Where $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$*

Also, by Corollary 5.1, Corollary 6.1 and Theorem 4.6, we have:

**Corollary 6.6** *Where $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$.*

In Section C.3 of Appendix C, it is shown that $\mathbf{HAL}^{\mathbf{m}}$ and $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ are in general incomparable, i.e. for some logics $\mathbf{HAL}^{\mathbf{m}}$ and $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$, there are premise sets $\Gamma$ and $\Gamma'$ such that (i) $Cn_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma) \not\subseteq Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$ and (ii) $Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma') \not\subseteq Cn_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma')$. As will become clear there, this argument does not depend on the particularities of the concrete systems, and can easily be generalized to other superpositions of ALs and hierarchic ALs.

## 6.4 Equivalence Results

In the preceding, we saw that $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ and $\mathbf{HAL}^{\mathbf{m}}$ are sound with respect to their semantics characterizations, and that those characterizations are both equivalent to the $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$-semantics. As I will now show, the *syntactic consequence relations* of these logics are also often equivalent. I will start with an equivalence result that follows immediately from preceding theorems and corollaries, and illustrate it by means of a simple example (Section 6.4.1). From Section 6.4.2 onwards, this result is generalized to a much broader class of premise sets – how this is done precisely, will be explained below.[4]

[4]Sections 6.4.2-6.4.5 are based on the joint paper with Christian Straßer, cited at the start of this chapter. I added 6.4.1 to clarify the relation with other properties established in this thesis, and to illustrate the equivalence results from this section.

### 6.4.1 A Simple Equivalence Result

**A Surprising Corollary** In Chapters 3 and 4, it was shown that whenever $\Phi(\Gamma)$ is finite, then all logics **SAL** and **HAL** are sound and complete with respect to their semantics (see Corollary 3.2, resp. Corollary 4.8). In view of Corollary 6.1, this means that when $\Phi(\Gamma)$ is finite, $\mathbf{SAL}^{\mathbf{m}}_{\mathbf{(I)}}$ and $\mathbf{HAL}^{\mathbf{m}}$ are equivalent to $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$:

**Corollary 6.7** *If* $\Gamma \subseteq \mathcal{W}_s$ *and* $\Phi(\Gamma)$ *is finite, then* $Cn_{\mathbf{SAL}^{\mathbf{m}}_{\mathbf{(I)}}}(\Gamma) = Cn_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$

This result is fairly surprising, in view of the structural differences between these three formats. So let me insert an example that may help to understand it better, and that at the same time illustrates the various proof theories from preceding chapters.

Recall $\mathbf{SK2}^{\mathbf{m}}_{\mathbf{(2)}}$ and $\mathbf{HK2}^{\mathbf{m}}$. The first of these two logics is obtained by the sequential superposition of $\mathbf{K}^{\mathbf{m}}_{\mathbf{(2)}}$ on $\mathbf{K}^{\mathbf{m}}_{\mathbf{(1)}}$; the second by the hierarchic combination of the same two flat ALs. Let $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$ be the lexicographic AL defined by the triple $\langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_1, \Omega^{\mathbf{K}}_2 \rangle, \mathbf{m} \rangle$.

Consider

$$\Gamma_{\mathsf{eq}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \Diamond\Diamond s, \neg p \vee \neg q, \neg p \vee \neg r, \neg q \vee \neg r, \neg r \vee \neg s\}$$

To get a bit more grip on it, note that this premise set has exactly four minimal Dab-consequences $Dab(\Delta)$ ($\Delta \subseteq \Omega^{\mathbf{K}}_{(2)}$):

$$\begin{aligned}
Dab(\Delta_1) &= \; !^1 p \check{\vee} !^1 q \text{ (from } \Diamond p, \Diamond q, \neg p \vee \neg q) \\
Dab(\Delta_2) &= \; !^1 p \check{\vee} !^2 r \text{ (from } \Diamond p, \Diamond\Diamond r, \neg p \vee \neg r) \\
Dab(\Delta_3) &= \; !^1 p \check{\vee} !^2 r \text{ (from } \Diamond q, \Diamond\Diamond r, \neg q \vee \neg r) \\
Dab(\Delta_4) &= \; !^2 r \check{\vee} !^2 s \text{ (from } \Diamond\Diamond r, \Diamond\Diamond s, \neg r \vee \neg s)
\end{aligned}$$

Note that $\Sigma(\Gamma_{\mathsf{eq}}) = \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ has exactly three ($\subset$-)minimal choice sets:

$$\begin{aligned}
\varphi_1 &= \{!^1 p, !^1 q, !^2 s\} \\
\varphi_2 &= \{!^1 p, !^2 r\} \\
\varphi_3 &= \{!^1 q, !^2 r\}
\end{aligned}$$

I will now discuss the behavior of each of the three aforementioned logics one by one, for this specific example.

First of all, there is the lexicographic AL $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$. Note that $\varphi_2 \sqsubset \varphi_1$, and also $\varphi_3 \sqsubset \varphi_1$. Hence $\Phi^{\sqsubset}(\Gamma_{\mathsf{eq}}) = \{\varphi_2, \varphi_3\}$. So intuitively, $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$ selects only two interpretations of $\Gamma_{\mathsf{eq}}$ as those that are optimal according to the order $\sqsubset$ — I will explain right away what this means, both at the syntactic and semantic level.

The following is a $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma_{\mathsf{eq}}$:[5]

[5] I omit reference to the stage in the ✓-symbols. The marks represent marking for $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$ at stage 20.

| | | | | |
|---|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\emptyset$ | |
| 2 | $\Diamond q$ | PREM | $\emptyset$ | |
| 3 | $\Diamond\Diamond r$ | PREM | $\emptyset$ | |
| 4 | $\Diamond\Diamond s$ | PREM | $\emptyset$ | |
| 5 | $\neg p \vee \neg q$ | PREM | $\emptyset$ | |
| 6 | $\neg p \vee \neg r$ | PREM | $\emptyset$ | |
| 7 | $\neg q \vee \neg r$ | PREM | $\emptyset$ | |
| 8 | $\neg r \vee \neg s$ | PREM | $\emptyset$ | |
| 9 | $p$ | 1;RC | $\{!^1 p\}$ | ✓ |
| 10 | $q$ | 2;RC | $\{!^1 q\}$ | ✓ |
| 11 | $p \vee q$ | 9;RU | $\{!^1 p\}$ | |
| 12 | $p \vee q$ | 10;RU | $\{!^1 q\}$ | |
| 13 | $r$ | 3;RC | $\{!^2 r\}$ | ✓ |
| 14 | $\neg r$ | 6,9;RU | $\{!^1 p\}$ | |
| 15 | $\neg r$ | 7,10;RU | $\{!^1 q\}$ | |
| 16 | $s$ | 4;RC | $\{!^2 s\}$ | |
| 17 | $!^1 p \check{\vee} !^1 q$ | 1,2,5;RU | $\emptyset$ | |
| 18 | $!^1 p \check{\vee} !^2 r$ | 1,3,6;RU | $\emptyset$ | |
| 19 | $!^1 q \check{\vee} !^2 r$ | 2,3,7;RU | $\emptyset$ | |
| 20 | $!^2 r \check{\vee} !^2 s$ | 3,4,8;RU | $\emptyset$ | |

Note that at stage 20 of the above proof, all minimal Dab-consequences of $\Gamma_{\mathsf{eq}}$ are derived (see lines 17-20). As a result, $\Phi^{\sqsubset}_{20}(\Gamma_{\mathsf{eq}}) = \Phi^{\sqsubset}(\Gamma_{\mathsf{eq}}) = \{\varphi_2, \varphi_3\}$.

Line 9 is marked in the above proof, since $p$ is not derived on a condition that does not overlap with $\varphi_2$. Likewise, line 10 is marked since $q$ is not derived on a condition that has an empty intersection with $\varphi_3$. Line 13 is marked because its condition overlaps with both $\varphi_2$ and $\varphi_3$. Since all minimal Dab-consequences of $\Gamma_{\mathsf{eq}}$ have been derived at stage 20, lines 11, 12 and 14-16 remain unmarked in every extension of this proof. Hence $p \vee q$, $\neg r$ and $s$ are finally $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$-derivable from $\Gamma_{\mathsf{eq}}$.

It can also be easily verified that $p$ and $q$ are not $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$-consequences of $\Gamma_{\mathsf{eq}}$, in view of the $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$-semantics. By Theorem 5.4, there are two kinds of $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$-models of $\Gamma_{\mathsf{eq}}$: models $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{eq}})$ for which $Ab(M) = \varphi_2$, and models $M' \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{eq}})$ for which $Ab(M') = \varphi_3$. Models of the first kind verify $!^1 p$ and hence falsify $p$; those of the second kind verify $!^1 q$ and hence falsify $q$.

Let us now turn to the second logic, i.e. the superposition-logic $\mathbf{SK2}^{\mathbf{m}}_{(\mathbf{2})}$. Note that $!^1 p \check{\vee} !^1 q$ is the only minimal $\mathrm{Dab}_{(1)}$-consequence of $\Gamma_{\mathsf{eq}}$. Hence, $\Phi^{(1)}(\Gamma_{\mathsf{eq}}) = \{\{!^1 p\}, \{!^1 q\}\}$. This means that, just as with $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$, we can finally $\mathbf{K}^{\mathbf{m}}_{(\mathbf{1})}$-derive $p \vee q$ and $\neg r$ from $\Gamma_{\mathsf{eq}}$ (both formulas are finally derivable on the conditions $\{!^1 p\}$ and $\{!^1 q\}$). Hence also $!^2 r \in Cn_{\mathbf{K}^{\mathbf{m}}_{(1)}}(\Gamma_{\mathsf{eq}})$.

As a result, $!^2 r \check{\vee} !^2 s$ is not a *minimal* Dab-consequence of $Cn_{\mathbf{K}^{\mathbf{m}}_{(1)}}(\Gamma_{\mathsf{eq}})$. The only minimal $\mathrm{Dab}_{(2)}$-consequences of $Cn_{\mathbf{K}^{\mathbf{m}}_{(1)}}(\Gamma_{\mathsf{eq}})$ are $!^1 p \check{\vee} !^1 q$ and $!^2 r$. Thus, $\Phi^{(2)}(Cn_{\mathbf{K}^{\mathbf{m}}_{(1)}}(\Gamma_{\mathsf{eq}})) = \{\{!^1 p, !^2 r\}, \{!^1 q, !^2 r\}\} = \Phi^{\sqsubset}(\Gamma_{\mathsf{eq}})$, and hence $s$ can be finally $\mathbf{K}^{\mathbf{m}}_{(\mathbf{2})}$-derived from $Cn_{\mathbf{K}^{\mathbf{m}}_{(1)}}(\Gamma_{\mathsf{eq}})$ on the condition $\{!^2 s\}$.

The following proof shows how $p \vee q, \neg r$ and $s$ can be finally $\mathbf{SK2}^{\mathbf{m}}_{(\mathbf{2})}$-derived from $\Gamma_{\mathsf{eq}}$.

| | | | |
|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 2 | $\Diamond q$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 3 | $\Diamond\Diamond r$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 4 | $\Diamond\Diamond s$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 5 | $\neg p \vee \neg q$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 6 | $\neg p \vee \neg r$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 7 | $\neg q \vee \neg r$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 8 | $\neg r \vee \neg s$ | PREM | $\langle\emptyset,\emptyset\rangle$ |
| 9 | $p$ | 1;RC | $\langle\{!^1p\},\emptyset\rangle \checkmark_1$ |
| 10 | $q$ | 2;RC | $\langle\{!^1q\},\emptyset\rangle \checkmark_1$ |
| 11 | $p \vee q$ | 9;RU | $\langle\{!^1p\},\emptyset\rangle$ |
| 12 | $p \vee q$ | 10;RU | $\langle\{!^1q\},\emptyset\rangle$ |
| 13 | $!^1p \check{\vee} !^1q$ | 1,2,5;RU | $\langle\emptyset,\emptyset\rangle$ |
| 14 | $!^2r \check{\vee} !^1p$ | 1,3,6;RU | $\langle\emptyset,\emptyset\rangle$ |
| 15 | $!^2r$ | 14;RC | $\langle\{!^1p\},\emptyset\rangle$ |
| 16 | $!^2r \check{\vee} !^1q$ | 2,3,7;RU | $\langle\emptyset,\emptyset\rangle$ |
| 17 | $!^2r$ | 16;RC | $\langle\{!^1q\},\emptyset\rangle$ |
| 18 | $r$ | 3;RC | $\langle\emptyset,\{!^2r\}\rangle \checkmark_2$ |
| 19 | $s$ | 4;RC | $\langle\emptyset,\{!^2s\}\rangle$ |
| 20 | $\neg r$ | 1,6;RC | $\langle\{!^1p\},\emptyset\rangle$ |
| 21 | $\neg r$ | 2,7;RC | $\langle\{!^1q\},\emptyset\rangle$ |

Note that the Dab-formula on line 13 is the only minimal $\mathrm{Dab}_{(1)}$-formula at stage 20. Hence $\Phi^{(1)}_{20}(\Gamma_{\mathsf{eq}}) = \{\{!^1p\},\{!^1q\}\}$, whence lines 9 and 10 are 1-marked at this stage, and remain marked in every further extension of the proof. Since lines 15 and 17 are not 1-marked, $!^2r$ is a minimal $\mathrm{Dab}_{(2)}$-formula at stage 20, and hence $\Phi^{(2)}_{20}(\Gamma_{\mathsf{eq}}) = \{\{!^1p,!^2r\},\{!^1q,!^2r\}\}$. As a result, line 18 is marked, but lines 19-21 are not.

Since the only minimal $\mathrm{Dab}_{(1)}$-consequence of $\Gamma_{\mathsf{eq}}$ is derived at stage 21, and since the only two minimal $\mathrm{Dab}_{(2)}$-consequences of $Cn_{\mathbf{K^m_1}}(\Gamma_{\mathsf{eq}})$ are derived at stage 21, the marks remain stable in every further extension of the above proof. As a result, $p \vee q$, $\neg r$ and $s$ are finally $\mathbf{SK2^m_{(2)}}$-derived from $\Gamma_{\mathsf{eq}}$ in the above proof.

Semantically, the picture is as follows for $\mathbf{SK2^m_{(2)}}$: first, all models $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{eq}})$ are selected for which either $Ab(M) \cap \Omega^{\mathbf{K}}_{(1)} = \{!^1p\}$, or $Ab(M) \cap \Omega^{\mathbf{K}}_{(1)} = \{!^1q\}$. As a result, all models selected after the first round verify $!^2r$. In the second selection, only those models $M$ from the previous step are retained, that do not validate any level 2-abnormality other than $!^2r$. So by the sequential selection procedure, we again end up with those $\mathbf{K}$-models of $\Gamma_{\mathsf{eq}}$ whose abnormal part equals either $\varphi_2$ or $\varphi_3$.

Third and last, there is the logic $\mathbf{HK2^m}$. By the same reasoning as above, we can show that $p \vee q$ and $\neg r$ are in the $\mathbf{K^m_{(1)}}$-consequence set of $\Gamma_{\mathsf{eq}}$. Hence, in view of the definition of $\mathbf{HK2^m}$, $p \vee q, \neg r, s \in Cn_{\mathbf{HK2^m}}(\Gamma_{\mathsf{eq}})$. But how can we finally $\mathbf{HK2^m}$-derive $s$ from $\Gamma_{\mathsf{eq}}$?

Note that $r \vee s$ can be derived from $\Gamma_{\mathsf{eq}}$ on two conditions, i.e. $\{!^2r\}$ and $\{!^2s\}$. The first of these two conditions has an empty intersection with $\varphi_1$; the second with both $\varphi_2$ and $\varphi_3$. Hence, $r \vee s \in Cn_{\mathbf{K^m_{(2)}}}(\Gamma_{\mathsf{eq}})$.

Recall that $Cn_{\mathbf{HK2^m}}(\Gamma)$ is obtained by closing the union of $Cn_{\mathbf{K^m_{(1)}}}(\Gamma)$ and $Cn_{\mathbf{K^m_{(2)}}}(\Gamma)$ under **K**. But then, since $\neg r \in Cn_{\mathbf{K^m_{(1)}}}(\Gamma_{\mathsf{eq}})$ and $r \vee s \in Cn_{\mathbf{K^m_{(2)}}}(\Gamma_{\mathsf{eq}})$, it follows that $s \in Cn_{\mathbf{HK2^m}}(\Gamma_{\mathsf{eq}})$.

The fact that the closure under **K** is needed to obtain $s$, is illustrated by the following proof:

| | | | | |
|---|---|---|---|---|
| 1 | $\Diamond p$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 2 | $\Diamond q$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 3 | $\Diamond\Diamond r$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 4 | $\Diamond\Diamond s$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 5 | $\neg p \vee \neg q$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 6 | $\neg p \vee \neg r$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 7 | $\neg q \vee \neg r$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 8 | $\neg r \vee \neg s$ | PREM | $\emptyset$ | $-_{1,2}$ |
| 9 | $!^1 p \,\check{\vee}\, !^1 q$ | 1,2,5;RU | $\emptyset$ | $-_{1,2,*}$ |
| 10 | $!^1 p \,\check{\vee}\, !^2 r$ | 1,3,6;RU | $\emptyset$ | $-_{1,2,*}$ |
| 11 | $!^1 q \,\check{\vee}\, !^2 r$ | 2,3,7;RU | $\emptyset$ | $-_{1,2,*}$ |
| 12 | $!^1 r \,\check{\vee}\, !^1 s$ | 3,4,8;RU | $\emptyset$ | $-_{1,2,*}$ |
| 13 | $p$ | 1;RC | $\{!^1 p\}$ | ✓ |
| 14 | $q$ | 2;RC | $\{!^1 q\}$ | ✓ |
| 15 | $p \vee q$ | 13;RU | $\{!^1 p\}$ | $-_{1,*}$ |
| 16 | $p \vee q$ | 14;RU | $\{!^1 q\}$ | $-_{1,*}$ |
| 17 | $r$ | 3;RC | $\{!^2 r\}$ | ✓ |
| 18 | $\neg r$ | 1,6;RU | $\{!^1 p\}$ | $-_{1,*}$ |
| 19 | $\neg r$ | 2,7;RU | $\{!^1 q\}$ | $-_{1,*}$ |
| 20 | $r \vee s$ | 3;RC | $\{!^2 r\}$ | $-_{2}$ |
| 21 | $r \vee s$ | 4;RC | $\{!^2 s\}$ | $-_{2}$ |
| 22 | $s$ | 19,21;RU | $\{!^1 p, !^2 r\}$ | $-_{*}$ |

Note that $\Phi^{(1)}_{17}(\Gamma_{\mathsf{eq}}) = \{\{!^1 p\}, \{!^1 q\}\}$ and $\Phi^{(2)}_{17}(\Gamma_{\mathsf{eq}}) = \{\varphi_1, \varphi_2, \varphi_3\}$. As a result, lines 13 and 14 are 1-unmarked, and lines 15 and 16 are 2-unmarked at stage 17. Since line 17 is derived by the rule RU from lines 13 and 15, line 17 is *-unmarked at stage 17. Again, the marks remain stable in every extension of this proof, since all minimal Dab-consequences of $\Gamma_{\mathsf{eq}}$ are derived in it (see lines 9-12).

The set of $\mathbf{HK2^m}$-models is obtained by taking the intersection of two sets of models:

(i) $\mathcal{M}_{\mathbf{K^m_{(1)}}}(\Gamma_{\mathsf{eq}}) = \{M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{eq}}) \mid Ab(M) \cap \Omega^{\mathbf{K}}_{(1)} = \{!^1 p\}$ or $Ab(M) \cap \Omega^{\mathbf{K}}_{(1)} = \{!^1 q\}\}$, and

(ii) $\mathcal{M}_{\mathbf{K^m_{(2)}}}(\Gamma_{\mathsf{eq}}) = \{M \in \mathcal{M}_{\mathbf{K}}(\Gamma_{\mathsf{eq}}) \mid Ab(M) \cap \Omega^{\mathbf{K}}_{(2)} \in \{\varphi_1, \varphi_2, \varphi_3\}\}$

Since those $M \in \mathcal{M}_{\mathbf{K^m_{(2)}}}(\Gamma_{\mathsf{eq}})$ with $Ab(M) = \varphi_1$ are not members of $\mathcal{M}_{\mathbf{K^m_{(1)}}}(\Gamma_{\mathsf{eq}})$, we again end up with exactly those **K**-models of $\Gamma_{\mathsf{eq}}$ that correspond to either $\varphi_2$ or $\varphi_3$.

**Weaker Equivalence Criteria** In the remainder of this section, weaker conditions are shown to be sufficient for the soundness and completeness of the class of logics $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$, and for *all* logics **HAL**. In other words, the restricted completeness results from Chapters 3 and 4 are generalized to a broader range of premise sets. This generalization is made possible by the introduction of the sets ${}^{\mathsf{c}}\Phi(\Gamma)$ and ${}^{\mathsf{c}}\Phi^{\sqsubset}(\Gamma)$ – see below. In addition, we present a hierarchy of conditions for the soundness and completeness of $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$, resp. **HAL**.

The relations between the various equivalence criteria are depicted in Figure 6.1 on page 148. At the end of Section 6.4.5, we give some simple examples that show the results from this section are truly generalizations (and not just more technical formulations) of Corollary 6.7 above.

### 6.4.2 The Basic Criteria for Equivalence

Note that the following is the case:

**Theorem 6.6** *Where* $\Gamma \subseteq \mathcal{W}$ *and* $\mathbf{PAL} \in \{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}, \mathbf{HAL}^{\mathbf{m}}\}$*: if*

$$\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{PAL}}(\Gamma)) = \mathcal{M}_{\mathbf{PAL}}(\Gamma) \tag{6.2}$$

*then*

1. $\Gamma \models_{\mathbf{PAL}} A$ *iff* $A \in Cn_{\mathbf{PAL}}(\Gamma)$*, and*
2. $Cn_{\mathbf{PAL}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$*.*

*Proof. Ad 1.* ($\Rightarrow$) If $\Gamma \models_{\mathbf{PAL}} A$ then $A$ is true in every $M \in \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{PAL}}(\Gamma))$. By the completeness of **LLL** and the fact that $Cn_{\mathbf{PAL}}(\Gamma)$ is closed under **LLL** (in case $\mathbf{PAL} = \mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ see Theorem 3.1.2, in case $\mathbf{PAL} = \mathbf{HAL}^{\mathbf{m}}$ this holds by definition), $A \in Cn_{\mathbf{PAL}}(\Gamma)$. ($\Leftarrow$) See Corollary 6.4, resp. Theorem 4.6.

*Ad 2.* Immediate in view of item 1, Corollary 6.1 and the soundness and completeness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$. ■

Equation (6.2) expresses that the set of **PAL**-models is characterized by means of the **PAL**-consequence set: the models of the prioritized adaptive logic are exactly those **LLL**-models that verify the **PAL**-consequences. This is a central criterion since it is sufficient for both, the soundness and completeness of **PAL** (point 1.), and for the equivalence of the syntactic consequence relations of the three prioritized adaptive logics that are discussed in this chapter (point 2.).

The criteria for soundness and equivalence are defined by means of sets of complements of minimal choice sets. Where $\prec \in \{\subset, \sqsubset\}$, let

$${}^{\mathsf{c}}\Phi^{\prec}(\Gamma) =_{\mathsf{def}} \{\Omega - \varphi \mid \varphi \in \Phi^{\prec}(\Gamma)\}$$

Likewise, let

$${}^{\mathsf{c}}\Phi^{\sqsubset_{(i)}}(\Gamma) =_{\mathsf{def}} \{\Omega_{(i)} - \varphi \mid \varphi \in \Phi^{\sqsubset_{(i)}}(\Gamma)\}$$

and let

$${}^{\mathsf{c}}\Phi^{(i)}(\Gamma) =_{\mathsf{def}} \{\Omega_{(i)} - \varphi \mid \varphi \in \Phi^{(i)}(\Gamma)\}$$

In Sections 6.4.3 and 6.4.4 we will give syntactic criteria in terms of these sets, that warrant (6.2) for $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$, resp. $\mathbf{HAL}^{\mathbf{m}}$. But first, let us show that this holds for flat and lexicographic ALs.

**Lemma 6.3** *Where $\Gamma \subseteq \mathcal{W}_s$: if ${}^{\mathsf{c}}\Phi^{\prec}(\Gamma)$ has no infinite minimal choice sets, then $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)$.*

*Proof.* Suppose ${}^{\mathsf{c}}\Phi^{\prec}(\Gamma)$ has no infinite minimal choice sets. That $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma))$ is immediate in view of Definition 6.2 and the soundness of $\mathbf{AL}^{\mathbf{m}}_{\preceq}$. So assume that $M \in \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)) - \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)$. By Theorem 6.2, for every $\varphi \in \Phi^{\prec}(\Gamma)$, there is an $A_\varphi \in \Omega - \varphi$ such that $M \Vdash A_\varphi$. Note that $\{A_\varphi \mid \varphi \in \Phi^{\prec}(\Gamma)\}$ is a choice set of ${}^{\mathsf{c}}\Phi^{\prec}(\Gamma)$. Hence by the supposition, there is a finite $\Theta \subseteq \{A_\varphi \mid \varphi \in \Phi^{\prec}(\Gamma)\}$, such that $\Theta$ is a choice set of ${}^{\mathsf{c}}\Phi^{\prec}(\Gamma)$. It follows by Theorem 6.2 that $\Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\preceq}} \check{\neg} \check{\bigwedge} \Theta$, and hence by the completeness of $\mathbf{AL}^{\mathbf{m}}_{\preceq}$, also $\Gamma \vdash_{\mathbf{AL}^{\mathbf{m}}_{\preceq}} \check{\neg} \check{\bigwedge} \Theta$. But then $M \notin \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma))$ — a contradiction. ■

**Lemma 6.4** *If $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)$, then ${}^{\mathsf{c}}\Phi^{\prec}(\Gamma)$ has no infinite minimal choice sets.*

*Proof.* Let $\Theta$ be an infinite minimal choice set of ${}^{\mathsf{c}}\Phi^{\prec}(\Gamma)$. Assume there is no **LLL**-model of $Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma) \cup \Theta$. By the compactness of **LLL**, there is a finite $\{A_j \mid j \in J\} \subset \Theta$ such that $Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma) \models_{\mathbf{LLL}} \check{\bigvee}_{j \in J} \check{\neg} A_j$. Hence by the soundness[6] of $\mathbf{AL}^{\mathbf{m}}_{\preceq}$, every $\mathbf{AL}^{\mathbf{m}}_{\preceq}$-model of $\Gamma$ falsifies an $A_j$ ($j \in J$). By Theorem 6.2, for every $\varphi \in \Phi^{\prec}(\Gamma)$, there is a $j \in J$ such that $A_j \notin \varphi$. But then $\{A_j \mid j \in J\}$ is a choice set of ${}^{\mathsf{c}}\Phi^{\prec}(\Gamma)$ — a contradiction to the minimality of $\Theta$. So there is a **LLL**-model $M$ of $Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma) \cup \Theta$.

Assume $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)$. By Theorem 6.2, there is a $\varphi \in \Phi^{\prec}(\Gamma)$ such that $Ab(M) = \varphi$. However, since $\Theta$ is a choice set of ${}^{\mathsf{c}}\Phi^{\prec}(\Gamma)$, there is an $A \in (\Omega - \varphi) \cap \Theta$ — a contradiction. Hence $M \in \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)) - \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)$. ■

**Corollary 6.8** *Where $\Gamma \subseteq \mathcal{W}_s$: ${}^{\mathsf{c}}\Phi^{\prec}(\Gamma)$ has no infinite minimal choice sets iff $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\preceq}}(\Gamma)$.*

Where $\prec = \subset$, the same result can be obtained for a specific class of premise sets $\Gamma \subseteq \check{\mathcal{W}}_s$:

**Lemma 6.5** *Where $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$: ${}^{\mathsf{c}}\Phi(\Gamma)$ has no infinite minimal choice sets iff $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}}(\Gamma)) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}}(\Gamma)$.*

*Proof.* ($\Rightarrow$) Immediate in view of the proof for Lemma 6.3 – replace Theorem 5.1.2 by Theorem 2.23. ($\Leftarrow$) Immediate in view of Lemma 6.4. ■

The above results are of crucial importance for the completeness and equivalence results of both $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ and $\mathbf{HAL}^{\mathbf{m}}$, which we shall present subsequently. The following additional lemmas will also be useful in the remainder:

**Lemma 6.6** *For every $\varphi \in \Phi^{\sqsubset_{(i)}}(\Gamma)$, there is a $\psi \in \Phi^{\sqsubset}(\Gamma)$ such that $\psi \cap \Omega_{(i)} = \varphi$.*

*Proof. Case 1.* $\Gamma$ is not **LLL**-satisfiable. In that case, $\Gamma \vdash_{\mathbf{LLL}} A$ for every $A \in \Omega$, whence $\Phi^{\sqsubset_{(i)}}(\Gamma) = \{\Omega_{(i)}\}$ and $\Phi^{\sqsubset}(\Gamma) = \{\Omega\}$. Hence the lemma follows immediately.

[6]Note that every $\mathbf{AL}^{\mathbf{m}}_{\preceq}$-model of $\Gamma$ is an **LLL**-model.

*Case 2.* $\Gamma$ is **LLL**-satisfiable. Suppose $\varphi \in \Phi^{\sqsubset_{(i)}}(\Gamma)$ for an $i \in I$. By Theorem 5.4, there is an $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$ such that $Ab(M) \cap \Omega_{(i)} = \varphi$. Note that $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. If $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, then by Theorem 5.3, $Ab(M) \in \Phi^{\sqsubset}(\Gamma)$, whence the lemma follows immediately. So suppose $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. Then by Theorem 6.1, there is an $M' \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$.

Assume (†) $Ab(M') \cap \Omega_{(i)} \neq Ab(M) \cap \Omega_{(i)}$. In view of Definitions 5.1 and 6.1, there is a $j \leq i$ such that $Ab(M') \sqsubset_{(j)} Ab(M)$. By Fact 6.1.3, $Ab(M') \sqsubset_{(i)} Ab(M)$. But then $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$ — a contradiction. Hence (†) fails: $Ab(M') \cap \Omega_{(i)} = Ab(M) \cap \Omega_{(i)}$. Since by Theorem 5.3, $Ab(M') \in \Phi^{\sqsubset}(\Gamma)$, the lemma follows immediately. ■

**Lemma 6.7** *For every $\varphi \in \Phi^{\sqsubset}(\Gamma)$, $\varphi \cap \Omega_{(i)} \in \Phi^{\sqsubset_{(i)}}(\Gamma)$.*

*Proof.* Assume that $\varphi \in \Phi^{\sqsubset}(\Gamma)$, but $\varphi \cap \Omega_{(i)} \notin \Phi^{\sqsubset_{(i)}}(\Gamma)$. Note that since $\varphi$ is a choice set of $\Sigma(\Gamma)$, $\varphi \cap \Omega_{(i)}$ is a choice set of $\Sigma^{(i)}(\Gamma)$. Hence there is a $\psi \in \Phi^{\sqsubset_{(i)}}(\Gamma)$ such that $\psi \sqsubset_{(i)} \varphi$. By Lemma 6.6, there is a $\psi' \in \Phi^{\sqsubset}(\Gamma)$ such that $\psi' \cap \Omega_{(i)} = \psi$. But then by Fact 6.1.2, $\psi' \sqsubset \varphi$ — a contradiction. ■

**Corollary 6.9** $\Phi^{\sqsubset_{(i)}}(\Gamma) = \{\varphi \cap \Omega_{(i)} \mid \varphi \in \Phi^{\sqsubset}(\Gamma)\}$.

**Lemma 6.8** *If ${}^{\mathsf{c}}\Phi^{\sqsubset}(\Gamma)$ has no infinite minimal choice sets, then for every $i \in I$, ${}^{\mathsf{c}}\Phi^{\sqsubset_{(i)}}(\Gamma)$ has no infinite minimal choice sets.*

*Proof.* Let $\Theta$ be an infinite minimal choice set of ${}^{\mathsf{c}}\Phi^{\sqsubset_{(i)}}(\Gamma)$. By Corollary 6.9, ($\dagger_1$) $\Theta$ is a minimal choice set of $\{\Omega_{(i)} - (\varphi \cap \Omega_{(i)}) \mid \varphi \in \Phi^{\sqsubset}(\Gamma)\} = \{\Omega_{(i)} - \varphi \mid \varphi \in \Phi^{\sqsubset}(\Gamma)\}$.

Assume that for some $\varphi \in \Phi^{\sqsubset_{(i)}}(\Gamma)$, $\Omega_{(i)} - \varphi = \emptyset$. But then $\varphi = \Omega_{(i)}$ and whence $\Phi^{\sqsubset_{(i)}}(\Gamma) = \{\Omega_{(i)}\}$. Hence ${}^{\mathsf{c}}\Phi^{\sqsubset_{(i)}}(\Gamma) = \{\emptyset\}$ which is a contradiction to the minimality of $\Theta$. Thus:

($\dagger_2$) for all $\varphi \in \Phi^{\sqsubset_{(i)}}(\Gamma)$, $\Omega_{(i)} - \varphi \neq \emptyset$

By ($\dagger_1$) and ($\dagger_2$), for all $\varphi \in \Phi^{\sqsubset}(\Gamma)$, $\Omega_{(i)} - \varphi \neq \emptyset$. By ($\dagger_1$) and since $\Omega_{(i)} - \varphi \subseteq \Omega - \varphi$, $\Theta$ is a choice set of ${}^{\mathsf{c}}\Phi^{\sqsubset}(\Gamma)$.

Assume there is a $\Theta' \subset \Theta$ which is a choice set of ${}^{\mathsf{c}}\Phi^{\sqsubset}(\Gamma)$. Since $\Theta \subseteq \Omega_{(i)}$, also ($\dagger_3$) $\Theta' \subset \Omega_{(i)}$. Note that ($\dagger_4$) for each $\varphi \in \Phi^{\sqsubset}(\Gamma)$, $((\Omega - \varphi) - (\Omega_{(i)} - \varphi)) \cap \Omega_{(i)} = \emptyset$. By ($\dagger_3$) and ($\dagger_4$), $\Theta'$ is a choice set of ${}^{\mathsf{c}}\Phi^{\sqsubset_{(i)}}(\Gamma)$, which contradicts the minimality of $\Theta$. Hence $\Theta$ is a minimal choice set of ${}^{\mathsf{c}}\Phi^{\sqsubset}(\Gamma)$. ■

### 6.4.3 Restricted Completeness and Equivalence for $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$

The basic completeness/equivalence criterion for $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ reads as follows:

($\bigstar_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}$) ${}^{\mathsf{c}}\Phi^{\sqsubset}(\Gamma)$ has no infinite minimal choice sets

In the remainder, we will show that $\bigstar_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}$ is equivalent to equation (6.2) from Theorem 6.6, for $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$.

**Lemma 6.9** *If for all $i \in I$, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m_i}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL^m_i}}(\Gamma)$, then*

$$\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL^m}}(\Gamma)$$

*Proof.* We have: $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m}}(\Gamma)) = \mathcal{M}_{\mathbf{LLL}}(\bigcup_{i\in I} Cn_{\mathbf{SAL^m_i}}(\Gamma)) = \bigcap_{i\in I} \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m_i}}(\Gamma)) = \bigcap_{i\in I} \mathcal{M}_{\mathbf{SAL^m_i}}(\Gamma) = \mathcal{M}_{\mathbf{SAL^m}}(\Gamma)$. ■

**Lemma 6.10** *Where $\Gamma \subseteq \mathcal{W}_s$: if $\Gamma$ satisfies $\bigstar_{\mathbf{SAL^m_{(I)}}}$, then $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL^m}}(\Gamma)$.*

*Proof.* Suppose $\Gamma \subseteq \mathcal{W}_s$ and $\Gamma$ satisfies $\bigstar_{\mathbf{SAL^m_{(I)}}}$. Thus, ${}^{\mathsf{c}}\Phi^{\sqsubset}(\Gamma)$ has no infinite minimal choice sets.

If $\Gamma$ is not **LLL**-satisfiable, then by Fact 3.1.5 and the monotonicity of **LLL**, $Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma)$ is not **LLL**-satisfiable for every $i \in I$. Also, by Definition 3.4, $\Gamma$ is not $\mathbf{SAL^m_{(i)}}$-satisfiable for every $i \in I$. Hence $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL^m_{(i)}}}(\Gamma) = \emptyset$, whence the lemma follows immediately. So suppose that $\Gamma$ is **LLL**-satisfiable. We will prove by induction that for every $i \in I$, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL^m_{(i)}}}(\Gamma)$, whence by Lemma 6.9, the property follows immediately.

($i = 1$) By the supposition, Fact 6.2.1 and Lemma 6.8, ${}^{\mathsf{c}}\Phi^{(1)}(\Gamma)$ has no infinite minimal choice sets. Hence by Lemma 6.3, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^m_{(1)}}}(\Gamma)) = \mathcal{M}_{\mathbf{AL^m_{(1)}}}(\Gamma)$. The rest is immediate in view of Facts 6.3 and 3.1.1.

($i \Rightarrow i+1$) Let $\Gamma' = Cn_{\mathbf{SAL^m_{(i)}}}(\Gamma)$. By Definition 6.2, $\mathcal{M}_{\mathbf{AL^m_{(i+1)}}}(\Gamma') = \{M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma') \mid$ there is no $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma')$ such that $Ab(M') \cap \Omega_{(i+1)} \subset Ab(M) \cap \Omega_{(i+1)}\}$. By the induction hypothesis and Definition 3.4,

$$\mathcal{M}_{\mathbf{AL^m_{(i+1)}}}(\Gamma') = \mathcal{M}_{\mathbf{SAL^m_{(i+1)}}}(\Gamma) \tag{6.3}$$

By Theorem 6.3 and (6.3), $\mathcal{M}_{\mathbf{AL^m_{(i+1)}}}(\Gamma') = \mathcal{M}_{\mathbf{AL^m_{\sqsubset(i+1)}}}(\Gamma)$. By Theorems 2.1 and 5.4, we obtain that $\Phi^{(i+1)}(\Gamma') = \Phi^{\sqsubset(i+1)}(\Gamma)$, whence also

$${}^{\mathsf{c}}\Phi^{(i+1)}(\Gamma') = {}^{\mathsf{c}}\Phi^{\sqsubset(i+1)}(\Gamma) \tag{6.4}$$

By the supposition and Lemma 6.8, ${}^{\mathsf{c}}\Phi^{\sqsubset(i+1)}(\Gamma)$ has no infinite minimal choice sets. Hence in view of (6.4), ${}^{\mathsf{c}}\Phi^{(i+1)}(\Gamma')$ has no infinite minimal choice sets. By Theorem 3.1.1 and Lemma 6.5,

$$\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^m_{(i+1)}}}(\Gamma')) = \mathcal{M}_{\mathbf{AL^m_{(i+1)}}}(\Gamma') \tag{6.5}$$

Hence in view of Definition 3.1, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m_{(i+1)}}}(\Gamma)) = \mathcal{M}_{\mathbf{AL^m_{(i+1)}}}(\Gamma')$. By (6.3), $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m_{(i+1)}}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL^m_{(i+1)}}}(\Gamma)$. ■

**Lemma 6.11** *Where $\Gamma \subseteq \mathcal{W}_s$: if $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m_{(I)}}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL^m_{(I)}}}(\Gamma)$ then $\Gamma$ satisfies $\bigstar_{\mathbf{SAL^m_{(I)}}}$.*

*Proof.* Suppose ${}^{\mathsf{c}}\Phi^{\sqsubset}(\Gamma)$ has an infinite minimal choice set. By Lemma 6.4, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^m_{\sqsubset}}}(\Gamma)) \neq \mathcal{M}_{\mathbf{AL^m_{\sqsubset}}}(\Gamma)$. By the soundness of $\mathbf{AL^m_{\sqsubset}}$, $\mathcal{M}_{\mathbf{AL^m_{\sqsubset}}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^m_{\sqsubset}}}(\Gamma))$. It follows that there is an $M \in \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^m_{\sqsubset}}}(\Gamma)) - \mathcal{M}_{\mathbf{AL^m_{\sqsubset}}}(\Gamma)$. By Corollary 6.5 and the monotonicity of **LLL**, $M \in \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m_{(I)}}}(\Gamma))$. By Corollary 6.1, $M \notin \mathcal{M}_{\mathbf{SAL^m_{(I)}}}(\Gamma)$. ■

**Corollary 6.10** *Where* $\Gamma \subseteq \mathcal{W}$*:* $\Gamma$ *satisfies* $\bigstar_{\mathbf{SAL^m_{(I)}}}$ *iff* $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{SAL^m_{(I)}}}(\Gamma)) = \mathcal{M}_{\mathbf{SAL^m_{(I)}}}(\Gamma)$*.*

In view of Theorem 6.6, we immediately obtain:[7]

**Corollary 6.11** *Where* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma$ *satisfies* $\bigstar_{\mathbf{SAL^m_{(I)}}}$ *, then each of the following holds:*

1. $A \in Cn_{\mathbf{SAL^m_{(I)}}}(\Gamma)$ *iff* $\Gamma \models_{\mathbf{SAL^m_{(I)}}} A$
2. $Cn_{\mathbf{SAL^m_{(I)}}}(\Gamma) = Cn_{\mathbf{AL^m_{\sqsubset}}}(\Gamma)$

### 6.4.4 Restricted Completeness and Equivalence for HAL$^\mathbf{m}$

As we will now show, every logic **HAL** is sound and complete whenever it obeys the following criterion:

($\bigstar_{\mathbf{HAL}}$) for every $i \in I$, ${}^{\mathsf{c}}\Phi^{(i)}(\Gamma)$ has no infinite minimal choice sets

**Lemma 6.12** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Gamma$ *satisfies* $\bigstar_{\mathbf{HAL}}$*, then* $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{HAL}}(\Gamma)) = \mathcal{M}_{\mathbf{HAL}}(\Gamma)$*.*

*Proof.* Suppose the antecedent holds. By Lemma 6.8, it follows that (†) for every $i \in I$, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma)) = \mathcal{M}_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma)$. By Definition 4.1, (†) and Definition 4.2 consecutively, we have $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{HAL}}(\Gamma)) = \mathcal{M}_{\mathbf{LLL}}(\bigcup_{i\in I} Cn_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma)) = \bigcap_{i\in I} \mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma)) = \bigcap_{i\in I} \mathcal{M}_{\mathbf{AL^{x_i}_{(i)}}}(\Gamma) = \mathcal{M}_{\mathbf{HAL}}(\Gamma)$. ■

Unlike for $\mathbf{SAL^m_{(I)}}$, the right-left direction of the above lemma fails. To see why, consider $\Theta = \{!^1p_i \vee !^1p_j \vee !^2q \mid i, j \in \mathbb{N}, i \neq j\}$. Although ${}^{\mathsf{c}}\Phi^{(2)}(\Theta)$ has one infinite minimal choice set (i.e. the set $\{!^1p_i \mid i \in \mathbb{N}\}$), it can be shown that $\mathcal{M}_{\mathbf{K^+}}(Cn_{\mathbf{HK2^m}}(\Theta)) = \mathcal{M}_{\mathbf{HK2^m}}(\Theta)$.

By Theorem 6.6, we immediately obtain:[8]

**Corollary 6.12** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Gamma$ *satisfies* $\bigstar_{\mathbf{HAL}}$*, then each of the following holds:*

1. $A \in Cn_{\mathbf{HAL^m}}(\Gamma)$ *iff* $\Gamma \models_{\mathbf{HAL^m}} A$
2. $Cn_{\mathbf{HAL^m}}(\Gamma) = Cn_{\mathbf{AL^m_{\sqsubset}}}(\Gamma)$

### 6.4.5 Some Weaker Completeness and Equivalence Criteria

In the preceding, we saw two sufficient syntactic criteria for the completeness results for $\mathbf{SAL^m_{(I)}}$, resp. **HAL**. As we will now show, several more straightforward criteria can be listed, each of which imply that either one or both of the

[7] $\mathbf{SAL^m_{(I)}}$ is not complete for all premise sets – we refer to Section C.3 for a counterexample. Notably, this example also illustrates that in some cases, $\mathbf{HAL^m}$ may yield more consequences that $\mathbf{SAL^m_{(I)}}$.

[8] As shown in Section C.1, unrestricted completeness fails for $\mathbf{HAL^m}$.

conditions for completeness are obeyed. Hence in concrete applications, there are various ways to establish that e.g. $Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma) = Cn_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, or that $\Gamma \models_{\mathbf{HAL}} A$ iff $A \in Cn_{\mathbf{HAL}}(\Gamma)$. The following is proven at the end of this section:

**Theorem 6.7** *Each of the following holds for every $\Gamma \subseteq \mathcal{W}_s$:*

1. *$\Sigma(\Gamma)$ is finite iff every $\varphi \in \Phi(\Gamma)$ is finite.*
2. *If every $\varphi \in \Phi(\Gamma)$ is finite, then $\Phi(\Gamma)$ is finite.*
3. *If $\Phi(\Gamma)$ is finite, then for all $i \in I$, $\Phi^{(i)}(\Gamma)$ is finite.*
4. *If $\Phi^{(i)}(\Gamma)$ is finite for every $i \in I$, then $\Gamma$ satisfies $\bigstar_{\mathbf{HAL}}$.*
5. *If $\Phi(\Gamma)$ is finite, then $\Phi^{\sqsubset}(\Gamma)$ is finite.*
6. *If $\Phi^{\sqsubset}(\Gamma)$ is finite, then for all $i \in I$, $\Phi^{\sqsubset(i)}(\Gamma)$ is finite.*
7. *If $\Phi^{\sqsubset}(\Gamma)$ is finite, then $\Gamma$ satisfies $\bigstar_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}$.*
8. *$\Gamma$ satisfies $\bigstar_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}$ iff for no $i \in I$, ${}^{\mathsf{c}}\Phi^{\sqsubset(i)}(\Gamma)$ has infinite minimal choice sets.*

$$\begin{tikzcd}
\text{$\Sigma(\Gamma)$ is finite} \arrow[d, Leftrightarrow] & & \\
\begin{array}{c}\text{f.a. } \varphi \in \Phi(\Gamma),\\ \varphi \text{ is finite}\end{array} \arrow[d, Rightarrow] & & \\
\text{$\Phi(\Gamma)$ is finite} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c}\text{f.a. } i \in I,\\ \Phi^{(i)}(\Gamma) \text{ is finite}\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\Gamma \text{ satisfies}\\ \bigstar_{\mathbf{HAL}}\end{array} \\
\text{$\Phi^{\sqsubset}(\Gamma)$ is finite} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c}\text{f.a. } i \in I,\\ \Phi^{\sqsubset(i)} \text{ is finite}\end{array} & \\
\begin{array}{c}\Gamma \text{ satisfies}\\ \bigstar_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}\end{array} \arrow[r, Leftrightarrow] & \begin{array}{c}\text{f.a. } i \in I,\ {}^{\mathsf{c}}\Phi^{\sqsubset(i)}(\Gamma)\\ \text{has no infinite}\\ \text{minimal choice sets}\end{array} &
\end{tikzcd}$$

Figure 6.1: Syntactic criteria for completeness and equivalence – "f.a." abbreviates "for all".

Figure 6.1 illustrates the relation between the criteria listed in Theorem 6.7 and the criteria $\bigstar_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}$ and $\bigstar_{\mathbf{HAL}}$.

**Proof of Theorem 6.7** For the proof of Theorem 6.7, we will rely on two facts, a lemma about minimal choice sets, and a lemma that states that where $i \in I$, $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ is a conservative extension of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}$. The first fact was proven in [137] (Lemma 3.2.4), the second is an immediate consequence of Theorem 6.3 and Definition 3.4.

**Fact 6.5** *If every $\varphi \in \Phi(\Gamma)$ is finite, then $\Phi(\Gamma)$ is finite.*

**Fact 6.6** $\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma) = \bigcap_{i\in I} \mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma) = \bigcap_{i\in I} \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$.

**Lemma 6.13** *If $\Sigma$ is a finite set of sets, then $\Sigma$ has no infinite minimal choice sets.*

*Proof.* Let $\Sigma = \{\Theta_i \mid i \leq n\}$ and let $\varphi$ be an infinite choice set of $\Sigma$. For every $i \leq n$, let $A_i$ be an arbitrary element of $\varphi \cap \Theta_i$, and let $\varphi' = \{A_1, \ldots, A_n\}$. Note that since $\varphi'$ is finite, $\varphi' \subset \varphi$. Since $\varphi'$ is a choice set of $\Sigma$, $\varphi$ is not a minimal choice set of $\Sigma$. ■

**Lemma 6.14** *Each of the following holds:*

$\mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$
*Where* $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$.

*Proof. Ad 1.* Assume that $M \in \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) - \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$. By Definition 5.2, $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$. Hence By Theorem 5.17, there is an $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $Ab(M') \sqsubset_{(i)} Ab(M)$. But then by Fact 6.1.2, $Ab(M') \sqsubset Ab(M)$, and hence $M \notin \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ — a contradiction.

*Ad 2.* Immediate in view of item 1, the soundness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}$, and the completeness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ for all $\Gamma \subseteq \mathcal{W}_s$. ■

*Proof of Theorem 6.7.* Let $\Gamma \subseteq \mathcal{W}_s$. *Ad 1.* ($\Leftarrow$) Immediate in view of Lemma 6.13. ($\Rightarrow$) Suppose that every $\varphi \in \Phi(\Gamma)$ is finite. By Fact 6.5, $\Phi(\Gamma)$ is finite, whence also $\bigcup \Phi(\Gamma)$ and $\wp(\bigcup \Phi(\Gamma))$ are finite. As explained in Chapter 2, Section 2.2, $\bigcup \Sigma(\Gamma) = U(\Gamma) = \bigcup \Phi(\Gamma)$. It follows that $\Sigma(\Gamma) \subseteq \wp(\bigcup \Phi(\Gamma))$, whence $\Sigma(\Gamma)$ is finite.

*Ad 2.* This is Fact 6.5.

*Ad 3.* Immediate in view of Lemma 3.6.

*Ad 4 and 7.* Immediate in view of Lemma 6.13.

*Ad 5.* Immediate in view of Theorem 5.2.

*Ad 6.* Let $i \in I$. Let $\varphi, \psi$ be arbitrary members of $\Phi^{\sqsubset_{(i)}}(\Gamma)$ such that $\varphi \neq \psi$. By Lemma 6.6, there are $\varphi', \psi' \in \Phi^{\sqsubset}(\Gamma)$ such that $\varphi' \cap \Omega_{(i)} = \varphi$ and $\psi' \cap \Omega_{(i)} = \psi$. Hence also $\varphi' \neq \psi'$. So the cardinality of $\Phi^{\sqsubset}(\Gamma)$ is at least as great as that of $\Phi^{\sqsubset_{(i)}}(\Gamma)$.

*Ad 8.* ($\Rightarrow$) This is Lemma 6.8. ($\Leftarrow$) Suppose that for no $i \in I$, ${}^{\mathsf{c}}\Phi^{\sqsubset_{(i)}}(\Gamma)$ has infinite minimal choice sets. Hence (†) for every $i \in I$, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$. By (†), Lemma 6.14 and the monotonicity of **LLL**, for every $i \in I$, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)) \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$. By Fact 6.6, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)) \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. Also, by the soundness of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$, $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)) \supseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. Hence $\mathcal{M}_{\mathbf{LLL}}(Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. By Lemma 6.4, ${}^{\mathsf{c}}\Phi^{\sqsubset}(\Gamma)$ has no infinite minimal choice sets. ■

**Counterexamples for the converses** To finish this section, let us briefly show that the converses of items 2-7 of Theorem 6.7 fail. We will not give full proofs for these claims, but simply list the counterexamples and some of their most salient properties.

*Ad 2.* Let $\Theta_1 = \{!^1 p \vee !^1 q_i \mid i \in \mathbb{N}\}$. Note that there is an infinite minimal choice sets of $\Sigma(\Theta_1)$, i.e. the set $\varphi = \{!^1 q_i \mid i \in \mathbb{N}\}$. Still, $\Phi(\Theta_1) = \{\{!^1 p\}, \varphi\}$ is finite.

*Ad 3 and 6.* Let $\Theta_2 = \{!^i p_{i_1} \vee !^i p_{i_2} \mid i \in \mathbb{N}\}$. Let $\Phi(\Theta_2)$ be the set of minimal choice sets with respect to the flat adaptive logic $\mathbf{K}^{\mathbf{m}}_{\cup} = \langle \mathbf{K}, \bigcup_{i \in \mathbb{N}} \Omega^{\mathbf{K}}_{(i)}, \mathbf{m}\rangle$ and let $\Phi^{\sqsubset}(\Theta_2)$ be the set of minimal choice sets with respect to the prioritized adaptive logic $\mathbf{K}^{\mathbf{m}}_{\sqsubset} = \langle \mathbf{K}, \langle \Omega^{\mathbf{K}}_{(i)}\rangle_{i \in \mathbb{N}}, \mathbf{m}\rangle$. Note that for every $i \in \mathbb{N}$, $\Sigma^{(i)}(\Theta_2)$ is finite, whence $\Phi^{(i)}(\Theta_2)$ and $\Phi^{\sqsubset(i)}(\Theta_2)$ have only finitely many minimal choice sets. However, $\Phi(\Theta_2) = \Phi^{\sqsubset}(\Theta_2)$ is infinite.

*Ad 4 and 7.* Let $\Theta_3 = \{!^1 p_{2n} \vee !^1 p_{2n+1} \mid n \in \mathbb{N}\}$. Note that $\Phi^{(1)}(\Theta_3) = \Phi^{\sqsubset(1)}(\Theta_3)$ is infinite. Nevertheless, every minimal choice set of ${}^{\mathsf{c}}\Phi^{(1)}(\Theta_3) = {}^{\mathsf{c}}\Phi^{\sqsubset(1)}(\Theta_3)$ is a couple: ${}^{\mathsf{c}}\Phi^{(1)}(\Theta_3) = \{\{!^1 p_2, !^1 p_3\}, \{!^1 p_4, !^1 p_5\}, \ldots\}$.

*Ad 5.* Let $\Theta_4 = \{!^1 p_i \vee !^1 p_j \vee !^2 q \mid i, j \in \mathbb{N}, i \neq j\}$. Let $\Psi = \{\{!^1 p_i \mid i \in \mathbb{N} - \{k\}\} \mid k \in \mathbb{N}\}$. Note that $\Phi^{(2)}(\Theta_4) = \{\{!^2 q\}\} \cup \Psi$, whereas $\Phi^{\sqsubset(2)}(\Theta_4) = \{\{!^2 q\}\}$.

## 6.5 Derived Results

In this brief section, we show that a restricted version of Cumulative Transitivity, Fixed Point and the Deduction Theorem can easily be obtained for $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ and $\mathbf{HAL}^{\mathbf{m}}$, in view of the soundness and equivalence results from the two preceding sections. In the remainder of this section, let $\mathbf{PAL} \in \{\mathbf{HAL}^{\mathbf{m}}, \mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}\}$.

**Theorem 6.8** *Where $\Gamma \subseteq \mathcal{W}_s$ and $\Gamma$ satisfies $\bigstar_{\mathbf{PAL}}$: if $\Gamma' \subseteq Cn_{\mathbf{PAL}}(\Gamma)$, then $Cn_{\mathbf{PAL}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{PAL}}(\Gamma)$. (Restricted Cumulative Transitivity)*

*Proof.* Suppose the antecedent holds. By the soundness of **PAL** and Corollary 6.1, $\Gamma' \subseteq \{A \mid \Gamma \models_{\mathbf{PAL}} A\} = \{A \mid \Gamma \models_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}} A\}$. Hence by Corollary 6.1 and Lemma 6.2:

$$\mathcal{M}_{\mathbf{PAL}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma \cup \Gamma') = \mathcal{M}_{\mathbf{PAL}}(\Gamma \cup \Gamma') \tag{6.6}$$

Suppose that $A \in Cn_{\mathbf{PAL}}(\Gamma \cup \Gamma')$. By the soundness of **PAL**, $A$ is true in every $M \in \mathcal{M}_{\mathbf{PAL}}(\Gamma \cup \Gamma')$. Hence by (6.6), $A$ is true in every $M \in \mathcal{M}_{\mathbf{PAL}}(\Gamma)$. Since $\Gamma$ obeys $\bigstar_{\mathbf{PAL}}$, it follows that $A \in Cn_{\mathbf{PAL}}(\Gamma)$. ■

**Theorem 6.9** *Where $\Gamma \subseteq \mathcal{W}_s$ and $\Gamma$ satisfies $\bigstar_{\mathbf{PAL}}$: $Cn_{\mathbf{PAL}}(\Gamma) = Cn_{\mathbf{PAL}}(Cn_{\mathbf{PAL}}(\Gamma))$. (Restricted Fixed Point)*

*Proof.* Suppose the antecedent holds. ($Cn_{\mathbf{PAL}}(\Gamma) \subseteq Cn_{\mathbf{PAL}}(Cn_{\mathbf{PAL}}(\Gamma))$) Immediate in view of the reflexivity of **PAL**.

($Cn_{\mathbf{PAL}}(Cn_{\mathbf{PAL}}(\Gamma)) \subseteq Cn_{\mathbf{PAL}}(\Gamma)$) By the reflexivity of **PAL**, $Cn_{\mathbf{PAL}}(\Gamma) = \Gamma \cup Cn_{\mathbf{PAL}}(\Gamma)$. But then $Cn_{\mathbf{PAL}}(Cn_{\mathbf{PAL}}(\Gamma)) = Cn_{\mathbf{PAL}}(\Gamma \cup Cn_{\mathbf{PAL}}(\Gamma))$, whence by the restricted cumulative transitivity of **PAL**, $Cn_{\mathbf{PAL}}(Cn_{\mathbf{PAL}}(\Gamma)) \subseteq Cn_{\mathbf{PAL}}(\Gamma)$. ■

The following Corollary summarizes Corollaries 6.5 and 6.6, and is used in the proof of Theorem 6.10 below:

**Corollary 6.13** *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $Cn_{\mathbf{PAL}}(\Gamma) \subseteq Cn_{\mathbf{AL^m_\sqsubset}}(\Gamma)$.

**Theorem 6.10** *Where* $\Gamma \subseteq \mathcal{W}_s$ *and* $\Gamma$ *satisfies* $\bigstar_{\mathbf{PAL}}$*: if* $B \in Cn_{\mathbf{PAL}}(\Gamma \cup \{A\})$*, then* $A \check{\supset} B \in Cn_{\mathbf{PAL}}(\Gamma)$*. (Restricted Deduction Theorem)*

*Proof.* Suppose the antecedent holds. By Corollary 6.13, $B \in Cn_{\mathbf{AL^m_\sqsubset}}(\Gamma \cup \{A\})$. Hence, since the Deduction Theorem holds for $\mathbf{AL^m_\sqsubset}$, $A \check{\supset} B \in Cn_{\mathbf{AL^m_\sqsubset}}(\Gamma)$. By Corollary 6.11.2 (for $\mathbf{SAL^m_{(I)}}$), resp. Corollary 6.12.2 (for $\mathbf{HAL^m}$) and the supposition, $A \check{\supset} B \in Cn_{\mathbf{PAL}}(\Gamma)$. ■

## 6.6 $\mathbf{SAL^m}$ Versus $\mathbf{SAL^m_{(I)}}$, $\mathbf{HAL^m}$ and $\mathbf{AL^m_\sqsubset}$

**Theorem 6.11** $\mathcal{M}_{\mathbf{SAL^m}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{SAL^m_{(I)}}}(\Gamma)$.

*Proof.* Assume that $M \in \mathcal{M}_{\mathbf{SAL^m}}(\Gamma) - \mathcal{M}_{\mathbf{SAL^m_{(I)}}}(\Gamma)$. By Definition 3.4, there is an $i \in I$ such that $M \notin \mathcal{M}_{\mathbf{SAL^m_{(i)}}}(\Gamma)$. By Corollary 6.1, $M \notin \mathcal{M}_{\mathbf{AL^m_{\sqsubset_{(i)}}}}(\Gamma)$. Let $j \leq i$ be the smallest $j \in I$ such that $M \notin \mathcal{M}_{\mathbf{AL^m_{\sqsubset_{(j)}}}}(\Gamma)$. By Definition 6.2, there is an $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that each of the following holds:

(i) $Ab(M') \cap \Omega_j \subset Ab(M) \cap \Omega_j$
(ii) $Ab(M') \cap \Omega_{(j-1)} = Ab(M) \cap \Omega_{(j-1)}$

Since $M \in \mathcal{M}_{\mathbf{SAL^m}}(\Gamma)$, by Definition 3.4, $M \in \mathcal{M}_{\mathbf{SAL^m_{j-1}}}(\Gamma)$. Hence by (ii) and the fact that $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$, $M' \in \mathcal{M}_{\mathbf{SAL^m_{j-1}}}(\Gamma)$. But then by Definition 3.4, $M \notin \mathcal{M}_{\mathbf{SAL^m_j}}(\Gamma)$, such that $M \notin \mathcal{M}_{\mathbf{SAL^m}}(\Gamma)$ — a contradiction. ■

In Section 3.3.2 of Chapter 3, it was shown that whenever $\Phi(\Gamma)$ is finite, then $\mathbf{SAL^m}$ and $\mathbf{SAL^m_{(I)}}$ are sound and complete with respect to their semantics (see Corollary 3.2). In Chapter 4, it was shown that given the same restriction, $\mathbf{HAL^m}$ is at least as strong as $\mathbf{AL^m}$. From these facts and Corollary 6.7, we can derive:

**Corollary 6.14** *Where* $\Gamma \subseteq \mathcal{W}_s$*: if* $\Phi(\Gamma)$ *is finite, then* $Cn_{\mathbf{AL^m}}(\Gamma) \subseteq Cn_{\mathbf{SAL^m_{(I)}}}(\Gamma) = Cn_{\mathbf{HAL^m}}(\Gamma) = Cn_{\mathbf{AL^m_\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{SAL^m}}(\Gamma)$.

In Section C.7 of Appendix C, examples are presented which illustrate that Corollary 6.14 cannot be generalized to all premise sets $\Gamma \subseteq \mathcal{W}_s$.

It can easily be shown that logics of the format $\mathbf{SAL^m}$ are often *stronger* than the corresponding logic $\mathbf{SAL^m_{(I)}}$ in case $\Phi(\Gamma)$ is finite. Recall that $\mathbf{SK2^m}$ is obtained by the superposition of the logic $\mathbf{K_2^m} = \langle \mathbf{K}, \Omega_2^{\mathbf{K}}, \mathbf{m}\rangle$ on $\mathbf{K_1^m} = \langle \mathbf{K}, \Omega_1^{\mathbf{K}}, \mathbf{m}\rangle$.

Let $\Gamma_{\mathsf{e2}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg p \vee \neg r\}$ — this set was introduced in Example 3.2 in Section 3.1.2 of Chapter 3. As explained there, $r \in Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{e2}})$. The reason is that $Cn_{\mathbf{K^m_1}}(\Gamma_{e2})$ has no Dab$_2$-consequences.[9] Hence $r$ can be finally $\mathbf{K^m_2}$-derived from $Cn_{\mathbf{K^m_1}}(\Gamma_{e2})$, on the condition $\{!^2 r\}$.

However, $r \notin Cn_{\mathbf{K2^m_\sqsubset}}(\Gamma_{e2})$. That is, the set of $\sqsubset$-minimal choice sets of $\Gamma_{e2}$ is $\Phi^{\sqsubset_2}(\Gamma_{e2}) = \Phi^{(2)}(\Gamma_{e2}) = \{\{!^1 p\}, \{!^1 q, !^2 r\}\}$. The lexicographic AL does not allow us to choose between either of these choice sets, since they are incomparable with regards to the level 1-abnormalities they contain. At the semantic level, this means $!^2 r$ is true in some $\sqsubset$-minimal abnormal models of $\Gamma_{e2}$, and hence $r$ is false in an $M \in \mathcal{M}_{\mathbf{K2^m_\sqsubset}}(\Gamma_{e2})$.

Note that $\Phi^{(2)}(\Gamma_{e2})$ is finite. Hence $Cn_{\mathbf{K2^m_\sqsubset}}(\Gamma_{e2}) = Cn_{\mathbf{HK2^m}}(\Gamma_{e2}) = Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma_{e2})$ in view of the equivalence results from the current chapter. So we can derive:

**Proposition 6.1** *There are $\Gamma \subseteq \mathcal{W}_s$, such that each of the following holds:*

1. $\Phi^{(2)}(\Gamma)$ *is finite*
2. $Cn_{\mathbf{SK2^m}}(\Gamma) \nsubseteq Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma)$
3. $Cn_{\mathbf{SK2^m}}(\Gamma) \nsubseteq Cn_{\mathbf{HK2^m}}(\Gamma)$
4. $Cn_{\mathbf{SK2^m}}(\Gamma) \nsubseteq Cn_{\mathbf{K2^m_\sqsubset}}(\Gamma)$

## 6.7 The Reliability-variants

In this section, I prove each of the claims (v)-(vii) from the introduction of this chapter (see page 132). For reasons of space, I confine myself to the proofs, and will only briefly indicate why the converse of certain theorems fails – these are mostly technical results which are called upon in Section 6.8. The main results of this section are summarized by Corollaries 6.15 and 6.16 below. In the remainder, I will rely freely on the following two properties:

(i) Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$, $Cn_{\mathbf{SAL_i}}(\Gamma)$ is **LLL**-closed (see Theorem 3.1.1).
(ii) Every logic $\mathbf{AL}, \mathbf{SAL}, \mathbf{HAL}, \mathbf{AL}_\sqsubset$ is reflexive.

### 6.7.1 $\mathbf{SAL^r_{(I)}}$ is at least as strong as $\mathbf{HAL^r}$

**Lemma 6.15** *Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$: if $A \in Cn_{\mathbf{SAL^r_i}}(\Gamma)$, then $\Gamma \vdash_{\mathbf{LLL}} A \vee Dab(\Delta)$ for a $\Delta \subset \Omega_{(i)}$ such that $\check{\neg} Dab(\Delta) \in Cn_{\mathbf{SAL^r_i}}(\Gamma)$.*

*Proof.* ($i = 1$) Suppose $A \in Cn_{\mathbf{SAL^r_1}}(\Gamma) = Cn_{\mathbf{AL^r_1}}(\Gamma)$. By Theorem 2.6, there is a $\Delta \subset \Omega_1$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U^1(\Gamma) = \emptyset$. By Theorem 2.6, $\check{\neg} Dab(\Delta) \in Cn_{\mathbf{AL^r_1}}(\Gamma) = Cn_{\mathbf{SAL^r_1}}(\Gamma)$.

($i \Rightarrow i+1$) Suppose $A \in Cn_{\mathbf{SAL^r_{i+1}}}(\Gamma)$. By Theorem 2.6 and Definition 3.1, there is a $\Theta \subset \Omega_{i+1}$ such that (†) $Cn_{\mathbf{SAL^r_i}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta)$, and $\Theta \cap U^{i+1}(Cn_{\mathbf{SAL^r_i}}(\Gamma)) = \emptyset$. By Theorem 2.6, (‡) $\check{\neg} Dab(\Theta) \in Cn_{\mathbf{AL^r_{i+1}}}(Cn_{\mathbf{SAL^r_i}}(\Gamma)) = Cn_{\mathbf{SAL^r_{i+1}}}(\Gamma)$.

By (†) and the fact that $Cn_{\mathbf{SAL^r_i}}(\Gamma)$ is **LLL**-closed, $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL^r_i}}(\Gamma)$. By the induction hypothesis, there is a $\Theta' \subseteq \Omega_{(i)}$, such that $\Gamma \vdash_{\mathbf{LLL}}$

[9] $!^2 r \check{\vee} !^1 p$ is not a Dab$_2$-formula, and we cannot derive $!^2 r$ on a condition that does not overlap with the minimal choice set $\{!^1 p\} \in \Phi^1(\Gamma_{e2})$).

$A \check{\vee} Dab(\Theta) \check{\vee} Dab(\Theta')$ and $\check{\neg} Dab(\Theta') \in Cn_{\mathbf{SAL^r_i}}(\Gamma)$. By the reflexivity of $\mathbf{AL^r_{i+1}}$, $\check{\neg} Dab(\Theta') \in Cn_{\mathbf{SAL^r_{i+1}}}(\Gamma)$. Together with (‡) and the fact that $Cn_{\mathbf{SAL^r_{i+1}}}(\Gamma)$ is **LLL**-closed, this implies that $\check{\neg} Dab(\Theta \cup \Theta') \in Cn_{\mathbf{SAL^r_{i+1}}}(\Gamma)$. ■

**Lemma 6.16** *Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$, each of the following holds:*

1. $Cn_{\mathbf{AL^r_{(i)}}}(\Gamma) \subseteq Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$.
2. $\bigcup_{j \leq i} Cn_{\mathbf{AL^r_{(i)}}}(\Gamma) \subseteq Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$.

*Proof. Ad 1.* Let $i \in I$ and suppose $A \in Cn_{\mathbf{AL^r_{(i)}}}(\Gamma)$. By Theorem 2.6, there is a $\Delta \subset \Omega_{(i)}$ such that (1) $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and (2) $\Delta \cap U^{(i)}(\Gamma) = \emptyset$. Note that since $\mathbf{SAL^r_{(i-1)}}$ is reflexive, (3) $A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$.

Assume that

$(\star)\quad \Delta \cap U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)) \neq \emptyset$

Hence there is a minimal Dab$_{(i)}$-consequence $Dab(\Theta)$ of $Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$, such that $\Theta \cap \Delta \neq \emptyset$. By Lemma 6.15, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta) \check{\vee} Dab(\Lambda)$ for a $\Lambda \subset \Omega_{(i-1)}$, such that $(\star\star)$ $\check{\neg} Dab(\Lambda) \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$.[10] Let $\Lambda'$ be a minimal subset of $\Lambda$, such that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta) \check{\vee} Dab(\Lambda')$. Assume moreover that $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta') \check{\vee} Dab(\Lambda')$ for a $\Theta' \subset \Theta$. In that case, by $(\star\star)$ and the reflexivity and **LLL**-closure of $Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$, $Dab(\Theta') \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$, which contradicts the assumption that $\Theta$ is a minimal Dab$_{(i)}$-consequence of $Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$.

Hence, there is no $\Theta' \subset \Theta$ for which $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta') \check{\vee} Dab(\Lambda')$. Together with the minimality of $\Lambda'$, this implies that $Dab(\Theta \cup \Lambda')$ is a minimal Dab$_{(i)}$-consequence of $\Gamma$. But then $\Theta \cup \Lambda' \subseteq U^{(i)}(\Gamma)$, and hence $\Delta \cap U^{(i)}(\Gamma) \neq \emptyset$, which contradicts (2).

Hence, $(\star)$ is false, which means that $\Delta \cap U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)) = \emptyset$. By (3) and Theorem 2.6, $A \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$.

*Ad 2.* Immediate in view of item 1 and Theorem 3.1.2. ■

**Theorem 6.12** *Where $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{HAL^r}}(\Gamma) \subseteq Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$.*

*Proof.* Suppose $A \in Cn_{\mathbf{HAL^r}}(\Gamma)$. Hence, there are $B_1, \ldots, B_n$ such that (1) each $B_j \in \bigcup_{i \in I} Cn_{\mathbf{AL^r_{(i)}}}(\Gamma)$ and (2) $\{B_1, \ldots, B_n\} \vdash_{\mathbf{LLL}} A$. By (i), there is a $k \in I$ such that each $B_j \in \bigcup_{i \leq k} Cn_{\mathbf{AL^r_{(i)}}}(\Gamma)$. Hence, in view of Lemma 6.16.2, each $B_j \in Cn_{\mathbf{SAL^r_{(k)}}}(\Gamma)$. Since $Cn_{\mathbf{SAL^r_{(k)}}}(\Gamma)$ is closed under **LLL**, we can derive from (ii) that $A \in Cn_{\mathbf{SAL^r_{(k)}}}(\Gamma)$. By Definition 3.1, $A \in Cn_{\mathbf{SAL^r_{(I)}}}(\Gamma)$. ■

We can use a simple example to show that the converse of Theorem 6.12 fails. Recall that the logic $\mathbf{SK2^r_{(2)}}$ is obtained by the superposition of the logic $\mathbf{K^r_{(2)}} = \langle \mathbf{K}, \Omega^{\mathbf{K}}_{(2)}, \mathbf{r} \rangle$ on $\mathbf{K^r_{(1)}} = \langle \mathbf{K}, \Omega^{\mathbf{K}}_{(1)}, \mathbf{r} \rangle$, and that $\mathbf{HK2^r}$ is obtained by the hierarchic combination of these two logics.

[10] Note that every logic $\mathbf{SAL^r_{(i)}}$ is a logic in the format of superpositions of ALs, whence we can also apply Lemma 6.15 to $\mathbf{SAL^r_{(i)}}$.

Consider the set $\Gamma_{\mathrm{t}} = \{\Diamond p, \Diamond\Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg r \vee \neg q\}$ from page 105. There it was explained that $r \notin Cn_{\mathbf{HK2^r}}(\Gamma_{\mathrm{t}})$, in view of $!^2 q \check{\vee} !^2 r$, which is a minimal $\mathrm{Dab}_{(2)}$-consequence of $\Gamma_{\mathrm{t}}$.

Note that $\Gamma_{\mathrm{t}} \vdash_{\mathbf{K^r_1}} \neg !^1 p$, since $\Gamma_{\mathrm{t}}$ has no $Dab_{(1)}$-consequences. This means that $!^2 q$ is a $\mathrm{Dab}_{(2)}$-consequence of $Cn_{\mathbf{K^r_1}}(\Gamma_{\mathrm{t}})$, whence $!^2 q \check{\vee} !^2 r$ is not a *minimal* $\mathrm{Dab}_{(2)}$-consequence of $Cn_{\mathbf{K^r_1}}(\Gamma_{\mathrm{t}})$. As a result, $!^2 r \notin U^{(2)}(Cn_{\mathbf{K^r_1}}(\Gamma_{\mathrm{t}}))$, and hence $r \in Cn_{\mathbf{K^r_{(2)}}}(Cn_{\mathbf{K^r_1}}(\Gamma_{\mathrm{t}})) = Cn_{\mathbf{SK2^r_{(2)}}}(\Gamma_{\mathrm{t}})$. So we have:

**Proposition 6.1** *There are $\Gamma \subseteq \mathcal{W}_s$ such that $Cn_{\mathbf{SK2^r_{(2)}}}(\Gamma) \not\subseteq Cn_{\mathbf{HK2^r}}(\Gamma)$.*

### 6.7.2 SAL$^{\mathrm{r}}$ is at least as strong as SAL$^{\mathrm{r}}_{(\mathrm{I})}$

**Lemma 6.17** *Where $\Gamma \subseteq \mathcal{W}_s$, $i \in I$ and $A \in \Omega_{(i)}$: if $A \notin U^{(i+1)}(Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma))$, then each of the following holds:*

1. $A \notin U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma))$
2. $\check{\neg} A \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$.

*Proof. Ad 1.* Suppose $A \in U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma))$. It follows that (†) there is a minimal $\mathrm{Dab}_{(i)}$-consequence $Dab(\Delta)$ of $Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$, such that $A \in \Delta$.

Assume now that $A \notin U^{(i+1)}(Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma))$. By the reflexivity of $\mathbf{AL^r_{(i)}}$, $Dab(\Delta) \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$. Hence there is a $\Delta' \subseteq \Delta - \{A\}$ such that $Dab(\Delta') \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$. However, note that since $\Delta' \subset \Delta$, also $\Delta' \subseteq \Omega_{(i)}$. By Lemma 3.17 (see page 84), $Dab(\Delta') \in Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma)$, which contradicts (†).
*Ad 2.* Immediate in view of item 1, Theorem 2.6 and Definition 3.1. ■

**Lemma 6.18** *Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$, each of the following holds*

1. $U^i(Cn_{\mathbf{SAL^r_{i-1}}}(\Gamma)) \subseteq U^{(i)}(Cn_{\mathbf{SAL^r_{(i-1)}}}(\Gamma))$
2. $Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma) \subseteq Cn_{\mathbf{SAL^r_i}}(\Gamma)$.

*Proof.* ($i = 1$) *Ad 1 and 2.* Immediate in view of Definition 3.1 and the fact that $\Omega_{(1)} = \Omega_1$.

($i \Rightarrow i+1$) *Ad 1.* Suppose $A \in U^{i+1}(Cn_{\mathbf{SAL^r_i}}(\Gamma))$. Hence there is a minimal $\mathrm{Dab}_{i+1}$-consequence $Dab(\Delta)$ of $Cn_{\mathbf{SAL^r_i}}(\Gamma)$, with $A \in \Delta$. Since $Cn_{\mathbf{SAL^r_i}}(\Gamma)$ is **LLL**-closed, $Dab(\Delta) \in Cn_{\mathbf{SAL^r_i}}(\Gamma)$. By Lemma 6.15, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta) \check{\vee} Dab(\Theta)$ for a $\Theta \subset \Omega_{(i)}$, such that (†) $\check{\neg} Dab(\Theta) \in Cn_{\mathbf{SAL^r_i}}(\Gamma)$. Since $\mathbf{SAL^r_{(i)}}$ is at least as strong as **LLL** (see Fact 3.1.6), (‡) $Dab(\Delta) \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL^r_{(i)}}}$.

Assume that (⋆) $A \notin U^{(i+1)}(Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma))$. Hence by (‡) and the fact that $Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$ is **LLL**-closed, $Dab(\Delta' \cup \Theta) \in Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma)$ for a $\Delta' \subseteq (\Delta - A)$. By item 2 of the induction hypothesis, $Dab(\Delta' \cup \Theta) \in Cn_{\mathbf{SAL^r_i}}(\Gamma)$. By (†) and the fact that $Cn_{\mathbf{SAL^r_i}}(\Gamma)$ is **LLL**-closed, $Dab(\Delta') \in Cn_{\mathbf{SAL^r_i}}(\Gamma)$. But then $Dab(\Delta)$ is not a minimal $\mathrm{Dab}_{i+1}$-consequence of $Cn_{\mathbf{SAL^r_i}}(\Gamma)$ — a contradiction. Hence (⋆) is false: $A \in U^{(i+1)}(Cn_{\mathbf{SAL^r_{(i)}}}(\Gamma))$.

*Ad 2.* Suppose that $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i+1})}}(\Gamma)$. By Theorem 2.6, $Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, for a $\Delta \subseteq \Omega_{(i+1)} - U^{(i+1)}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma))$. By the **LLL**-closure of $Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma)$ and the induction hypothesis,

$$A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\mathbf{i}}}(\Gamma) \tag{6.7}$$

Let $\Delta_{i+1} = \Delta \cap \Omega_{i+1}$, and let $\Delta_{(i)} = \Delta - \Omega_{i+1}$. Note that for every $B \in \Delta_{(i)}$, $B \in \Omega_{(i)} - U^{(i+1)}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma))$. By Lemma 6.17.2, for every $B \in \Delta_{(i)}$, $\check{\neg} B \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma)$. Hence by the induction hypothesis, for every $B \in \Delta_{(i)}$, $\check{\neg} B \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\mathbf{i}}}(\Gamma)$. Since $Cn_{\mathbf{SAL}^{\mathbf{r}}_{\mathbf{i}}}(\Gamma)$ is **LLL**-closed, and by (6.7), we can derive that

$$A \check{\vee} Dab(\Delta_{i+1}) \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\mathbf{i}}}(\Gamma) \tag{6.8}$$

By item 1, $U^{i+1}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\mathbf{i}}}(\Gamma)) \subseteq U^{(i+1)}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma))$. It follows that $\Delta_{i+1} \subseteq \Omega_{i+1} - U^{i+1}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\mathbf{i}}}(\Gamma))$. Hence by (6.8), Theorem 2.6 and Definition 3.1, $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\mathbf{i+1}}}(\Gamma)$. ■

**Theorem 6.13** *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}}(\Gamma) \subseteq Cn_{\mathbf{SAL}^{\mathbf{r}}}(\Gamma)$.

*Proof.* Suppose $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}}(\Gamma)$. By Definition 3.1, $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma)$ for an $i \in I$. By Lemma 6.18.2, $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\mathbf{i}}}(\Gamma)$. By Definition 3.1, $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}}(\Gamma)$. ■

Recall the logic $\mathbf{SK2^r}$, which was obtained by the superposition of $\mathbf{K^r_2}$ on $\mathbf{K^r_1}$. By Theorem 6.13, it follows that for every $\Gamma \subseteq \mathcal{W}_m$, $Cn_{\mathbf{SK2}^{\mathbf{r}}_{(\mathbf{2})}}(\Gamma) \subseteq Cn_{\mathbf{SK2^r}}(\Gamma)$. To see that the set inclusion is sometimes proper, consider the set $\Gamma_{\mathsf{e2}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg p \vee \neg r\}$. This set was introduced in Example 3.2 in Section 3.1.2 of Chapter 3. There it was already shown that $r \in Cn_{\mathbf{SK2^r}}(\Gamma_{\mathsf{e2}}) - Cn_{\mathbf{SK2}^{\mathbf{r}}_{(\mathbf{2})}}(\Gamma_{\mathsf{e2}})$.

Hence the converse of Theorem 6.13 fails:

**Proposition 6.2** *There are* $\Gamma \subseteq \mathcal{W}_s$ *such that* $Cn_{\mathbf{SK2^r}}(\Gamma) \not\subseteq Cn_{\mathbf{SK2}^{\mathbf{r}}_{(\mathbf{2})}}(\Gamma)$.

The following corollary summarizes Theorems 4.20, 6.12 and 6.13:

**Corollary 6.15** *Where* $\Gamma \subseteq \mathcal{W}_s$*:* $Cn_{\mathbf{AL^r}}(\Gamma) \subseteq Cn_{\mathbf{HAL^r}}(\Gamma) \subseteq Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}}(\Gamma) \subseteq Cn_{\mathbf{SAL^r}}(\Gamma)$.

## 6.7.3 Lexicographic ALs versus Superpositions (1)

In this section, I prove that every logic $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$ is at least as strong as the corresponding logic $\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}$, and I illustrate the fact that many logics $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$ are actually stonger than the corresponding logics $\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}$. For the proof, I use a specific format, which is obtained by superposing lexicographic ALs. As far as I can see, this format is not really interesting in itself, but it allows me to bridge the gap between superpositions of ALs and lexicographic ALs.

Let $\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{0})} =_{\mathsf{df}} \mathbf{LLL}$. For every $i \in I$, define the logic $\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}$ as follows:

$$Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\ldots Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{2})}}(Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{1})}}(\Gamma))\ldots)$$

Since $Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$ is **LLL**-closed for all $\Gamma \subseteq \mathcal{W}_s$ (see Theorem 5.30), we immediately have:

**Fact 6.7** *Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$: $Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)) = Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$.*

**Lemma 6.19** *For every $i \in I$, if $A \in \Omega_{(i)} - U^{\sqsubset(i)}(\Gamma)$, then $A \in \Omega - U^{\sqsubset}(\Gamma)$.*

*Proof.* Let $i \in I$. Suppose $A \in \Omega_{(i)} - U^{\sqsubset(i)}(\Gamma)$. Hence $A \in \Omega$, and $A \notin \bigcup \Phi^{\sqsubset(i)}(\Gamma)$. It follows that for every $\varphi \in \Phi^{\sqsubset(i)}(\Gamma)$, $A \notin \varphi$.

Assume that $A \in U^{\sqsubset}(\Gamma)$. Hence $A \in \psi$ for a $\psi \in \Phi^{\sqsubset}(\Gamma)$. However, by Lemma 6.7, $\psi \cap \Omega_{(i)} \in \Phi^{\sqsubset(i)}(\Gamma)$, and hence $A \in \varphi$ for a $\varphi \in \Phi^{\sqsubset(i)}(\Gamma)$ — a contradiction. ■

**Lemma 6.20** *Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$, $Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$.*

*Proof.* Suppose $A \in Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$. By Lemma 5.11, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega_{(i)} - U^{\sqsubset(i)}(\Gamma)$. But then by Lemma 6.19, $\Delta \subseteq \Omega - U^{\sqsubset}(\Gamma)$. Hence by Lemma 5.10, $A \in Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$. ■

**Lemma 6.21** *Where $\Gamma \subseteq \mathcal{W}_s$: for all $i \in I$, $Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$.*

*Proof.* ($i = 1$) Immediate in view of Lemma 6.20 and the fact that $Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{1})}}(\Gamma) = Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{1})}}(\Gamma)$.

($i \Rightarrow i+1$) Note that by the definition of $\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}$,

$$Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma))$$

By Lemma 6.20, we have:

$$Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma))$$

By the cumulative indifference of $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$ (see Corollary 5.3) and the induction hypothesis,

$$Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)) = Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$$

■

**Lemma 6.22** *Where $\Gamma \subseteq \mathcal{W}_s$: if $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$, then there is a $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} \check{\neg} Dab(\Delta)$.*

*Proof.* Suppose $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} A$. By Lemma 5.11, there is a $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. Note that also $\Gamma \vdash_{\mathbf{LLL}} \check{\neg} Dab(\Delta) \check{\vee} Dab(\Delta)$, whence by Lemma 5.10, $\Gamma \vdash_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}} \check{\neg} Dab(\Delta)$. ■

**Lemma 6.23** *Where $\Gamma \subseteq \mathcal{W}_s$ and $i \in I$: if $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$, then there is a $\Delta \subseteq \Omega_{(i)}$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Gamma \vdash_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}} \check{\neg} Dab(\Delta)$.*

*Proof.* ($i = 1$) Suppose $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{1})}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma)$. By Lemma 5.11, there is a $\Delta \subset \Omega_1$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U^1(\Gamma) = \emptyset$. By Lemma 5.10, $\check{\neg} Dab(\Delta) \in Cn_{\mathbf{AL}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma) = Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{1})}}(\Gamma)$.

($i \Rightarrow i+1$) Suppose $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(\Gamma)$. By Lemma 5.11 and Definition 3.1, there is a $\Theta \subset \Omega_{i+1}$ such that (†) $Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma) \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta)$, and $\Theta \cap U^{i+1}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)) = \emptyset$. By Lemma 5.10,

$$(\ddagger)\ \check{\neg} Dab(\Theta) \in Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)) = Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(\Gamma)$$

By (†) and Fact 6.7, $A \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$. By the induction hypothesis, there is a $\Theta' \subseteq \Omega_{(i)}$ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Theta) \check{\vee} Dab(\Theta')$ and $\check{\neg} Dab(\Theta') \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$. By the reflexivity of $\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}$, $\check{\neg} Dab(\Theta') \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(\Gamma)$. Together with (‡) and Fact 6.7, this implies that $\check{\neg} Dab(\Theta \cup \Theta') \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(\Gamma)$. ■

**Lemma 6.24** *Where $\Gamma \subseteq \mathcal{W}_s$, each of the following holds for all $i \in I$:*

1. $U^{\sqsubset(i)}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i-1})}}(\Gamma)) \subseteq U^{(i)}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i-1})}}(\Gamma))$
2. $Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma) \subseteq Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$.

*Proof.* ($i = 1$) *Ad 1.* Immediate in view of Fact 6.2.4, and the fact that $\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{0})} =_{\mathsf{df}} \mathbf{LLL}$ and $\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{0})} =_{\mathsf{df}} \mathbf{LLL}$.

*Ad 2.* Immediate in view of the fact that $\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{1})} = \mathbf{AL}^{\mathbf{r}}_{\mathbf{1}} = \mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{1})} = \mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{1})}$.

($i \Rightarrow i+1$) *Ad 1.* Suppose $A \in U^{\sqsubset(i+1)}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma))$. Hence there is a $\Delta \subseteq \Omega_{(i+1)}$ such that $Dab(\Delta)$ is a minimal Dab-consequence of $Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$. By Lemma 6.23, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta) \check{\vee} Dab(\Theta)$, for a $\Theta \subseteq \Omega_{(i)}$ such that (†) $\check{\neg} Dab(\Theta) \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$.

Assume now that $A \notin U^{(i)}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i-1})}}(\Gamma))$. Note that since $\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}$ is at least as strong as $\mathbf{LLL}$, $Dab(\Delta) \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma)$. Hence there is a $\Delta' \subseteq \Delta - \{A\}$ such that

$$Dab(\Delta') \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma)$$

By the induction hypothesis,

$$Dab(\Delta') \check{\vee} Dab(\Theta) \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$$

But then by (†) and the fact that $Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$ is **LLL**-closed, we can derive that $Dab(\Delta') \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$. Hence $Dab(\Delta)$ is not a minimal $\mathrm{Dab}_{(i+1)}$-consequence of $Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$ — a contradiction.

*Ad 2.* Suppose $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i+1})}}(\Gamma)$. Hence by Lemma 5.11 and Definition 3.1, there is a $\Delta \subset \Omega_{(i+1)}$ such that (1) $A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma))$ and (2) $\Delta \cap U^{(i+1)}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma)) = \emptyset$.

By the induction hypothesis, (1) and the monotonicity of **LLL**, $A \check{\vee} Dab(\Delta) \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma))$. By item 1 and (2), $\Delta \cap U^{\sqsubset(i+1)}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)) = \emptyset$. Hence by Lemma 5.10, $A \in Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)) = Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i+1})}}(\Gamma)$. ■

**Theorem 6.14** *Where $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$.*

*Proof.* Suppose $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}}(\Gamma)$. By Definition 3.1, $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{i})}}(\Gamma)$ for an $i \in I$. Hence by Lemma 6.24.2, $A \in Cn_{\mathbf{SAL}^{\mathbf{r}}_{\sqsubset(\mathbf{i})}}(\Gamma)$. By Lemma 6.21, $A \in Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$. ■

By Theorems 4.20, 6.12 and 6.14, we have:

**Corollary 6.16** *Where $\Gamma \subseteq \mathcal{W}_s$: $Cn_{\mathbf{AL}^{\mathbf{r}}}(\Gamma) \subseteq Cn_{\mathbf{HAL}^{\mathbf{r}}}(\Gamma) \subseteq Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$.*

Also, in view of Corollary 6.11 and the fact that $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ is always at least as strong as $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$, we can infer:[11]

**Corollary 6.17** *If $\Gamma \subseteq \mathcal{W}_s$ and $\Gamma$ satisfies $\bigstar_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}$, then $Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}}(\Gamma) \subseteq Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma)$.*

To see that $\mathbf{K2}^{\mathbf{r}}_{\sqsubset}$ is sometimes stronger than $\mathbf{SK2}^{\mathbf{r}}_{(\mathbf{2})}$, we can use the premise set introduced on page 139: $\Gamma_{\mathsf{eq}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \Diamond\Diamond s, \neg p \vee \neg q, \neg p \vee \neg r, \neg q \vee \neg r, \neg r \vee \neg s\}$. As explained there, $\Phi^{\sqsubset}(\Gamma_{\mathsf{eq}}) = \{\{!^1 p, !^2 r\}, \{!^1 q, !^2 r\}\}$, and hence $U^{\sqsubset}(\Gamma_{\mathsf{eq}}) = \{!^1 p, !^1 q, !^2 r\}$. It follows that $!^2 s$ is a $\sqsubset$-reliable abnormality in view of $\Gamma_{\mathsf{eq}}$. So we can finally derive $s$ on the condition $\{!^2 s\}$ in a $\mathbf{K2}^{\mathbf{r}}_{\sqsubset}$-proof from $\Gamma_{\mathsf{eq}}$.

To see that $s \notin Cn_{\mathbf{SK2}^{\mathbf{r}}_{(\mathbf{2})}}(\Gamma_{\mathsf{eq}})$, note that $!^2 r \notin Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma_{\mathsf{eq}})$ — we can only derive this formula on the conditions $\{!^1 p\}$ and $\{!^1 q\}$, and $U^{(1)}(\Gamma_{\mathsf{eq}}) = \{!^1 p, !^1 q\}$. Hence $!^2 r \check{\vee} !^2 s$ is a minimal Dab-consequence of $Cn_{\mathbf{K}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma_{\mathsf{eq}})$, which implies that $!^2 s$ is not a reliable abnormality for the second logic in the superposition. As a result:

**Proposition 6.3** *There are $\Gamma \subseteq \mathcal{W}_s$ such that $Cn_{\mathbf{K2}^{\mathbf{r}}_{\sqsubset}}(\Gamma) \not\subseteq Cn_{\mathbf{HK2}^{\mathbf{r}}}(\Gamma)$.*

### 6.7.4 Lexicographic ALs versus Superpositions (2)

As stated in the introduction, logics $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$ and $\mathbf{SAL}^{\mathbf{r}}$ are in general incomparable. So let me briefly illustrate that $\mathbf{SK2}^{\mathbf{r}}$ sometimes yields more consequences than $\mathbf{K2}^{\mathbf{r}}_{\sqsubset}$, and vice versa.

[11] It is not yet clear whether this result can be generalized to all premise sets $\Gamma \subseteq \mathcal{W}_s$ of all premise sets $\Gamma \subseteq \check{\mathcal{W}}_s$.

The set $\Gamma_{\mathsf{eq}}$ from the preceding section can also be used to show that $\mathbf{SK2^r}$ is not always at least as strong as $\mathbf{K2^r_\sqsubset}$. By the same reasoning as the one above, we can show that $s \notin Cn_{\mathbf{SK2^r}}(\Gamma_{\mathsf{eq}})$.

To see that $\mathbf{SK2^r}$ and $\mathbf{K2^r_\sqsubset}$ are incomparable, consider the premise set $\Gamma_{\mathsf{e2}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg p \vee \neg r\}$ from Section 3.1.2 of Chapter 3. As noted there, $r \in Cn_{\mathbf{SK2^r}}(\Gamma_{\mathsf{e2}})$.

Note that $\Gamma_{\mathsf{e2}}$ has exactly two minimal Dab-consequences:

$!^1 p \,\check{\vee}\, !^1 q$
$!^1 p \,\check{\vee}\, !^2 r$

Hence $\Phi^\sqsubset(\Gamma_{\mathsf{e2}}) = \{\{!^1 p\}, \{!^1 q, !^2 r\}\}$. It follows that $!^2 r \in U^\sqsubset(\Gamma_{\mathsf{e2}})$, and hence $r \notin Cn_{\mathbf{K2^r_\sqsubset}}(\Gamma_{\mathsf{e2}})$. So we have:

**Proposition 6.4** *There are* $\Gamma, \Gamma' \subseteq \mathcal{W}_s$ *such that:*

1. $Cn_{\mathbf{K2^m_\sqsubset}}(\Gamma) \nsubseteq Cn_{\mathbf{SK2^m}}(\Gamma)$
2. $Cn_{\mathbf{SK2^m}}(\Gamma') \nsubseteq Cn_{\mathbf{K2^m_\sqsubset}}(\Gamma')$

## 6.8 In Conclusion

### 6.8.1 Overview of the metatheoretic results

Let me briefly recapitulate the main results of Part I of this thesis. I have presented and studied three formats for prioritized ALs: sequential superpositions of flat ALs, hierarchic combinations of flat ALs, and lexicographic ALs. For the first two, I started from a definition of their syntactic consequence relation, and next considered semantic and proof-theoretic characterizations of this consequence relation. For lexicographic ALs, the consequence relation was defined directly in terms of the proof theory; also for this format, a semantic characterization was provided. For all three formats, I checked a fixed list of metatheoretic properties that are known to hold for all ALs in standard format. Finally, in the current chapter, several formats were compared in terms of the selection of models they use, and of their logical strength. The following paragraphs summarize my main conclusions:[12]

**Semantics.**

- Some superpositions of ALs are neither sound nor complete with respect to their semantics. In some cases, even Semantic Reassurance fails for the semantics. On the positive side, it was shown that all superpositions of ALs that use the Reliability Strategy are sound and complete; also, when $\Phi(\Gamma)$ is finite, then all logics **SAL** are sound and complete. Finally, in the current chapter, we saw that all logics $\mathbf{SAL^m_{(I)}}$ are sound and satisfy a specific kind of Strong Reassurance, given by SR3 – see page 81.

[12] As explained in Chapter 2, Section 2.7, completeness results are always restricted to premise sets $\Gamma \subseteq \mathcal{W}_s$. The same holds for several other properties, such as Cumulative Transitivity, Idempotence, **LLL**-Closure and **LLL**-Invariance. I refer to the respective theorems where this restriction is always made explicit.

- Although all hierarhic ALs are sound with respect to their semantics, they are not in general complete. Hierarchic combinations of logics $\langle \mathbf{AL}^{\mathbf{r}}_{(\mathbf{i})}\rangle_{i\in I}$ are complete. Also, whenever $\Phi(\Gamma)$ is finite (or given weaker restrictions – see the current chapter), all logics **HAL** are sound and complete. In contrast to the semantics of superpositions, the **HAL**-semantics satisfies SR3 and hence also Semantic Reassurance.
- It was shown that lexicographic ALs are always sound and complete (for all $\Gamma \subseteq \mathcal{W}_s$) with respect to their semantics. Also, the $\mathbf{AL}_{\sqsubset}$-semantics satisfies SR3.
- The semantic consequence relations of $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$, $\mathbf{HAL}^{\mathbf{m}}$ and $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ are equivalent. Moreover, it was shown that for all $\Gamma$, $\mathcal{M}_{\mathbf{SAL}^{\mathbf{m}}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}^{\mathbf{m}}}(\Gamma)$.

**Proof Theories.**

- Every prioritized AL considered in this thesis can be fully characterized by a dynamic proof theory, which uses exactly the same notion of a proof and final derivability as ALs in standard format, and hence only differs in its marking definition. This result is particularly important since, as explained in Chapter 1, it is precisely their proof theory that turns ALs into very good candidates to capture the external and internal dynamics of (prioritized) defeasible reasoning methods. Also, since the proof formats of each of the prioritized ALs are identical to that of the standard format, we can shift between these formats within one and the same proof (see Chapter 3 where this was explained).
- For superpositions, we saw that it was only possible to obtain a proof theory in the proof format of flat ALs, if we give up some intuitive desiderata concerning the derivability of formulas at a stage. However, it was shown that if we slightly change the notion of a condition – replacing sets by sequences of sets –, a much more appealing proof theory can be used.

**Metatheoretic properties.**

- There are a number of properties which all the prioritized adaptive logics considered in this thesis share with flat ALs: Reflexivity, **LLL**-Closure, **LLL**-Invariance, Syntactic Reassurance. They all oscillate between their respective **LLL** and **ULL**, and are equivalent to their $i$th upper limit logic ($\mathbf{ULL}_{(\mathbf{i})}$) for premise sets that are normal up to level $i$.
- Lexicographic ALs score better than both superpositions of ALs and hierarchic ALs in a number of respects: every logic $\mathbf{AL}_{\sqsubset}$ is cumulatively transitive, cautiously monotonic, and hence also idempotent – each of these properties fail for the other formats in the general case. Also, it has been shown that the Maximality of **LLL** and the three criteria for equivalence (see Section 2.5) can easily be generalized from **AL** to $\mathbf{AL}_{\sqsubset}$.[13] Finally, $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ satisfies the Deduction Theorem.

[13]For superpositions and hierarchic ALs, criterion (C1) of equivalence fails – the counterexamples are the same as those for the Fixed Point property. It is not yet clear whether criteria (C2) and (C3), and the Maximality of **LLL** hold for these two formats of prioritized ALs.

- Superpositions of logics $\langle \mathbf{AL}^{\mathbf{r}}_{(\mathbf{i})} \rangle_{i \in I}$ are cumulatively transitive, cautiously monotonic, and idempotent.
- Given certain restrictions on the premise sets (spelled out in the current chapter), $\mathbf{HAL}^{\mathbf{m}}$ and $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ are also cumulatively transitive, idempotent and satisfy the Deduction Theorem. It remains an open question whether, under the same restrictions, these logics are also cautiously monotonic.

**Logical Strength.**

- Each of the following holds for all $\Gamma$:

  $Cn_{\mathbf{HAL}^{\mathbf{r}}}(\Gamma) \subseteq Cn_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma)$
  $Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$

  $\mathbf{SAL}^{\mathbf{r}}$ and $\mathbf{SAL}^{\mathbf{m}}$ are in general incomparable. However, given weak restrictions, $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$ is at least as strong as $\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}$ (see the penultimate section of this chapter).
- Each of the following holds for all $\Gamma \subseteq \mathcal{W}_s$:

  $Cn_{\mathbf{AL}^{\mathbf{r}}}(\Gamma) \subseteq Cn_{\mathbf{HAL}^{\mathbf{r}}}(\Gamma) \subseteq Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}}(\Gamma) \subseteq Cn_{\mathbf{AL}^{\mathbf{r}}_{\sqsubset}}(\Gamma)$
  $Cn_{\mathbf{AL}^{\mathbf{r}}}(\Gamma) \subseteq Cn_{\mathbf{HAL}^{\mathbf{r}}}(\Gamma) \subseteq Cn_{\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}}(\Gamma) \subseteq Cn_{\mathbf{SAL}^{\mathbf{r}}}(\Gamma)$

  $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$ and $\mathbf{SAL}^{\mathbf{r}}$ are incomparable in terms of logical strength.
- The logics $\mathbf{SAL}^{\mathbf{m}}, \mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}, \mathbf{HAL}^{\mathbf{m}}, \mathbf{AL}^{\mathbf{m}}_{\sqsubset}$ are in general incomparable. However, if $\Phi(\Gamma)$ is finite, then $Cn_{\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}}(\Gamma) = Cn_{\mathbf{HAL}^{\mathbf{m}}}(\Gamma) = Cn_{\mathbf{AL}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathbf{SAL}^{\mathbf{m}}}(\Gamma)$ for all $\Gamma \subseteq \mathcal{W}_s$.

One additional noteworthy result that does not fit into any of these categories, concerns the logics $\mathbf{HAL}^{\mathbf{r}}$: it was shown that these systems have an upper bound complexity of $\Sigma^0_3$, which is the same upper bound as that of $\mathbf{AL}^{\mathbf{r}}$.

In view of these results, none of the three formats (superpositions, hierarchic ALs and lexicographic ALs), nor any of their subformats (e.g. $\mathbf{HAL}^{\mathbf{r}}$ or $\mathbf{SAL}^{\mathbf{m}}_{(\mathbf{I})}$) can be discarded. One possible exception is $\mathbf{HAL}^{\mathbf{m}}$, which is merely interesting in view of its quasi-equivalence to $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$. For all the other subformats, we can at most say that it depends on other criteria, which of them is to be preferred. I will specify some of these in the next paragraph.

### 6.8.2 A More General Evaluation

Logics and generic formats of logics can be chosen on the basis of various desiderata. Let me briefly consider some desiderata here, and show how these can help us to evaluate the formats at hand.

First, some formats may be either too weak or too strong to capture a given reasoning form.[14] For instance, in Chapter 8, a superposition of ALs is discussed which models a specific kind of abduction. This logic combines singular fact abduction (i.e. to derive $A\beta$ from $\forall\alpha(A\alpha \supset B\alpha)$ and $B\beta$) and inductive generalization (to derive $\forall\alpha A\alpha$ from $\exists\alpha A\alpha$). As argued there, singular fact abduction

[14] This point is similar to the idea that in some contexts, Reliability is more suitable, whereas in others, Minimal Abnormality is the preferable strategy – see Chapter 2.

needs to be prioritized over inductive generalization – otherwise the logic is too weak to serve its purpose. However, for the same reason, also a hierarchic AL in the format of $\mathbf{HAL^r}$ would be too weak. An interesting topic for further research would be to see whether, in some contexts, there are formats which are too strong, in that the logics defined in those formats allow us to derive unjustified conclusions from a given premise set.

Secondly, one may ask whether the ultimate aim of an adaptive logic is to maximally approximate a given standard of normality – hence its upper limit logic –, or rather to approximate it in such a way that the resulting consequence relation has a number of Tarski-properties or properties that closely resemble them, e.g. Idempotence and Cautious Monotonicity.

For instance, in the context of belief revision, we can distinguish between two sorts of operations: single revisions of a prioritized belief base, and iterations of such revisions. In the former case, sequential superpositions of the type of $\mathbf{SK^m}$ yield a very powerful consequence relation. However, when applied to iterations of revision, such systems can lead to counterintuitive results. Suppose for instance that, after revising a set of formulas $\Gamma$ in view of the new information $A$, we end up with the set of beliefs $\Gamma'$. Suppose moreover that $B \in \Gamma'$. In that case, it would be strange that revising $\Gamma'$ with $B$ results in a still different set $\Gamma''$ (hence, $\Gamma'' \neq \Gamma'$). In other words, if $B$ is part of the "output" of a given revision operation, then revising our revision set once more with $B$ should make no difference. Hence, if prioritized ALs are used to model iterative belief revision (of possibly infinite, complex premise sets), the formats $\mathbf{AL}_{\sqsubset}$ and $\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}$ seem more suitable candidates than $\mathbf{SAL^m}$ or $\mathbf{SAL^r}$.

This relates to a third desideratum, i.e. the ability to deal with very complex premise sets. For example, when a logic is intended to explicate our reasoning about obligations in a simple propositional language, then it seems hard to think of concrete examples where there are infinitely many minimal choice sets of $\Sigma(\Gamma)$. Hence a deontic logician may consider the case where $\Phi(\Gamma)$ is infinite as irrelevant. On the other hand, most propositional logics are merely propositional for the sake of simplicity, and their extension to the predicative level is often (implicitly) taken for granted. As shown in [157], at the predicative level, finite premise sets $\Gamma$ can already lead to the maximal upper bound complexity of $Cn_{\mathbf{AL^m}}(\Gamma)$. Likewise, the negative results for superpositions and hierarchic ALs can be obtained from finite premise sets at the predicative level.[15] Hence, more is required to argue one's way out of the metatheoretic problems of e.g. logics like $\mathbf{HAL^m}$.

A fourth question concerns the distinction between semantic and syntax-based approaches in logic. Ideally, syntax and semantics should go hand in hand, but in view of the above results, the adaptive logician's job is not always ideal. If the aim of a logician is to explicate dynamic reasoning forms, then the proof theory of sequential superpositions seems very suitable, since it forces the user to explicate (at the object-level) the way priorities allow us to block certain derivations, and to let other derivations pass. If on the other hand, one wants

[15] For instance, consider the infinite set $\{!^1p_i \vee !^1p_j \mid i,j \in \mathbb{N}, i \neq j\}$, which was used to generate an infinite $\Phi(\Gamma)$ in all the critical examples from this and previous chapters. If we move to the predicative level, with abnormalities of the type $!^1Px = \exists x(\Diamond^1 Px \wedge \neg Px)$, we can obtain a similar result from the singleton $\{\forall x \forall x(x = y \vee !^1Px \vee !^1Py)\}$.

to provide an intuitive and meaningful semantics for the syntactic consequence relation at hand, then lexicographic ALs are the best candidates (recall that the semantics of superpositions suffers from various problems, i.a. that it does not have the Reassurance property).

Finally, there is the complexity-issue. Although hierarchic ALs of the type $\mathbf{HAL^r}$ have certain metatheoretic disadvantages, they have one clear advantage, i.e. their low complexity. In Appendix D, I give an outline for a proof that, where $n \in \mathbb{N}$, superpositions of the logics $\langle \mathbf{K_i^r} \rangle_{i \leq n}$ have a worst case complexity of at least $\Sigma^0_{2n+1}$. Also, since the formats of logics $\mathbf{SAL^m}, \mathbf{HAL^m}$ and $\mathbf{AL^m_\sqsubset}$ are extensions of the standard format, it seems evident that prioritized logics that have the Minimal Abnormality strategy are at least $\Pi^1_1$-complex.[16] Finally, also sets $Cn_{\mathbf{AL^r_\sqsubset}}(\Gamma)$ can be $\Pi^1_1$-complex, in view of a simple variant of the example $\Gamma_R$ from [157]. I did not have the time to spell out full-blown proofs for these claims, and neither did I find ways to minimize the upper bound complexity of the aforementioned formats – this is work in progress. Nevertheless, it is very likely that the various formats differ with respect to their complexity, and hence, that the results of this research are relevant for our choice among several formats.

### 6.8.3 Topics for Further Research

There is another, perhaps even more compelling reason to be pluralistic concerning the formats for prioritized ALs. Rather than a museum of systems and their properties, the current study is intended to be a starting point of new research lines. Some of these were already mentioned in Chapter 1. I will reconsider them here, showing that the various formats for prioritized ALs allow for various interesting hypotheses to be tackled in future work.

**Partial Orders** First and foremost, there is the question of how to deal with partial orders within the adaptive logic framework. Recall that so far, no tools were provided to do this, just as there is no generic way to capture prioritized defeasible reasoning by means of ALs in standard format. However, relying on the current results, it seems that at least two generic formats for reasoning with partially ordered information can be developed. The first is based on the alternative semantics and proof theory of $\mathbf{HAL^r}$; the second on the selection semantics of $\mathbf{AL^m_\sqsubset}$. Let me briefly try to explain the rough idea behind each of these proposals separately, for the sake of argument.

As indicated above, the distinctive feature of the alternative characterization of $\mathbf{HAL^r}$ consists in the fact that abnormalities are avoided by minimizing the unreliable part of Dab-formulas. That is, for any given minimal Dab-consequence $Dab(\Delta)$ of $\Gamma$, a preferred (non-empty) fragment $\Delta' \subseteq \Delta$ is selected, and (only) the abnormalities in $\Delta'$ are unreliable in view of this Dab-consequence. The preferred fragment consists of those abnormalities that are "the least harmful", in view of their priority level.

[16] As shown in [157], the flat AL $\mathbf{CLuN^m}$ already has a worst case complexity of $\Pi^1_1$ (see also Chapter 2). This fact does not rely on specific aspects of $\mathbf{CLuN^m}$, and can easily be generalized to e.g. the $\mathbf{K}$-based flat ALs from the preceding chapters.

In a similar vein, we can use any strict partial order $\prec$ on $\Omega$, in order to select preferred fragments $\mathsf{pf}_{\prec}(\Delta) = \{A \in \Delta \mid \text{ there is no } B \in \Delta : B \prec A\}$. Whenever the order $\prec$ is acyclic, it can easily be shown that for every non-empty $\Delta$, $\mathsf{pf}_{\prec}(\Delta)$ is also non-empty.[17] The set of unreliable formulas in view of $\Gamma$ could then be defined as follows:

$$U^{\prec}(\Gamma) = \bigcup_{\Delta \in \Sigma(\Gamma)} \mathsf{pf}_{\prec}(\Delta)$$

Likewise, we can define the set of unreliable formulas at stage $s$ as

$$U_s^{\prec}(\Gamma) = \bigcup_{\Delta \in \Sigma_s(\Gamma)} \mathsf{pf}_{\prec}(\Delta)$$

Finally, we can define a syntax and semantics on the basis of both sets – this is but a matter of replacing every occurrence of $U(\Gamma)$ and $U_s(\Gamma)$ in the definition of $\mathbf{AL^r}$ with $U^{\prec}(\Gamma)$, resp. $U_s^{\prec}(\Gamma)$. Although this is only a first idea, it seems that the resulting consequence relation warrants reassurance and has the same upper bound complexity as $\mathbf{HAL^r}$ and $\mathbf{AL^r}$.

An altogether different way to deal with partial orders, is by translating them into a partial order $\prec$ on the *power set* of $\Omega$.[18] Note that $\subset$ and $\sqsubset$ are such partial orders. Recall that both in the syntax and semantics of $\mathbf{AL}^{\mathbf{m}}_{\sqsubset}$, the order $\sqsubset$ is crucial. Syntactically, it allows us to select a set of $\sqsubset$-minimal choice sets of $\Sigma_s(\Gamma)$, which determines the marking at stage $s$. Semantically, it marks off the set of selected models, viz. those that are minimal according to the order $\sqsubset$. Upon inspection of the proofs in Chapter 5, it turns out that one can easily replace $\sqsubset$ with any partial order $\prec$ on $\wp(\Omega)$, provided that $\prec$ extends $\subset$, and that it is smooth for all sets $\{Ab(M) \mid M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)\}$. This idea is thus a very straightforward generalization of the idea of lexicographic adaptive logics.[19]

To describe, explain and study each of these two proposals in detail, would require yet another thesis. However, the main point I wanted to make is that, on the basis of the current results, we can think of several ways to deal with partial orders in the framework of adaptive logics.

**Dynamic Priorities** It was mentioned in Chapter 1 that so far, no one has attempted to capture prioritized reasoning in which the priorities themselves are defeasible or dynamic, within the adaptive logic framework. Again, various ways to deal with this matter seem promising – I will explain two of them here, restricting myself to reasoning on the basis of prioritized beliefs.

Suppose that we want to model the fact that the plausibility degree of a certain belief is itself defeasible. The most obvious way to do this, is by means of an additional layer of logical operators, that allow one to express such things as "it is plausible that $A$ has plausibility degree $i$". The idea is that the adaptive logic first allows us to derive "$A$ has plausibility degree $i$" from this, and only

[17] An order $\prec$ on $\Delta$ is *acyclic* iff there are no $A_1, \ldots, A_n \in \Delta$ such that $A_1 \prec \ldots \prec A_n \prec A_1$.

[18] From any strict partial order $\prec$ on the set $\Omega$, one can obtain a partial order $\prec'$ on the set $\wp(\Omega)$ by a so-called "lifting criterion". For example, one can let $\Delta \prec' \Delta'$ iff for every $A \in \Delta' - \Delta$, there is a $B \in \Delta - \Delta'$ such that $B \prec A$ – this criterion was introduced by Brass in his [40].

[19] It was Christian Straßer who first came up with a general proposal that relies on this idea – heretofore, none of it has been published however.

afterwards derive $A$ from the latter. If at some point, the reasoner realizes that in fact, $A$ should receive a lower or higher priority degree, then he may extend his premises accordingly. Although this idea seems to be fairly intuitive, it remains to be seen how the various parameters of the resulting logic should be defined in order to achieve a sensible and well-behaved system, and how the two notions of plausibility in this construction should relate to each other.

There is also another, perhaps less technically involving way to deal with the dynamics of priorities. For the sake of simplicity, let us consider the case where a proposition $A$ can have two levels of likelihood: either $A$ is "very plausible", or it is "rather plausible". Take a bi-modal logic **BM** with operators $\Diamond_a, \Diamond_b$, where each of these operators behave like the $\Diamond$-operator of **K** and are independent of each other.[20] In that case, there are three possible cases to consider:

(i) $\Diamond_a A$ and $\Diamond_b A$ both mean that $A$ is very plausible
(ii) $\Diamond_a A$ means that $A$ is very plausible; $\Diamond_b A$ means that $A$ is rather plausible
(iii) $\Diamond_b A$ means that $A$ is very plausible; $\Diamond_a A$ means that $A$ is rather plausible

Accordingly, we can define three adaptive logics: (i) a flat AL which treats abnormalities of the form $\Diamond_a A \wedge \neg A$ ($A \in \mathcal{W}_c^l$) and those of the form $\Diamond_b A \wedge \neg A$ ($A \in \mathcal{W}_c^l$) in the same way; (ii) an AL which prioritizes abnormalities of the first kind over those of the second kind; and (iii) an AL which prioritizes abnormalities of the second kind over those of the first kind. Note that, in view of the results from this thesis, adaptive proofs in each of these three logics can have exactly the same format. So we can switch between marking in view of each of these logics to model changes in the priority degree of certain propositions.

**Other Approaches to Non-monotonic Logic** A different question is whether we may use the same results to shed new light on other approaches to non-monotonic logic. For instance, given the general well-behavedness of superpositions, can we use similar ideas in e.g. (prioritized) belief revision, default logic, or circumscription? As I will show in Chapter 10, the idea of superposing revision operations has quite a few interesting applications. Similarly, we may ask whether it is possible to merge non-monotonic consequence relations as this is done in hierarchic ALs, and what the resulting systems look like.

**Computational Complexity** A last topic for further research was already mentioned before in this section, i.e. the computational complexity of (prioritized and flat) adaptive logics. As explained there, more work is needed to determine exactly what the computational complexity of prioritized ALs in the various formats is. However, another question is whether certain restrictions on premise sets allow us to minimize the upper complexity bound of the systems. The hierarchy of conditions spelled out in Section 6.4.5 of this chapter seems to be a good starting point for this endeavor. For instance, it can easily be shown that in those cases where $\Gamma$ has only finitely many minimal Dab-consequences – and hence $\Sigma(\Gamma)$ is finite –, every flat adaptive logic has an upper bound complexity

[20] More precisely: $\nvdash_{\mathbf{BK}} \Diamond aA \supset \Diamond bA$ and $\nvdash_{\mathbf{BK}} \Diamond bA \supset \Diamond aA$.

of $\Sigma^0_2$.[21] This leads to the question whether weaker restrictions allow us to minimize the upper bound complexity of (flat or prioritized) ALs that use the Minimal Abnormality Strategy.

[21] In such cases, $A \in Cn_{\mathbf{AL}}(\Gamma)$ iff there is a finite proof from $\Gamma$, such that $A$ is derived on an unmarked line $l$ in this proof, and in every finite extension of the proof, line $l$ remains unmarked. This is so since whenever $\Sigma(\Gamma)$ is finite, we can derive all minimal Dab-consequences of $\Gamma$ in a finite proof.

# Part II

# Applications

# Chapter 7

# Prioritized Normative Reasoning

*This chapter is based on the paper "A Logic for Prioritized Normative Reasoning" (Journal of Applied Logic, to appear), which was co-authored by Christian Straßer. We are indebted to Joke Meheus, Mathieu Beirlaen and the anonymous referees for their comments and suggestions that helped improve that paper.*

## 7.1 Introduction

### 7.1.1 Deontic Conflicts and Adaptive Logics

Deontic conflicts have been the subject of much debate in philosophical logic and computer science.[1] Roughly speaking, a deontic conflict occurs if two or more obligations[2] cannot be mutually realized – we will present a more precise characterization of deontic conflicts in Section 7.1.3. In the face of such conflicts, Standard Deontic Logic (henceforth **SDL**) leads to triviality in view of the rule (D): from $OA$, infer $\neg O \neg A$. Giving up (D) is necessary but not sufficient to allow for deontic conflicts: whenever the premises feature a deontic conflict, the other rules of **SDL** still cause *deontic explosion*, i.e. the conclusion that everything is obligatory.

To solve this problem, various conflict-tolerant deontic logics have been developed over the last few decades. As Lou Goble points out in his [64, pp. 462-465], there are basically three options to avoid deontic explosion in the face of conflicting obligations: (i) reject the modal inheritance rule: from $OA$ and $A \vdash B$ infer $OB$ (see e.g. [63, 64]); (ii) reject the axiom of aggregation (AND): $OA \wedge OB \supset O(A \wedge B)$ (see e.g. [155, 60, 130]); (iii) weaken one or more of the non-modal connectives. Examples of the third option can be found in [44, 124], where the classical negation is replaced by a paraconsistent one, and in [98, 59], where a relevant implication is used.

[1] See e.g. [155, 60, 61, 62, 130]. See [100] for an overview of the literature on deontic conflicts.

[2] In this chapter, we use the generic term "obligations" to refer to duties, imperatives, rules, norms, (moral) laws, and so on.

Implementing any of these options in terms of a monotonic logic falls prey to the objection that a number of intuitive inferences are no longer valid. There is a variety of non-monotonic formalizations that are conflict-tolerant and give rise to stronger consequence relations (e.g. Input/Output logics with constraints [96] and Horty's [80]). However, these approaches typically lack a proof theory that explicates the (dynamics of) reasoning on the basis of deontic conflicts.

Recently, adaptive deontic logics have been developed that are satisfactory in all the discussed respects: while allowing for genuine deontic conflicts, they offer a strong consequence relation and a dynamic proof theory (see [35, 109, 110, 138, 139]). Every such adaptive logic is based on a monotonic conflict-tolerant deontic logic that is designed in terms of one of the three options (i)–(iii). The general idea of the adaptive logics is that the omitted rules of standard deontic logic are recuperated as much as possible – unless this leads to some form of explosion. In this way we obtain a significantly larger consequence set compared to the logic that defines the monotonic core of the adaptive logic. Moreover, given that the adaptive logics are defined in the standard format, an intuitive dynamic proof theory that is sound and complete with respect to a static semantics, and a great number of metatheoretic results are immediately available – see Chapter 2.

Take for example the (flat) adaptive logics $\mathbf{P2.2^r}$ and $\mathbf{P2.2^m}$ from [110]. These have as their lower limit logic the logic $\mathbf{SDL}_a\mathbf{P}_e$ from [60]. The semantics of $\mathbf{SDL}_a\mathbf{P}_e$ offers a way to interpret conflicts between prima facie obligations. The latter are expressed by the modal operator $O_e$. (AND) does not hold for these: $\nvdash_{\mathbf{SDL}_a\mathbf{P}_e} (O_eA \wedge O_eB) \supset O_e(A \wedge B)$. Goble's system also allows for the expression of actual, all-things-considered obligations: obligations that behave classically and are considered as guiding our actions. These are expressed with the aid of the operator $O_a$. Aggregation holds for such obligations: $\vdash_{\mathbf{SDL}_a\mathbf{P}_e} (O_aA \wedge O_aB) \supset O_a(A \wedge B)$, and also $\vdash_{\mathbf{SDL}_a\mathbf{P}_e} (O_aA \wedge O_eB) \supset O_e(A \wedge B)$.

The logics $\mathbf{P2.2^r}$ and $\mathbf{P2.2^m}$ from [110] make it possible to turn prima facie obligations into all-things-considered obligations, on the condition that they are not contradicted by other prima facie obligations. This is realized by allowing for conditional applications of the rule $(O_eO_a)$: $O_eA \supset O_aA$. As a consequence, we regain the rule of aggregation for all those obligations that are not involved in a conflict. The rules of **SDL** are thus recuperated by making the detour via all-things-considered obligations.

### 7.1.2 Ordered Sets of Obligations

In this chapter, the idea implemented in $\mathbf{P2.2^r}$ and $\mathbf{P2.2^m}$ will be applied to prioritized obligations. Our logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ allows one to derive from a prima facie obligation to bring about $A$ the all-things-considered obligation to bring about $A$, on the assumption that there is no conflicting obligation with at least the same priority. Note that in many cases, our norms come in different degrees of importance, specificity or urgency. For instance, in most countries, there is a fixed hierarchy between different kinds of traffic rules: those enforced by the signaling boards, by marks on the road, by traffic lights, or by a police officer's commands. In some specific situation, e.g. when we happen to be at the site of a car accident, the traffic rules may be overruled by more urgent and compelling obligations, such as taking an injured kid away from the site of the accident. When a conflict arises in these cases, the agents typically reason from their prima facie obligations and their respective degrees of priority, to find out what they ought to do.

The idea that prima facie obligations are to some extent ordered, and that this may help to resolve contradictions between them, was initiated by Ross in his [128]. Formal investigations of this idea started with [2], and are still ongoing, see e.g. [39, 68, 70]. Most authors in the field start from an ordered set of obligations, and provide a criterion to fix a set of all-things-considered obligations. However, they do not provide a proof theory that explicates the reasoning that could lead to such a selection. As we will argue, adaptive logics are a good candidate to fill this lacuna.

Our logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ allows us to solve the problem of prioritized conflicts within the format of lexicographic adaptive logics that was studied in Chapter 5. As a result, many important meta-theorems, as well as a full-blown proof theory and semantics are immediately available.

To completely settle the working ground, we need to make two restrictions. First of all, in line with the restrictions from Chapter 1, we will focus on sets of obligations that are ordered in a modular way. This means that the set of prima facie obligations can be represented by a tuple: $\mathbb{O} = \langle \mathcal{O}_1, \mathcal{O}_2, \ldots \rangle$, where $\mathcal{O}_1$ contains the most important obligations, $\mathcal{O}_2$ the obligations that are less important, $\mathcal{O}_3$ obligations that are still less important, and so on. We will sometimes write that an obligation $A$ has *priority level* $i$, by which we mean that $A \in \mathcal{O}_i$. Note, however, that the same obligation may occur in different sets $\mathcal{O}_i$. We will use $\mathcal{O}$ to refer to the set $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \ldots$, hence the set of all prima facie obligations (irrespective of their priority level). Second, we will restrict ourselves to the framework of monadic deontic logic in this chapter.[3] We will return to these restrictions in the concluding section.

### 7.1.3 Some Examples

Before turning to the formal system, let us present some concrete examples. These will help us to clarify the logic we present below, and to compare it to other approaches in the literature. Case 1 is inspired by [68, pp. 6-7], Case 2 by the visiting daughters example from [81, p. 581], and Case 3 is an extension of the Johnson example from [110, p. 8].

**Case 1.** Mary had a car accident, with some minor material damage as a result. She faces the obligation to stay at the site of the accident to fill in insurance papers ($S$). However, she also promised her mother to pick her up from the supermarket and take her home ($M$). Finally, her boss asked her to post an urgent letter this same morning ($P$). It is impossible to post the letter in time, and to fill in the entire insurance poll: $S$ excludes $P$. However, she can call her mother to notify her she will be a bit later, and hence do both $S$ and $M$ without too much trouble. The obligation to do $S$ has a higher priority than both the one to do $P$ and the one to do $M$, while the latter two are equally important.

[3] See [138, 139] for some adaptive logics based on dyadic deontic logics.

**Case 2.** Michael has promised his daughter to pay her a visit today ($D$). However, he receives the news that his uncle is very ill, and now faces the problem that he should also visit him ($U$). The obligation to do $U$ is more urgent than the one to do $D$. Moreover, $U$ and $D$ are practically incompatible. Finally, Michael also told his nephew he would drop by whenever he was around ($N$). The nephew lives close to the uncle, hence $U$ and $N$ can be combined easily, while $N$ and $D$ exclude each other. Visiting the nephew is less important than visiting the daughter.

**Case 3.** Tom's mother asks him to go to the shop and buy bread, cheese, and either pork or tofu ($B \wedge C \wedge (P \vee T)$). Since Tom (a fifteen year old boy) wants to be a vegan, he'd rather not buy either cheese or pork ($\neg C \wedge \neg P$). However, Tom also knows that his little sister is very fond of pork, and hence in order to please her, he should buy pork ($P$). Since it is after all his mother who does the cooking, her orders take priority over Tom's vegan principles. Satisfying his sister is however less important than those principles.

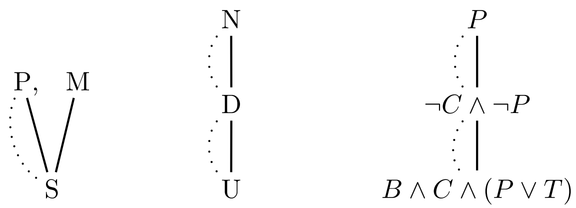


Figure 7.1: Illustrations for the priorities among the obligations in Cases 1–3. Dotted lines indicate incompatibilities while solid lines indicate priorities where obligations at the bottom of the graph have the highest priority.

Figure 7.1 represents the priority-relations and conflicts in the different cases. For now, we leave it to the reader to decide what the actual obligations of Mary, Michael and Tom are, given the above descriptions.

In the first two examples, we used a notion of impossibility, e.g. when we said in Case 1 that it is "impossible" for Mary to both stay at the site of the accident ($S$) and post the letter ($P$). Inspired by input-output logic [96], we will henceforth speak of "constraints" to refer to statements about practical or physical matters of fact that restrict the kind of actions we may perform.[4] We will assume the set of constraints $\mathcal{C}$ to be consistent.

We may now express in a more precise way what is meant by a deontic conflict. A deontic conflict is always relative to the set of prima facie obligations $\mathcal{O}$ and a set of constraints $\mathcal{C}$. That our prima facie obligations conflict with our constraints means that, in view of the latter, we cannot mutually realize each of the former.

[4]Constraints are hence strongly related to the concept of "derogation" as studied in e.g. [2]. To derogate $\mathcal{O}$ in $A$ means to interpret $\mathcal{O}$ in such a way that $A$ is no longer entailed by it. Although we do not discuss them here, one might also have deontic constraints on the set of actual obligations. For example, where $A$ is a basic human right, a judge may want to select a subset of $\mathcal{O}$ in such that the negation of $A$ does not follow from the remaining set of obligations.

More formally, it means that $\mathcal{O} \cup \mathcal{C}$ is inconsistent.[5] As we indicated before, our actual obligations should properly guide our actions, whence we want them to be conflict-free according to this general concept: we want it to be possible that each of them can be carried out, without violating the set of constraints at hand.

The remainder of this chapter is structured as follows. In Section 7.2, we will discuss how the various examples are translated into premise sets for the adaptive logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$. The latter is presented in Section 7.3. Section 7.4 presents some meta-theoretic results that are specific to the context of prioritized obligations. We will return to the examples in Section 7.5 to illustrate the proof theory of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ and to compare some existing criteria for preferred obligations to this new system. We mention some prospects for further research and loose ends in the concluding section.

## 7.2 The Logic $\mathbf{MP_s}$

In this section, we define the lower limit logic of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$, which is called $\mathbf{MP_s}$. The latter can be seen as **SDL** extended with a multi-modal variant of the logic **P** from [60].

### 7.2.1 The Language of $\mathbf{MP_s}$

The language of $\mathbf{MP_s}$ contains an infinite number of conflict-tolerant ought-operators: $O_1, O_2, O_3, \ldots$. The formula $O_iA$ should be read as: there is a prima facie obligation of priority level $i$ that tells us to do or bring about $A$. It is important to note that the priority of the normative standard gets higher as the priority index gets *lower*. $O_1$-obligations are thus the strongest, most compelling prima facie obligations, $O_2$-obligations are weaker, and so on. Each $O_i$-operator behaves exactly like the $O_e$-operator from $\mathbf{SDL}_a\mathbf{P}_e$ – see Section 7.1.1. Actual obligations will be denoted by the $O$-operator without index. This $O$-operator behaves just as the ordinary $O$-operator from **SDL**.

As before, let $\mathcal{L}_c$ refer to the standard language of classical propositional logic, and let $\mathcal{W}_c^l$ denote the set of literals (sentential letters and their negations) in $\mathcal{W}_c$. The language $\mathcal{L}_o$ is $\mathcal{L}_c$ extended with the modal operators $O$ and $\langle O_i \rangle_{i\in\mathbb{N}}$. Although it is possible to define the respective duals $P$ and $\langle P_i \rangle_{i\in\mathbb{N}}$, we will not do so here, since we are only interested in reasoning about obligations.

The set of well-formed formulas of $\mathcal{L}_o$ is defined as the smallest set $\mathcal{W}_o$ that satisfies the following conditions:[6]

(i) $\mathcal{W}_c \subset \mathcal{W}_o$
(ii) if $A \in \mathcal{W}_c$ then $OA \in \mathcal{W}_o$
(iii) if $A \in \mathcal{W}_c$ then $O_iA \in \mathcal{W}_o$ for every $i \in \mathbb{N}$
(iv) if $A \in \mathcal{W}_o$ then $\neg A \in \mathcal{W}_o$
(v) if $A, B \in \mathcal{W}_o$ then $A \vee B, A \wedge B, A \supset B, A \equiv B \in \mathcal{W}_o$

We will now explain how an ordered set of obligations, together with a set of constraints, is translated into a premise set $\Gamma \subseteq \mathcal{W}_o$. Where $\mathbb{O} = \langle \mathcal{O}_1, \mathcal{O}_2, \ldots \rangle$ is

[5] Since constraints may also be of a purely logical kind, our concept includes more basic deontic conflicts, such as having the obligation to do $A$ and the obligation to do $\neg A$.

[6] As is clear from the definition of $\mathcal{W}_o$, we do not allow for nested obligations.

a sequence of sets of propositions that represent obligatory states or actions, we first define:

$$\Gamma_{\mathbb{O}} =_{\mathsf{df}} \{O_i A \mid A \in \mathcal{O}_i, i \in \mathbb{N}\}$$

In our simplified formal framework, physical and practical constraints can be translated as follows: where $A$ is a constraint, $\neg O \neg A$ is introduced as a premise. Thus, the impossibility to do both $S$ and $P$ will be expressed by $\neg O \neg(\neg P \vee \neg S)$. This is justified by the observation that we want our actual obligations to be obeyable in a practical sense, as required by the "ought implies can"-principle (OIC): $OA \supset \Diamond A$ (where $\Diamond A$ expresses that $A$ is physically or practically possible). Note that (OIC) implies $\neg \Diamond A \supset \neg OA$ by contraposition. Of course, a more natural approach would make use of additional nomological modalities, but this would severely complicate the language of the system. Logical constraints will be dealt with solely by the logic $\mathbf{MP_s}$ itself, as we will explain in Section 7.2.2.

Recall that the set $\mathcal{C}$ represents the set of (physical and practical) constraints relative to the context in which we reason about our normative system(s). In order to apply this idea in a sensible way, it should be assumed that $\mathcal{C}$ is consistent and closed under conjunction. For instance, where $A, B \in \mathcal{C}$, we need to assume that $A \wedge B$ is also a constraint – compare this to the fact that when $A$ and $B$ are both physically necessary, then also $A \wedge B$ is physically necessary.

Although this seems in principle a reasonable assumption, it would be preferable if we could simply start with a set of constraints, and let the system, i.e. the translation in co-operation with the logic, close these under conjunction. This is what we will do in the following.

In general, we define

$$\Gamma_{\mathcal{C}} =_{\mathsf{df}} \{\neg O \neg(\bigwedge \Delta) \mid \Delta \text{ is a finite and non-empty subset of } \mathcal{C}\}$$

However, where $\mathcal{C}$ is finite, we can simply let $\Gamma_{\mathcal{C}} =_{\mathsf{df}} \{\neg O \neg(\bigwedge \mathcal{C})\}$ – it can easily be verified that this simpler formulation is equivalent to the above translation.[7] Finally, let

$$\Gamma_{\mathbb{O},\mathcal{C}} =_{\mathsf{df}} \Gamma_{\mathbb{O}} \cup \Gamma_{\mathcal{C}}$$

Let us illustrate this translation function by the examples from Section 7.1.3. These are translated as follows:

Case 1 : $\Gamma_1 = \{O_1 S, O_2 P, O_2 M, \neg O \neg(\neg S \vee \neg P)\}$

Case 2 : $\Gamma_2 = \{O_1 U, O_2 D, O_3 N, \neg O \neg((\neg U \vee \neg D) \wedge (\neg N \vee \neg D))\}$

Case 3 : $\Gamma_3 = \{O_1(B \wedge C \wedge (P \vee T)), O_2(\neg C \wedge \neg P), O_3 P\}$

[7] Recall that the $O$-operator of $\mathbf{MP_s}$ behaves exactly like the $O$-operator of $\mathbf{SDL}$. That the simpler formulation is entailed by the official formulation, is immediate. For the other direction, it suffices to see that $\vdash_{\mathbf{SDL}} \neg O \neg(A \wedge B) \supset \neg O \neg A$.

### 7.2.2 The Logic $\mathbf{MP_s}$

In order to allow for conflicting obligations, we will generalize Goble's multi-relational semantics for the system **P** – see e.g. [60]. The latter is itself a generalization of the semantics of **SDL**: not one, but many accessibility relations are in play. Goble defines a model in terms of a set of serial accessibility relations: $\mathcal{R} = \{R_1, R_2, \ldots\}$. To handle prioritized sets of obligations, two basic changes to Goble's **P**-system will be made.

First of all, we use a *set* of sets of serial accessibility relations $\boldsymbol{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots\}$. Where $i \in \mathbb{N}$, $\mathcal{R}_i$ refers to a set of prima facie obligations of priority level $i$, that all have the same priority level $i$. This way, two prima facie obligations with the same priority may still be conflicting. As may be expected, the $O_i$-operator is linked to the set $\mathcal{R}_i$.

The other change has to do with the additional $O$-operator. Since we want this operator to behave classically, it is stipulated that there is a single accessibility relation $R$ that corresponds to the accessibility relation of **SDL**. This implies that the logic itself takes care of the logical constraints on the set of actual obligations: for every self-contradictory formula $A$, $\neg OA$ is an $\mathbf{MP_s}$-theorem.

An $\mathbf{MP_s}$-model $M$ is a quintuple $\langle W, \boldsymbol{R}, R, v, w_0\rangle$, where $W$ is a set of possible worlds, $\boldsymbol{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots\}$ is a set of non-empty sets of serial accessibility relations on $W$, $R$ is a serial accessibility relation, $v : \mathcal{S} \times W \to \{0, 1\}$ is an assignment function and $w_0 \in W$ is the actual world.

The valuation $v_M$ defined by the model $M$ is characterized by:

C1 where $A \in \mathcal{S}$, $v_M(A, w) = v(A, w)$
C2 $v_M(\neg A, w) = 1$ iff $v_M(A, w) = 0$
C3 $v_M(A \vee B, w) = 1$ iff $v_M(A, w) = 1$ or $v_M(B, w) = 1$
C4 $v_M(A \wedge B, w) = 1$ iff $v_M(A, w) = 1$ and $v_M(B, w) = 1$
C5 $v_M(A \supset B, w) = 1$ iff $v_M(A, w) = 0$ or $v_M(B, w) = 1$
C6 $v_M(A \equiv B, w) = 1$ iff $v_M(A, w) = v_M(B, w)$
C7 $v_M(O_iA, w) = 1$ iff, for an $R_j \in \mathcal{R}_i$, $v_M(A, w') = 1$ for all $w'$ such that $R_jww'$
C8 $v_M(OA, w) = 1$ iff $v_M(A, w') = 1$ for all $w'$ such that $Rww'$

An $\mathbf{MP_s}$-model $M$ verifies $A$, $M \Vdash A$ iff $v_M(A, w_0) = 1$. $\models_{\mathbf{MP_s}} A$ iff all $\mathbf{MP_s}$-models verify $A$. We say that $M$ is an $\mathbf{MP_s}$-model of $\Gamma$, $M \Vdash \Gamma$ iff $M \Vdash A$ for every $A \in \Gamma$. We use $\mathcal{M}_{\mathbf{MP_s}}(\Gamma)$ to refer to the set of $\mathbf{MP_s}$-models of $\Gamma$. Finally, $\Gamma \models_{\mathbf{MP_s}} A$ iff all $\mathbf{MP_s}$-models of $\Gamma$ verify $A$.

A syntax for $\mathbf{MP_s}$ is obtained as follows. We extend an axiomatization of classical propositional logic (henceforth **CL**) with the following axioms (where $A, B \in \mathcal{W}_c$):

K $O(A \supset B) \supset (OA \supset OB)$
D $OA \supset \neg O \neg A$

and close it under modus ponens (MP) and the following rules (where $A, B \in \mathcal{W}_c$):

RN if $\vdash A$, then $\vdash OA$
$\text{RM}_i$ where $i \in \mathbb{N}$: if $\vdash A \supset B$, then $\vdash O_iA \supset O_iB$
$\text{P}_i$ where $i \in \mathbb{N}$: if $\vdash A$, then $\vdash \neg O_i \neg A$
$\text{RN}_i$ where $i \in \mathbb{N}$: if $\vdash A$, then $\vdash O_iA$

where $\vdash$ indicates membership in the set of $\mathbf{MP_s}$-axioms. The two axioms (K) and (D) together with the rules (MP) and (RN) deliver **SDL** for $O$. The rules (RM$_i$), (P$_i$) and (RN$_i$) deliver Goble's **P** for all operators $O_i$. Note that there are no bridging rules that link the different ought-operators: $OA$ does not imply that $O_iA$ or vice versa; $O_iA$ does not imply that $O_jA$ for any $j \neq i$. Note also that $O_iA \nvdash_{\mathbf{MP_s}} \neg O_i\neg A$ and $O_iA, O_j\neg A \nvdash O_kB$ for any $i, j, k \in \mathbb{N}$. In Section 7.4, we will discuss even more general results with regards to the conflict-tolerance of $\mathbf{MP_s}$.

We define $\Gamma \vdash_{\mathbf{MP_s}} A$ iff there are $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_{\mathbf{MP_s}} (B_1 \wedge \ldots \wedge B_n) \supset A$. Note that this definition entails that the consequence relation is compact. The proof of the following theorem is outlined in Section 7.2.3 below:

**Theorem 7.1** $\Gamma \vdash_{\mathbf{MP_s}} A$ *iff* $\Gamma \models_{\mathbf{MP_s}} A$ *(Soundness and Completeness for* $\mathbf{MP_s}$*)*

### 7.2.3 Soundness and Completeness of $\mathbf{MP_s}$

Soundness is proven by the usual inductive procedure and is safely left to the reader. To prepare for the completeness proof, a canonical model $M_c^{\Delta} = \langle W_c, \mathbf{R}, R, v_c, \Delta\rangle$ is defined as follows:[8]

(i) $W_c$ is the set of maximal consistent extensions of $\mathbf{MP_s}$;
(ii) $\mathbf{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots\}$, where $\mathcal{R}_i = \{R_i^A \mid A \in \mathcal{W}_c\}$ and $R_i^A = \{(w, w') \in W_c \times W_c \mid O_iA \notin w \text{ or } A \in w'\}$;
(iii) $R = \{(w, w') \in W_c \times W_c \mid \text{ either } OA \notin w \text{ or } A \in w'\}$ for an $A \in \mathcal{W}_c$;
(iv) for every $w \in W_c, p \in \mathcal{S}$: $v_c(p, w) = 1$ iff $p \in w$;
(v) and $\Delta$ is an arbitrary element of $W_c$.

Note that by the definition $\mathcal{R}_i$ is non-empty for each $i$. Moreover, each $R_i^A \in \mathcal{R}_i$ is serial. Suppose that $O_iA \notin w$. Then $R_i^A ww'$ for all $w' \in W_c$. Suppose that $O_iA \in w$. Assume there is no $w' \in W_c$ such that $A \in w'$. Then, $\vdash_{\mathbf{MP_s}} \neg A$. By (P$_i$), $\vdash_{\mathbf{MP_s}} \neg O_iA$. Hence, $O_iA \notin w$, — a contradiction. Hence there is a $w' \in W_c$ for which $A \in w'$. Thus, $R_i^A ww'$. By the same argument $R$ is serial. Altogether this shows that $M_c^{\Delta}$ is an $\mathbf{MP_s}$-model.

**Lemma 7.1** *For all* $A \in \mathcal{W}^M$ *and all* $w \in W_c$, $v_{M_c^{\Delta}}(A, w) = 1$ *iff* $A \in w$.

*Proof.* This is shown by an induction on the complexity of $A$. The argument is straightforward. We only show the induction step for $A = O_iB$, the rest is left to the reader. Suppose $O_iB \in w$. Note that $R_i^B \in \mathcal{R}_i$ is defined in such a way that if $R_i^B ww'$ then $B \in w'$ and whence by the induction hypothesis $v_{M_c^{\Delta}}(B, w') = 1$. Hence, $v_{M_c^{\Delta}}(O_iB, w) = 1$.

Suppose now that $v_{M_c^{\Delta}}(O_iB, w) = 1$. There is a $R_i^C \in \mathcal{R}_i$ such that if $R_i^C ww'$ then $v_{M_c^{\Delta}}(B, w') = 1$. Assume that $\{C, \neg B\}$ is consistent. Then there is a $w'' \in W_c$ for which $C, \neg B \in w''$. Hence $B \notin w''$. By the induction hypothesis $v_{M_c^{\Delta}}(B, w'') = 0$. Thus, $(w, w'') \notin R_i^C$. Whence, $O_iC \in w$ and $C \notin w''$,—a contradiction. Hence, $\vdash_{\mathbf{MP_s}} C \supset B$ and by (RM$_i$), $\vdash_{\mathbf{MP_s}} O_iC \supset O_iB$.

[8]This is very much inspired by the canonical model defined by Lou Goble in [60, p. 126].

Suppose $O_iC \in w$, then $O_iB \in w$ by (MP). Suppose (†) $O_iC \notin w$. Assume $\{\neg B\}$ is consistent. Hence, there is a maximal consistent extension $w'''$ of $\{\neg B\}$. Since by (†) $R_i^C ww'''$, also $v_{M_c^\Delta}(B, w''') = 1$. Thus, by the induction hypothesis, $B \in w'''$,—a contradiction. Hence, $\vdash_{\mathbf{MP_s}} B$. By ($\mathrm{RN}_i$), $\vdash_{\mathbf{MP_s}} O_iB$. Hence $O_iB \in w$. ■

**Lemma 7.2** *Every* $\mathbf{MP_s}$*-consistent set of sentences* $\Gamma$ *is satisfiable in* $\mathbf{MP_s}$.

*Proof.* By Lindenbaum's Lemma there is a maximal consistent extension $\Gamma'$ of $\Gamma$. By Lemma 7.1 and since $\Gamma \subseteq \Gamma'$, $M_c^{\Gamma'} \Vdash \Gamma$. ■

**Theorem 7.2** *If* $\Gamma \models_{\mathbf{MP_s}} A$ *then* $\Gamma \vdash_{\mathbf{MP_s}} A$.

*Proof.* Suppose $\Gamma \models_{\mathbf{MP_s}} A$. Hence, $\Gamma \cup \{\neg A\}$ is not satisfiable in $\mathbf{MP_s}$. Hence by Lemma 7.2, $\Gamma \cup \{\neg A\} \vdash_{\mathbf{MP_s}} A$. As a result, $\Gamma \vdash_{\mathbf{MP_s}} A$. ■

## 7.3 The Prioritized Adaptive Logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$

### 7.3.1 The Adaptive Approach

Recall that the central aim of this chapter is to capture how we can *reason* from an ordered set $\mathcal{O}$ of prima facie obligations and a set $\mathcal{C}$ of constraints towards a set of actual obligations. This can be realized by the dynamic proof theory of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$, which is discussed in Section 7.3.2. As $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ is defined in the format of a lexicographic AL, and uses the strategy $\sqsubset$-Minimal Abnormality, it suffices to specify the sequence of sets of abnormalities, which we will do here.

In our current formal framework, to derive actual obligations from prima facie obligations may be realized by the following rule (where $i \in \mathbb{N}$):

($\mathrm{O}_i\mathrm{O}$) if $O_iA$, then $OA$

Recall that this rule is not valid in $\mathbf{MP_s}$. Moreover, adding ($\mathrm{O}_i\mathrm{O}$) to the axioms of $\mathbf{MP_s}$ would result in plain triviality whenever our prima facie obligations are jointly incompatible.

The adaptive logic we will present uses the language and inference rules of $\mathbf{MP_s}$, but enhances it with the *defeasible* application of ($\mathrm{O}_i\mathrm{O}$). As soon as such a particular application turns out to be harmful, the logic ensures that it is retracted. However, other applications of the same rule may still be allowed for by the logic.

As explained in Chapter 1, the central motor behind adaptive logics is the assumption that abnormalities are false "until and unless proven otherwise". In the current case, abnormalities express that something is a prima facie obligation (of some priority level $i$), but not an actual obligation. Hence any formula of the form $O_iA \wedge \neg OA$ is an abnormality. Consider $\Gamma_1 = \{O_1S, O_2P, O_2M, \neg O(S \wedge P)\}$. The adaptive logic derives $OM$ on the assumption that $O_2M \wedge \neg OM$ is false. Note that $\Gamma_1 \vdash_{\mathbf{MP_s}} OM \vee (O_2M \wedge \neg OM)$. Hence, taking as our abnormalities all formulas of the form $O_iA \wedge \neg OA$, and prioritizing these abnormalities according to the index $i$, we seem to get a logic that allows us to model defeasible inferences on the basis of prioritized obligations.

However, simply taking $O_iA \wedge \neg OA$ for any $A \in \mathcal{W}_c$ as the form of the abnormalities leads to a flip-flop logic (see Section 2.4.2 in Chapter 2 for this concept). In the current context, this means that we would obtain a logic that considers every prima facie obligation as actual when there are no conflicts at all, but that reduces to $\mathbf{MP_s}$ as soon as a Dab-formula is derivable from the premise set. For instance, from $\{O_1p, O_1q, \neg O\neg p\}$, we can derive the following minimal disjunction of abnormalities:

$$(O_1q \wedge \neg q) \vee (O_1(p \vee \neg q) \wedge \neg O(p \vee \neg q))$$

Hence in this very simple example, the prima facie obligation to do $q$ would become suspicious, even though the real problem lies with the prima facie obligation to do $p$. This problem is overcome by using a slightly more complex form of the abnormalities, borrowed from [110]. The idea behind this form of the abnormalities is this: let $\Theta$ be a finite and non-empty set of literals and let $\bigvee \Theta$ be the disjunction of the members of $\Theta$ (if $\Theta = \{A\}$, $\bigvee \Theta = A$). The obligation to do $\bigvee \Theta$ is said to behave abnormally, if for some $i \in \mathbb{N}$, $O_i \bigvee \Theta \wedge \neg O \bigvee \Theta$ is true, *or* if for some non-empty $\Theta' \subset \Theta$, $O_i \bigvee \Theta' \wedge \neg O \bigvee \Theta'$ is true.

This warrants that if some part of the disjunction $\bigvee \Theta$ already behaves abnormally, then $\bigvee \Theta$ automatically becomes suspicious as well. In the prototypical example, we see that the prima facie obligation to do $p \vee \neg q$ can no longer "infect" the prima facie obligation to do $q$, since $p$ behaves abnormally in itself.

### 7.3.2 Some Definitions

In this section, we will present the formal apparatus that characterizes the logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$. We start with some general preliminary tools and briefly explain the $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-semantics. After that, we define the dynamic proof theory.

**The General Framework.** The logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ is a lexicographic adaptive logic, characterized by the triple $\langle \mathbf{MP}, \langle \Omega_i^{\mathbf{MP}} \rangle_{i \in \mathbb{N}}, \mathbf{m} \rangle$. In the remainder, we will briefly specify the two first elements of this triple.[9]

$\mathbf{MP}$ is obtained from $\mathbf{MP_s}$ by adding the checked connectives to $\mathcal{L}_o$, and by adding the usual classical axioms for these connectives to the set of $\mathbf{MP_s}$-axioms, closing the whole under the rule MP+.[10] In order to define the sequence of sets of abnormalities, we need some technical preparations. Where $\Theta$ is a non-empty and finite subset of $\mathcal{W}_c^l$, we will use the following abbreviation:

$$\sigma^i(\Theta) =_{\mathsf{df}} \bigvee \{O_i \bigvee \Theta' \wedge \neg O \bigvee \Theta' \mid \emptyset \neq \Theta' \subseteq \Theta\} \tag{7.1}$$

To avoid notational clutter, we will skip the set brackets for concrete sets of literals, e.g. we write $\sigma(p, \neg q)$ instead of $\sigma(\{p, \neg q\})$. To get better grip on the form of the abnormalities and their abbreviation, consider the following examples:

[9] In view of Chapter 5, $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ also has a Reliability-variant. We do not discuss this variant here for reasons of space, and since we believe that the Minimal Abnormality Strategy gives comparatively more intuitive results. For instance, if we have the obligation to save two twins but the constraint that we cannot save them both, the Reliability-variant does not allow us to derive the actual obligation to save at least one of them.

[10] Note that since $\mathbf{MP_s}$ is supraclassical, $\mathbf{MP}$ is obtained from $\mathbf{MP_s}$ in exactly the same way as $\mathbf{K}$ was obtained from $\mathbf{K_s}$ – see Chapter 2, Section 2.4.

$$\sigma^2(p) = \; O_2 p \wedge \neg O p$$

$$\sigma^2(\neg q) = \; O_2 \neg q \wedge \neg O \neg q$$

$$\sigma^3(\neg p, q) = \; (O_3 \neg p \wedge \neg O \neg p) \vee (O_3 q \wedge \neg O q) \vee (O_3(\neg p \vee q) \wedge \neg O(\neg p \vee q))$$

$$\sigma^1(p, q, r) = \; (O_1 p \wedge \neg O p) \vee (O_1 q \wedge \neg O q) \vee (O_1 r \wedge \neg O r) \vee (O_1(p \vee q) \wedge \neg O(p \vee q)) \vee (O_1(p \vee r) \wedge \neg O(p \vee r)) \vee (O_1(q \vee r) \wedge \neg O(q \vee r)) \vee (O_1(p \vee q \vee r) \wedge \neg O(p \vee q \vee r))$$

Note that the number of disjuncts of an abnormality $\sigma^i(\Theta)$ grows exponentially with the number of literals in $\Theta$. Also, where $\Theta'$ is a non-empty subset of $\Theta$, we have that $\vdash_{\mathbf{MP}} \sigma^i(\Theta') \supset \sigma^i(\Theta)$.

Where $i \in \mathbb{N}$, the $i$th set of abnormalities is defined as

$$\Omega_i^{\mathbf{MP}} =_{\mathsf{df}} \{\sigma^i(\Theta) \mid \emptyset \neq \Theta \subset \mathcal{W}_c^l\}$$

Note that $\Omega_i^{\mathbf{MP}} \cap \Omega_j^{\mathbf{MP}} = \emptyset$ for every $i, j \in \mathbb{N}, i \neq j$. We thus obtain the sequence of abnormalities $\langle \Omega_i^{\mathbf{MP}} \rangle_{i \in \mathbb{N}}$. Let henceforth $\Omega^{\mathbf{MP}} = \bigcup_{i \in \mathbb{N}} \Omega_i^{\mathbf{MP}}$, and let $Dab(\Delta)$ denote a disjunction of members of a finite $\Delta \subset \Omega^{\mathbf{MP}}$. In the remainder of this chapter, we skip the superscript **MP** in the names of the sets of abnormalities, to avoid notational clutter.

**Semantics.** Let us illustrate the $\mathbf{AL}_{\sqsubset}$-semantics once more, for the case of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$. The set of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-models of $\Gamma$ is a subset of the set of **MP**-models of $\Gamma$. As explained in Chapter 5, models are selected resp. deselected by the adaptive logic in view of their abnormal part and the lexicographic order $\sqsubset$. Consider the following example:

$\Delta = \{\sigma^1(p), \sigma^2(q, \neg r), \sigma^3(r), \sigma^4(\neg p, q)\}$
$\Delta' = \{\sigma^1(p), \sigma^2(q, \neg r), \sigma^2(s), \sigma^4(\neg p, q)\}$
$\Delta'' = \{\sigma^1(p), \sigma^2(q, \neg r), \sigma^2(s), \sigma^3(r)\}$
$\Delta''' = \{\sigma^1(\neg q), \sigma^2(q, \neg r), \sigma^2(s), \sigma^3(r)\}$

According to Definition 5.1, $\Delta \sqsubset \Delta' \sqsubset \Delta''$. That is, $\Delta$ beats $\Delta'$ at level 2, and $\Delta'$ beats $\Delta''$ at level 3. It follows immediately that $\Delta \sqsubset \Delta''$. However, $\Delta \not\sqsubset \Delta'''$, since the two are incomparable at level 1. All this becomes a lot more clear as soon as we represent these sets of abnormalities $\Theta$ in columns, where each separate column represents the intersection of $\Theta$ with an $\Omega_i$ – see Table 7.1.

| $\Omega$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | ... |
|---|---|---|---|---|---|
| $\langle \Delta \cap \Omega_i \rangle_{i \in \mathbb{N}} =$ | $\{\sigma^1(p)\}$ | $\{\sigma^2(q, \neg r)\}$ | $\{\sigma^3(r)\}$ | $\{\sigma^4(\neg p, q)\}$ | ... |
| $\langle \Delta' \cap \Omega_i \rangle_{i \in \mathbb{N}} =$ | $\{\sigma^1(p)\}$ | $\{\sigma^2(q, \neg r), \sigma^2(s)\}$ | $\emptyset$ | $\{\sigma^4(\neg p, q)\}$ | ... |
| $\langle \Delta'' \cap \Omega_i \rangle_{i \in \mathbb{N}} =$ | $\{\sigma^1(p)\}$ | $\{\sigma^2(q, \neg r), \sigma^2(s)\}$ | $\{\sigma^3(r)\}$ | $\emptyset$ | ... |
| $\langle \Delta''' \cap \Omega_i \rangle_{i \in \mathbb{N}} =$ | $\{\sigma^1(\neg q)\}$ | $\{\sigma^2(q, \neg r), \sigma^2(s)\}$ | $\{\sigma^3(r)\}$ | $\emptyset$ | ... |

Table 7.1: A representation of the sets $\Delta, \Delta', \Delta''$ and $\Delta'''$ as tuples of sets of abnormalities

In line with definition 5.2, the $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-models of $\Gamma$ are those **MP**-models of $\Gamma$ whose abnormal part is $\sqsubset$-minimal:

**Definition 7.1** *$M \in \mathcal{M}_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{MP}}(\Gamma)$ and there is no $M' \in \mathcal{M}_{\mathbf{MP}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$.*

**Definition 7.2** *$\Gamma \models_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} A$ iff $A$ is true in all $M \in \mathcal{M}_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$.*

**Proof Theory.** The $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-proof theory is defined according to the generic definitions from Chapter 5 – see page 115. In the current section, we will hence focus on the peculiarities of the proof theory of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$.

Recall that we wanted to allow for the defeasible application of the rule ($\mathrm{O}_i\mathrm{O}$) from Section 7.3.1. The third item of the following lemma shows how this rule can be applied in an $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-proof, using the conditional rule RC:

**Lemma 7.3** *Each of the following holds:*
1. *Where $\Theta$ is a finite set of literals, $O_i \bigvee \Theta \vdash_{\mathbf{MP}} O \bigvee \Theta \,\check{\vee}\, \sigma^i(\Theta)$.*
2. *Where $A \in \mathcal{W}_c$ and $\bigwedge_J \bigvee \Theta_j$ is a conjunctive normal form of $A$, $O_i A \vdash_{\mathbf{MP}} OA \,\check{\vee} \bigvee_J \sigma^i(\Theta_j)$.*
3. *Where $A \in \mathcal{W}_c$ and $O_i A$ is derived on a line $l$ of an $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma$, we can derive $OA$ on a line $l'$ in an extension of this proof.*

*Proof.* *Ad 1.* Suppose $O_i \bigvee \Theta$. Then by excluded middle, $O \bigvee \Theta \,\check{\vee} (O_i \bigvee \Theta \wedge \neg O \bigvee \Theta)$, and hence $O \bigvee \Theta \,\check{\vee}\, \sigma^i(\Theta)$.

*Ad 2.* Suppose $O_i A$. By ($\mathrm{RM}_i$), for each $j \in J$, $O_i \bigvee \Theta_j$. By item (i), $O \bigvee \Theta_j \,\check{\vee}\, \sigma^i(\Theta_j)$ for each $j \in J$. Hence, $\bigwedge_J (O \bigvee \Theta_j) \,\check{\vee} \bigvee_J \sigma^i(\Theta_j)$. Since aggregation holds for $O$, $O \bigwedge_J \bigvee \Theta_j \,\check{\vee} \bigvee_J \sigma^i(\Theta_j)$. Since inheritance holds for $O$, $OA \,\check{\vee} \bigvee_J \sigma^i(\Theta_j)$.

*Ad 3.* Suppose $O_i A$ is derived on a line $l$ of an $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma$. Let the condition of line $l$ be $\Delta$. By the conditional rule (RC) and item (ii), we can derive $OA$ on a line $l'$ in an extension of the proof, on the condition $\Delta \cup \{\bigvee_J \sigma^i(\Theta_j)\}$, where $\bigwedge_J \bigvee \Theta_j$ is a conjunctive normal form of $A$. ■

The $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-proof theory reflects a dynamic aspect of our reasoning about conflicting obligations: we may take some obligation as an actual obligation at some point, but only later on learn that this leads to a conflict on the level of the actual obligations. This may be due to the additional information, but it may also be the result of our reasoning about the same set of obligations. At that point, we have to retract some of our earlier conclusions. In Section 7.5, we will clarify this mechanism, using the canonical examples from Section 7.1.

### 7.3.3 Some Metatheoretic Properties

For the ease of reference, we restate some of the generic results for lexicographic ALs for to the logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ – we refer to Chapter 5 for more details.

**Theorem 7.3** *Each of the following holds:*

1. *Where $\Gamma \subseteq \mathcal{W}_o$: $\Gamma \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} A$ iff $\Gamma \models_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} A$ (Restricted Soundness and Completeness)*
2. *If $\Gamma \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} A$, then $\Gamma \models_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} A$ (Soundness)*

**Theorem 7.4** *If $M \in \mathcal{M}_{\mathbf{MP}}(\Gamma) - \mathcal{M}_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, then there is an $M' \in \mathcal{M}_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$.*

**Corollary 7.1** *If $\Gamma$ has* **MP***-models, then $\Gamma$ has* $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$*-models. (Reassurance)*

**Theorem 7.5** *Where $\Gamma \subseteq \mathcal{W}_o$: $Cn_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma) = Cn_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(Cn_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma))$. (Fixed Point)*

**Theorem 7.6** $\Gamma \subseteq Cn_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. *(Reflexivity)*

**Theorem 7.7** $Cn_{\mathbf{MP}}(\Gamma) \subseteq Cn_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$. *(*$\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ *strengthens* **MP***)*

Note that no premise set of the form $\Gamma_{\mathbb{O},\mathcal{C}}$ contains checked connectives, whence the $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-consequence relation is sound and complete and idempotent for all such premise sets.

## 7.4 Some Specific Properties of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$

### 7.4.1 The Properties

In this section, we present some properties that are more specific to the logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ and its application to premise sets of the form $\Gamma_{\mathbb{O},\mathcal{C}}$. The first theorem and corollary below indicate to what extent **MP** and $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ are conflict-tolerant. The other theorem and the subsequent corollaries express a lower bound on the set of actual obligations that are $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-derivable from any set $\Gamma_{\mathbb{O},\mathcal{C}}$. We refer to Section 7.4.2 for the proofs of Theorems 7.8 and 7.9. Recall that where $\mathbb{O} = \langle \mathcal{O}_1, \mathcal{O}_2, \ldots \rangle$, $\mathcal{O} =_{\mathsf{df}} \bigcup_{i \in \mathbb{N}} \mathcal{O}$.

**Theorem 7.8** $\Gamma_{\mathbb{O},\mathcal{C}}$ *has* **MP***-models iff (every $A \in \mathcal{O}$ is* **CL***-satisfiable and $\mathcal{C}$ is* **CL***-satisfiable).*

So $\Gamma_{\mathbb{O},\mathcal{C}}$ may contain any conflict, as long as the set of constraints is internally consistent and there are no prima facie obligations that are contradictory in themselves. A nice property of deontic adaptive logics in general is that they are just as conflict-tolerant as their lower limit logic. For $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$, this follows immediately from the property of Reassurance (see Corollary 7.1). Hence we obtain:

**Corollary 7.2** *If every $A \in \mathcal{O}$ is* **CL***-satisfiable and $\mathcal{C}$ is* **CL***-satisfiable, then $\Gamma_{\mathbb{O},\mathcal{C}}$ has* $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$*-models.*

Note that since the $O$-operator from $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ behaves according to the $O$-operator from **SDL**, it follows that whenever $\Gamma_{\mathbb{O},\mathcal{C}}$ is **MP**-satisfiable, then there is a $B \in \mathcal{W}_c$ such that $\Gamma_{\mathbb{O},\mathcal{C}} \nvdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OB$. Hence the logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ also avoids deontic explosion, as long as the antecedent of Corollary 7.2 holds.

To spell out a lower bound on the set of actual obligations that are $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-derivable from a set $\Gamma_{\mathbb{O},\mathcal{C}}$, we first introduce two more concepts:[11]

[11]Our notion of "conflict-freeness" is equivalent to the notion of "coherence" used in [71, p. 7], if we restrict the latter to unconditional obligations.

**Definition 7.3** *We call* $\Gamma_{\mathbb{O},\mathcal{C}}$

- conflict-free up to level $n$ *iff the set* $\{A \in \mathcal{O}_i \mid i \leq n\} \cup \mathcal{C}$ *is* **CL**-*satisfiable.*
- conflict-free *iff* $\mathcal{O} \cup \mathcal{C}$ *is* **CL**-*satisfiable.*

Note that $\Gamma_{\mathbb{O},\mathcal{C}}$ is conflict-free whenever the prima facie obligations are not in conflict with the set of constraints – see Section 7.1.3 where we explained our notion of a deontic conflict relative to a set of constraints.

**Theorem 7.9** *If* $\Gamma_{\mathbb{O},\mathcal{C}}$ *is conflict-free up to level n, then the following holds for all* $i \leq n$*: if* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} O_i A$*, then* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA$*.*

The following properties can easily be obtained from Theorem 7.9 and previously mentioned results. Corollary 7.3(i) follows from Theorem 7.9 in view of Theorem 7.7, Corollary 7.3(ii) follows from Theorem 7.9 in view of Theorem 7.6. Corollary 7.4 follows from Theorem 7.9 and Corollary 7.3 in view of Definition 7.3.

**Corollary 7.3** *If* $\Gamma_{\mathbb{O},\mathcal{C}}$ *is conflict-free up to level n, then each of the following holds for all* $i \leq n$*:*

*(i)* *if* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}} O_i A$*, then* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA$
*(ii)* *if* $O_i A \in \Gamma_{\mathbb{O},\mathcal{C}}$*, then* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA$

**Corollary 7.4** *If* $\Gamma_{\mathbb{O},\mathcal{C}}$ *is conflict-free, then each of the following holds for all* $i \in \mathbb{N}$*:*
*(i)* *if* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} O_i A$*, then* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA$
*(ii)* *if* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}} O_i A$*, then* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA$
*(iii)* *if* $O_i A \in \Gamma_{\mathbb{O},\mathcal{C}}$*, then* $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA$

Corollary 7.4 implies that if there is no conflict between any of the prima facie obligations in view of the constraints, then all prima facie obligations and their **CL**-consequences will be considered as actual obligations (irrespective of their priority). Note that Theorem 7.9 only serves as a lower bound on $Cn_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$; as we will see in the next section, $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ is actually a lot stronger than this lower bound.

### 7.4.2 Proof of Theorems 7.8 and 7.9

We prove the left-right direction and the right-left direction of Theorem 7.8 as two separate lemmas.

**Lemma 7.4** *If* $\Gamma_{\mathbb{O},\mathcal{C}}$ *has* **MP**-*models, then every* $A \in \mathcal{O}$ *is* **CL**-*satisfiable and* $\mathcal{C}$ *is* **CL**-*satisfiable.*

*Proof.* Assume that for some $A \in \mathcal{O}$, $A$ is not **CL**-satisfiable. Let $M = \langle W, \mathbf{R}, R, v, w_0 \rangle$ be an **MP**-model of $\Gamma_{\mathbb{O},\mathcal{C}}$ — we derive a contradiction. Note that $O_i A \in \Gamma_{\mathbb{O},\mathcal{C}}$, whence there is a $R_j \in \mathcal{R}_i$ such that, for all $w \in W$ such that $R_j w_0 w$, $v_M(A, w) = 1$. However, since $A$ is not **CL**-satisfiable, $v_M(A, w') = 0$ for all $w' \in W$. Since $R_j$ is serial, we can derive a contradiction.

Assume that $\mathcal{C}$ is not **CL**-satisfiable. Hence by the compactness of **CL**, there are $A_1, \ldots, A_n \in \mathcal{C}$ such that $A_1 \wedge \ldots \wedge A_n$ is not **CL**-satisfiable. Let $B = A_1 \wedge \ldots \wedge A_n$, and let $M = \langle W, \mathbf{R}, R, v, w_0 \rangle$ be an **MP**-model of $\Gamma_{\mathbb{O},\mathcal{C}}$ — we derive a contradiction. Note that $\neg O \neg B \in \Gamma_{\mathbb{O},\mathcal{C}}$, whence there is a $w \in W$ such that $Rw_0w$ and $v_M(B, w) = 1$. However, for every $w' \in W$, $v_M(B, w') = 0$ since $B$ is not **CL**-satisfiable. ■

**Lemma 7.5** *If every $A \in \mathcal{O}$ is* **CL***-satisfiable and $\mathcal{C}$ is* **CL***-satisfiable, then $\Gamma_{\mathbb{O},\mathcal{C}}$ has* **MP***-models.*

*Proof.* Suppose (1) every $A \in \mathcal{O}$ is **CL**-satisfiable and (2) $\mathcal{C}$ is **CL**-satisfiable. Let $M = \langle W, \mathbf{R}, R, v, w_0 \rangle$ be defined as follows:

(i) $W = \{w_A \mid A \in \mathcal{O}\} \cup \{w_\mathcal{C}, w_0\}$. For each $A \in \mathcal{O}$, $w_A$ is a maximal consistent set in $\mathcal{L}$ that contains $A$ – by (1), there is such a $w_A$ for every $A \in \mathcal{O}$. $w_\mathcal{C}, w_0$ are maximal consistent sets such that $\mathcal{C} \subseteq w_\mathcal{C}, w_0$ – by (2), there are such $w_\mathcal{C}, w_0$.
(ii) $\mathcal{R}_i = \{R_A \mid A \in \mathcal{O}_i\}$, where $R_A = \{(w_0, w_A), (w_A, w_A)\}$.
(iii) $R = \{(w_0, w_\mathcal{C}), (w_\mathcal{C}, w_\mathcal{C})\}$.
(iv) For every $w \in W, p \in \mathcal{S}$: $v(p, w) = 1$ iff $p \in w$.

It is easy to see that $M$ is a model of $\Gamma_{\mathbb{O},\mathcal{C}}$. Let $O_iA \in \Gamma_{\mathbb{O},\mathcal{C}}$. Then $A \in \mathcal{O}_i$. We have $M \Vdash O_iA$ iff $v_M(O_iA, w_0) = 1$ iff for some $R_i \in \mathcal{R}_i$, $v_M(A, w') = 1$ for all $w' \in W$ for which $R_iw_0w'$. Note that $R_A = \{(w_0, w_A), (w_A, w_A)\} \in \mathcal{R}_i$ and that by the construction and the induction hypothesis $v_M(A, w_A) = 1$.

Let $\neg O \neg A \in \Gamma_{\mathbb{O},\mathcal{C}}$. Hence $A = B_1 \wedge \ldots \wedge B_n$, for $B_1, \ldots, B_n \in \mathcal{C}$. We have $M \Vdash \neg O \neg A$ iff $v_M(O\neg A, w_0) = 0$ iff (there is a $w' \in W$ for which $Rw_0w'$ and $v_M(\neg A, w') = 0$) iff $v_M(\neg A, w_\mathcal{C}) = 0$ iff $v_M(A, w_\mathcal{C}) = 1$ iff $v_M(B_1 \wedge \ldots \wedge B_n, w_\mathcal{C}) = 1$ iff $v_M(B_1, w_\mathcal{C}) = 1$ and … and $v_M(B_n, w_\mathcal{C}) = 1$. The latter holds by the construction. ■

For the proof of Theorem 7.9, we first prove three lemmas:

**Lemma 7.6** *If $\Gamma_{\mathbb{O},\mathcal{C}}$ is* **MP***-satisfiable and conflict-free up to level $n$, then there is an* **MP***-model $M'$ of $\Gamma_{\mathbb{O},\mathcal{C}}$ such that $Ab(M') \cap (\Omega_1 \cup \ldots \cup \Omega_n) = \emptyset$.*

*Proof.* Suppose $\Gamma_{\mathbb{O},\mathcal{C}}$ is **MP**-satisfiable and conflict-free up to level $n$. By Lemma 7.4, every $A \in \mathcal{O}$ is **CL**-satisfiable and $\mathcal{C}$ is **CL**-satisfiable. Let $M$ be the model of $\Gamma_{\mathbb{O},\mathcal{C}}$ constructed in the proof of Lemma 7.5. We construct $M' = \langle W', \mathbf{R}', R', v', w_0 \rangle$ from $M$ in the following way:

(i) $W' = \{w_A \mid A \in \mathcal{O}_i \text{ where } i > n\} \cup \{w'_\mathcal{C}, w_0\}$. $w'_\mathcal{C}$ is a maximal consistent extension (with respect to **CL**) of $\mathcal{C} \cup \mathcal{O}_1 \cup \ldots \cup \mathcal{O}_n$ – since $\Gamma_{\mathbb{O},\mathcal{C}}$ is conflict-free up to level $n$, there is such a $w'_\mathcal{C}$.
(ii) $R' = \{(w_0, w'_\mathcal{C}), (w'_\mathcal{C}, w'_\mathcal{C})\}$.
(iii) $\mathbf{R}' = \{\mathcal{R}_i \mid i > n\} \cup \{\mathcal{R}'_i \mid i \leq n\}$ where $\mathcal{R}'_i = \{R'\}$ for each $i \leq n$.
(iv) For every $w \in W', p \in \mathcal{S}$: $v'(p, w) = 1$ iff $p \in w$.

By the construction, we can show that (1) $M'$ is an **MP**-model of $\Gamma_{\mathbb{O},\mathcal{C}}$, and (2) $M' \Vdash O_iA$ iff $M' \Vdash OA$ for all $i \leq n$. The proof of (1) is analogous to the one above. Let $O_iA \in \Gamma_{\mathbb{O},\mathcal{C}}$. Then $A \in \mathcal{O}_i$. We have $M \Vdash O_iA$ iff $v_M(O_iA, w_0) = 1$ iff for some $R_i \in \mathcal{R}_i$, $v_M(A, w') = 1$ for all $w' \in W$ for which $R_i w_0 w'$.

Suppose first that $i > n$. Note that $R_A = \{(w_0, w_A), (w_A, w_A)\} \in \mathcal{R}_i$ and that by the construction $v_M(A, w_A) = 1$. Suppose now that $i \leq n$. Note that $R' = \{(w_0, w'_\mathcal{C}), (w'_\mathcal{C}, w'_\mathcal{C})\} \in \mathcal{R}_i$ and that by the construction $v_M(A, w'_\mathcal{C}) = 1$.

Let $\neg O \neg A \in \Gamma_{\mathbb{O},\mathcal{C}}$. Then $A = B_1 \wedge \ldots \wedge B_n$, for $B_1, \ldots, B_n \in \mathcal{C}$. We have $M \Vdash \neg O \neg A$ iff $v_M(O\neg A, w_0) = 0$ iff (there is a $w' \in W$ for which $Rw_0w'$ and $v_M(\neg A, w') = 0$) iff $v_M(\neg A, w'_\mathcal{C}) = 0$ iff $v_M(A, w'_\mathcal{C}) = 1$ iff $v_M(B_1 \wedge \ldots \wedge B_n, w'_\mathcal{C}) = 1$ iff $v_M(B_1, w'_\mathcal{C}) = 1$ and $\ldots$ and $v_M(B_n, w'_\mathcal{C}) = 1$. The latter holds by the construction.

Ad (2): By the construction, $M \Vdash OA$ iff $v_M(OA, w_0) = 1$ iff $v_M(A, w'_\mathcal{C}) = 1$ iff $v_M(O_iA, w_0) = 1$ for all $i \leq n$. The latter holds since $\mathcal{R}_i = \{R'\}$ for all $i \leq n$. ■

**Lemma 7.7** *If $\Gamma_{\mathbb{O},\mathcal{C}}$ is* **MP***-satisfiable and conflict-free up to level $n$, then for every $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-model $M$ of $\Gamma_{\mathbb{O},\mathcal{C}}$: $Ab(M) \cap (\Omega_1 \cup \ldots \cup \Omega_n) = \emptyset$.*

*Proof.* Suppose $\Gamma_{\mathbb{O},\mathcal{C}}$ is **MP**-satisfiable and conflict-free up to level $n$. Let $M \in \mathcal{M}_{\mathbf{MP}}(\Gamma_{\mathbb{O},\mathcal{C}})$ be such that $Ab(M) \cap (\Omega_1 \cup \ldots \cup \Omega_n) \neq \emptyset$. Let $M'$ be the model constructed in Lemma 7.6. Then in view of Definition 5.1, $Ab(M') \sqsubset Ab(M)$, whence $M \notin \mathcal{M}_{\mathbf{MP}}(\Gamma_{\mathbb{O},\mathcal{C}})$. ■

**Lemma 7.8** *If $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} O_iA$, then $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega_i$.*

*Proof.* Let $\Gamma = \Gamma_{\mathbb{O},\mathcal{C}}$. Suppose $\Gamma \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} O_iA$. By the reflexivity of **MP**, $Cn_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \vdash_{\mathbf{MP}} O_iA$. Let $\bigwedge_J \bigvee \Theta_j$ be a conjunctive normal form of $A$. By Lemma 7.3.2, $Cn_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \vdash_{\mathbf{MP}} OA \check{\vee} \bigvee_J \sigma^i(\Theta_j)$ and by Theorem 7.7, $Cn_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma) \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA \check{\vee} \bigvee_J \sigma^i(\Theta_j)$. By Theorem 7.5, $\Gamma \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA \check{\vee} \bigvee_J \sigma^i(\Theta_j)$. Since $\{\sigma^i(\Theta_j) \mid j \in J\} \subset \Omega_i$, the lemma follows immediately. ■

**Proof of Theorem 7.9.** The last two lemmas make the proof of Theorem 7.9 rather short. Suppose $\Gamma_{\mathbb{O},\mathcal{C}}$ is conflict-free up to level $n$ and $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} O_iA$, where $i \leq n$. Then by Lemma 7.8, $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega_i$. By the soundness of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ (Theorem 7.3), for every $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-model $M$ of $\Gamma_{\mathbb{O},\mathcal{C}}$: (†) $M \Vdash OA \check{\vee} Dab(\Delta)$. There are two cases to consider:

*Case 1.* $\Gamma_{\mathbb{O},\mathcal{C}}$ is not **MP**-satisfiable. Then it immediately follows that $\Gamma_{\mathbb{O},\mathcal{C}}$ is $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-trivial, whence $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA$ for all $A \in \mathcal{W}_c$.

*Case 2.* $\Gamma_{\mathbb{O},\mathcal{C}}$ is **MP**-satisfiable. Then for every $M' \in \mathcal{M}_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}}(\Gamma)$, $M' \Vdash \neg Dab(\Delta)$ in view of Lemma 7.7. Hence $M \Vdash \neg Dab(\Delta)$, which implies by (†): $M \Vdash OA$. By the completeness of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ (Theorem 7.3), $\Gamma_{\mathbb{O},\mathcal{C}} \vdash_{\mathbf{MP}^{\mathbf{m}}_{\sqsubset}} OA$.

## 7.5 The Examples Reconsidered

### 7.5.1 Illustration of the Proof Theory

We will use the first example to illustrate the proof theory of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$. Recall that $\Gamma_1 = \{O_1S, O_2P, O_2M, \neg O \neg(\neg S \vee \neg P)\}$. Since replacement of equivalents holds within the scope of the $O$-operator, this premise set is equivalent to $\Gamma'_1 = \{O_1S, O_2M, O_2P, \neg O(S \wedge P)\}$ — we will use the latter set for the sake of simplicity. We start an $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma'_1$ with the use of the rule PREM:

| | | | |
|---|---|---|---|
| 1 | $O_1S$ | PREM | $\emptyset$ |
| 2 | $O_2M$ | PREM | $\emptyset$ |
| 3 | $O_2P$ | PREM | $\emptyset$ |
| 4 | $\neg O(S \wedge P)$ | PREM | $\emptyset$ |

At this stage, there is no marking at all: since no Dab-formulas have been derived, $\Phi^{\sqsubset}_4(\Gamma'_1) = \{\emptyset\}$. Mary may infer from this that $M$ is an actual obligation, on the conditional assumption that $O_2M \wedge \neg OM$ is false. Hence the proof continues like this:[12]

| | | | |
|---|---|---|---|
| 5 | $OM \vee \sigma^2(M)$ | 2; RU | $\emptyset$ |
| 6 | $OM$ | 5; RC | $\{\sigma^2(M)\}$ |

The crucial move is made between stage 5 and 6: we can see that by the application of the conditional rule RC, the abnormality $O_2M \wedge \neg OM$ is pushed to the condition. Below, we will skip the intermediary step represented at line 5 – note that this is perfectly in line with the definition of the conditional rule RC. Since $\Phi^{\sqsubset}_6(\Gamma'_1) = \{\emptyset\}$, line 6 is unmarked at this stage, which indicates that, at this stage, $OM$ is considered to follow from the premise set. To illustrate once more why the definition of final $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-derivability refers to extensions of a proof, consider the following continuation of the proof (we restate line 6):

| | | | |
|---|---|---|---|
| 6 | $OM$ | 5; RC | $\{\sigma^2(M)\}\checkmark^8$ |
| 7 | $\neg O(S \wedge P \wedge M)$ | 4; RU | $\emptyset$ |
| 8 | $\sigma^1(S) \vee \sigma^2(P) \vee \sigma^2(M)$ | 1,2,3,7; RU | $\emptyset$ |

At stage 8 of the proof, the second element of line 8 is the only Dab-formula in the proof, hence it is a minimal Dab-formula. According to the definitions from Section 7.3.2, $\Sigma_8(\Gamma'_1) = \{\{O_1S\wedge\neg OS, O_2P\wedge\neg OP, O_2M\wedge\neg OM\}\}$. There are two $\sqsubset$-minimal choice sets of $\Sigma_8(\Gamma'_1)$: $\Phi^{\sqsubset}_8(\Gamma'_1) = \{\{O_2P \wedge \neg OP\}, \{O_2M \wedge \neg OM\}\}$. This implies that line 6 is marked at stage 8, which is indicated by the $\checkmark^8$-sign.

However, we can extend the proof such that line 6 becomes unmarked at a later stage:

| | | | |
|---|---|---|---|
| 6 | $OM$ | 5; RC | $\{O_2M \wedge \neg OM\}$ |
| 7 | $\neg O(S \wedge P \wedge M)$ | 4; RU | $\emptyset$ |
| 8 | $\sigma^1(S) \vee \sigma^2(P) \vee \sigma^2(M)$ | 1,2,3,7; RU | $\emptyset$ |
| 9 | $\sigma^1(S) \vee \sigma^2(P)$ | 1,3,4; RU | $\emptyset$ |

As a result, the formula on line 8 is not a minimal Dab-formula anymore. We get that $\Phi^{\sqsubset}_9(\Gamma'_1) = \{\{O_2P \wedge \neg OP\}\}$. In this particular case, line 6 will remain unmarked in every extension of the proof. Along the same lines, we can extend the proof to finally derive $OS$:

| | | | |
|---|---|---|---|
| 10 | $OS$ | 1; RC | $\{\sigma^1(S)\}$ |

The formula at line 9 does not block this derivation, since $\sigma^1(S)$ is freed from suspicion, so to speak, by the abnormality $\sigma^2(P)$ – note that $\{\sigma^2(P)\} \sqsubset \{\sigma^1(S)\}$.

[12]Recall that according to the notational conventions from Section 7.3.2, $\sigma^2(M)$ abbreviates $O_2M \wedge \neg OM$.

### 7.5.2 Mediocrity Does Not Rule

Consider the following $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$-proof from $\Gamma_2 = \{O_1U, O_2D, O_3N, \neg O\neg((\neg U \vee \neg D) \wedge (\neg N \vee \neg D))\}$:

| | | | |
|---|---|---|---|
| 1 | $O_1U$ | PREM | $\emptyset$ |
| 2 | $O_2D$ | PREM | $\emptyset$ |
| 3 | $O_3N$ | PREM | $\emptyset$ |
| 4 | $\neg O\neg((\neg U \vee \neg D) \wedge (\neg N \vee \neg D))$ | PREM | $\emptyset$ |
| 5 | $OU$ | 1; RC | $\{\sigma^1(U)\}$ |
| 6 | $OD$ | 2; RC | $\{\sigma^2(D)\}$ $\checkmark^9$ |
| 7 | $ON$ | 3; RC | $\{\sigma^3(N)\}$ |
| 8 | $\sigma^1(U) \vee \sigma^2(D)$ | 1,2,4; RU | $\emptyset$ |
| 9 | $\sigma^2(D) \vee \sigma^3(N)$ | 2,3,4; RU | $\emptyset$ |

Note that $\Sigma_9(\Gamma_2') = \{\{\sigma^1(U), \sigma^2(D)\}, \{\sigma^2(D), \sigma^3(N)\}\}$, and hence $\Phi_9^{\sqsubset}(\Gamma_2') = \{\{\sigma^2(D)\}\}$ — the set $\varphi = \{\sigma^1(U), \sigma^3(N)\}$ is not a $\sqsubset$-minimal choice set of $\Sigma_9(\Gamma_2')$, since $\{\sigma^2(D)\} \sqsubset \{\sigma^1(U), \sigma^3(N)\}$. As a result, only line 6 is marked. Line 5 will remain unmarked in every extension of this proof, whence $OU$ is finally derivable: Michael has the obligation to visit his sick uncle. Since this removes the obligation to visit his daughter, we can also derive that Michael has to pass by his nephew (at line 7) – note that both $U$ and $N$ can be fulfilled, and all that kept Michael from visiting his nephew was the (now overridden) obligation to visit his daughter.

This example is instructive in that it shows a clear difference between $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ and the criterion of "Least Exposure" from Alchourrón and Makinson [2]. According to this criterion, an obligation $A$ is preferred if and only if it is an element of a maximal consistent subset $\Theta$ of the prima facie obligations, and every other maximal consistent subset $\Delta$ that does not contain $A$ is more exposed. In our terminology, that $\Delta$ is more exposed than $\Theta$ means that $\Delta$ contains obligations with a higher priority index than any of the obligations in $\Theta$.

Now consider the example. There are two maximal consistent subsets: $\{U, N\}$ and $\{D\}$. Since the latter is less exposed than the former, the criterion proposed by Alchourrón and Makinson yields $D$ as a preferred obligation: Michael ought to visit his daughter, which also means that he cannot visit his uncle and his nephew. Hansen refers to this as the "Mediocrity Rules"-problem in his [68], and sees it as a severe drawback of the criterion of Least Exposure. This problem is avoided by $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$: an obligation can only be suspended if it is involved in a deontic conflict, with obligations of the same or a higher priority level.

### 7.5.3 Analyzing Conflicts

Consider Case 3, and its translation into $\Gamma_3 = \{O_1(B \wedge C \wedge (P \vee T)), O_2(\neg C \wedge \neg P), O_3P\}$. As show in Figure 7.1, there is a conflict between the first and second obligation, and between the second and third one. However, a specific property of the logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ is that it reduces such conflicts to conflicts between disjunctions of literals. As we will argue in this section, this makes $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ more appropriate to deal with complex obligations than various other approaches in the literature.

Let us first explain what is going on for this particular example. Note that $\Gamma_3 \vdash_{\mathbf{MP}} O_1 B, O_1 C, O_1(P \vee T), O_2\neg C, O_2\neg P$. This has several important consequences. First of all, since the obligation to buy bread has the highest priority and does not contradict any of the other prima facie obligations, this becomes an actual obligation. The following proof shows how we can finally derive $OB$ from $\Gamma_3$:

| | | | |
|---|---|---|---|
| 1 | $O_1(B \wedge C \wedge (P \vee T))$ | PREM | $\emptyset$ |
| 2 | $O_2(\neg C \wedge \neg P)$ | PREM | $\emptyset$ |
| 3 | $O_3 P$ | PREM | $\emptyset$ |
| 4 | $O_1 B$ | 1;RU | $\emptyset$ |
| 5 | $OB$ | 4;RC | $\{\sigma^1 B\}$ |

The second consequence requires a little more explanation. Consider only the obligations of rank 1 and 2. For these, the real problem – so **MP** tells us – lies with $C$: Tom should buy cheese, but this runs against his vegan convictions. Since his mother's command takes priority, he will have to violate his principles for this particular item on the shopping list. However, the obligation to buy either pork or tofu can be fulfilled without any problem: Tom just has to buy tofu instead of pork. All this is illustrated by the following continuation of the proof:[13]

| | | | |
|---|---|---|---|
| 6 | $O_1 C$ | 1; RU | $\emptyset$ |
| 7 | $O_1(P \vee T)$ | 1;RU | $\emptyset$ |
| 8 | $O_2\neg C$ | 2;RU | $\emptyset$ |
| 9 | $O_2\neg P$ | 2;RU | $\emptyset$ |
| 10 | $OC$ | 6;RC | $\{\sigma^1(C)\}$ |
| 11 | $O\neg C$ | 8;RC | $\{\sigma^2(\neg C)\}$ $\checkmark^{12}$ |
| 12 | $\sigma^1(C) \check{\vee} \sigma^2(\neg C)$ | 6,8;RD | $\emptyset$ |
| 13 | $O(P \vee T)$ | 7;RC | $\{\sigma^1(P,T)\}$ |
| 14 | $O\neg P$ | 9;RC | $\{\sigma^2(\neg P)\}$ |
| 15 | $OT$ | 13,14;RU | $\{\sigma^1(P,T), \sigma^2(\neg P)\}$ |

Note that $\Phi^{\sqsubset}_{12}(\Gamma_3) = \ldots = \Phi^{\sqsubset}_{15}(\Gamma_3) = \{\{\sigma^2(\neg C)\}\}$ in view of the Dab-formula on line 12. So we have derived that Tom should buy bread, cheese and tofu. What then about his little sister? In the current case, she will clearly not get what she want. The following continuation of the proof shows how the derivation of $OP$ from $O_3 P$ is blocked:

| | | | |
|---|---|---|---|
| 16 | $O_3 P$ | PREM | $\emptyset$ |
| 17 | $OP$ | 16;RC | $\{\sigma^3(P)\}$ $\checkmark^{18}$ |
| 18 | $\sigma^2(\neg P) \check{\vee} \sigma^3(P)$ | 9,16;RU | $\emptyset$ |

Note that $\Sigma_{18}(\Gamma_3) = \{\{\sigma^1(C), \sigma^2(\neg C)\}, \{\sigma^2(\neg P), \sigma^3(P)\}\}$. Hence $\Phi^{\sqsubset}_{18}(\Gamma_3) = \{\{\sigma^2(\neg C), \sigma^3(P)\}\}$, which implies that at stage 18, line 17 is marked. It can easily be verified that the marking of lines 1-18 remain stable in every extension of the above proof. Hence $OB$, $OC$ and $OT$ are finally derived in the proof.

[13] As explained in Section 7.3.2, $\sigma^2(P,T)$ abbreviates $(O_2(P \vee T) \wedge \neg O(P \vee T)) \vee (O_2 P \wedge \neg OP) \vee (O_2 T \wedge \neg OT)$.

In general, $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ has the ability to analyze prima facie obligations, and thereby to save as much from a prima facie obligation as possible in case it is involved in a conflict. This is due to the fact that (i) **MP** validates modal inheritance ($\mathrm{RM}_i$) and (ii) $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ is a strengthening of **MP** (see Theorem 7.7). $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ has this ability in common with the logics $\mathbf{P2.2^r}$ and $\mathbf{P2.2^m}$ from [110].

Most of the existing criteria in the literature on prioritized information (beliefs, obligations, default rules) rely on (a selection among) the maximal consistent subsets from a (possibly inconsistent) base, and therefore depend quite heavily on the way this base is formulated. Examples are again Alchourrón and Makinson's Least Exposure, but also Brewka's preferred remainders, Nebel's prioritized removals, Prakken's criterion for hierarchic rebuttal and Sartor's "prevailing" relation.[14] More recently, the same basic idea was applied in Input/Output-logic by Boella and Van Der Torre, see [39]. Notwithstanding all the subtle differences between these systems, they have one thing in common: if an obligation (belief, default rule) is stated as a conjunction (such as e.g. $O_2(\neg C \wedge \neg P)$), and one of the conjuncts is involved in a conflict, this renders the whole obligation useless.

Consider Tom's prima facie obligations in their initial formulation: $\mathcal{O}_3 = \{B \wedge C \wedge (P \vee T), \neg C \wedge \neg P, P\}$. The following are the maximal consistent subsets of $\mathcal{O}_3$:

$$\Delta_3^a = \{B \wedge C \wedge (P \vee T), P\}$$
$$\Delta_3^b = \{\neg C \wedge \neg P\}$$

Note that $\Delta_3^a \cap \Delta_3^b = \emptyset$. Hence, if we consider only those obligations as actual that are in both $\Delta_3^a$ and $\Delta_3^b$, Tom can buy whatever he likes. If we consider $\Delta_3^a$ as the "best" maximal consistent subset of $\mathcal{O}_3$, then Tom should buy both cheese and pork, and hence totally abandon his vegan principles. Finally, if we consider $\Delta_3^b$ as the "best" maximal consistent subset of $\mathcal{O}_3$, then Tom is obliged to violate his mother's order to buy bread and cheese. It can easily be verified that if we replace the sets $\Delta_3^a$ and $\Delta_3^b$ by $Cn_{\mathbf{CL}}(\Delta_3^a)$ and $Cn_{\mathbf{CL}}(\Delta_3^b)$, the problems remain. Put differently, approaches that are based on a selection of maximal consistent subsets of $\mathcal{O}$ – possibly closing the selected sets by under **CL** – yield counterintuitive results in this specific case.

The fact that $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ analyzes conflicts by analyzing the prima facie obligations that cause them, makes this logic very suitable to deal with complex prima facie obligations. That is, the logic does not rely on the assumption that, before the reasoning process takes place, we first analyze all our obligations into very specific obligations – it is the logic itself that does this work. In Chapter 9, several logics for belief revision are presented which share this feature with $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$.

## 7.6 Conclusion

Let us briefly summarize the main results of this chapter. We have developed a logic $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ that deals with unconditional prioritized obligations and has a dynamic proof theory. We have described its main features and established a number of intuitive properties of it. Finally, concrete examples were presented,

[14]See [68] for an overview of these consequence relations.

which illustrate the proof theory and highlight some differences with other approaches in the literature.

We promised to say a bit more about the restrictions we made in the introduction. Removing one or more of these is a task for future research. First of all, as already pointed out in Chapter 1, it would be interesting to see if we can develop systems that give up the restriction that the order on the set of obligations has to be modular. To do so would imply that we cross the safe boundaries of the existing generic formats for adaptive logics. This work will hence require thorough investigations on the metatheoretical level.

We also restricted ourselves to a monadic framework. The extension to a dyadic deontic logic might lead to some problems, e.g. should obligations with more specific conditions receive a higher priority rank, or should contrary-to-duty obligations overrule conflicting unconditional obligations whenever the former are canceled? We refer to [138, 139, 70] for discussions of the various problems and paradoxes relating to conditional obligations. An extension of $\mathbf{MP}^{\mathbf{m}}_{\sqsubset}$ to the dyadic case should be able to cope with these issues to some extent.

# Chapter 8

# Two Logics for the Abduction of Generalizations

*This chapter is based on two joint papers with Tjerk Gauderis: "Abduction of Generalizations" (forthcoming in Theoria), and "Adaptive Logics for the Abduction of Generalizations" (in preparation). We thank Dagmar Provijn, Bert Leuridan, Erik Weber, Joke Meheus and two anonymous referees for their many helpful comments on the first of these two papers. I also thank Joke Meheus for her critical remarks on a previous draft of this chapter.*

## 8.1 Introduction

*Abduction* is generally defined as "the process of forming an explanatory hypothesis" [118, p. 216]. In this chapter we will focus on a specific "pattern of abduction" (to use a phrase introduced by Schurz [131]), which we call *abduction of generalizations* (henceforth AG). The following are prototypical examples of AG:[1]

$$\begin{array}{l}\text{All chocolate tastes sweet.}\\ \text{Everything that contains sugar, tastes sweet.}\\ \hline \text{All chocolate contains sugar.}\end{array}$$

$$\begin{array}{l}\text{All iron objects in this garage are corroded.}\\ \text{Whenever an iron object has been wet, it is corroded.}\\ \hline \text{All iron objects in this garage have been wet.}\end{array}$$

Schurz called this type of inference "law-abduction". The name "rule abduction" has also been used for a similar pattern [145]. But, as "law" and "rule"

[1]The first example is a variant of Schurz' pineapple example [131]. In the second example, we use "'has been wet" as a shortcut for more accurate descriptions of what causes the corrosion of an iron object, such as "has been left in a sufficiently humid environment for a sufficiently long period".

are heavily debated concepts in philosophy of science and philosophy in general, we will stick to the more neutral term *generalizations*. More examples and a general characterization of AG will be presented in Section 8.2. It will be argued that this pattern is ubiquitous in both everyday and scientific reasoning, and is commonly recognized as a useful – be it fallible – means to form hypotheses.

Notwithstanding the importance of AG, little effort has been made so far to study its characteristics, and to explicate it by means of a formal logic. As will be explained in Section 8.2.2, most scholars in AI and formal logic have focused on singular fact abduction, whereas philosophers of science have taken a more general, but informal point of view on abduction. It is our aim to treat AG as a distinct subject matter, and to see how one may understand and formalize it.

**Outline of this Chapter** A first analysis of AG is provided in Section 8.2. We describe this pattern informally, showing that it is a widespread inference pattern; secondly, we explain why it has been neglected in formal logic and philosophy of science; finally, we argue for the specific importance of AG in scientific contexts.

In Section 8.3, we turn our focus to problems that emerge when representing AG formally. We argue that a distinction in the object language is needed between what we call *mere generalizations* and the *explanatory framework*, for any logic that models AG. In addition, we show that this distinction is useful in any logic for abduction. We then present the lower limit logic $\mathbf{T}$, which allows us to express this distinction in a very simple way.

In Section 8.4, we will illustrate the idea of adaptive logics for abduction by means of a simple example, viz. the logic $\mathbf{LA}^{\mathbf{r}}_{\Box}$ for singular fact abduction. This logic is interesting for two reasons. Firstly, it allows us to show certain pitfalls for logics for abduction, and how the adaptive logics in this paper will evade these. Secondly, the set of abnormalities of $\mathbf{LA}^{\mathbf{r}}_{\Box}$ will play an important role in the definition of $\mathbf{SILA}^{\mathbf{r}}$, which we present in Section 8.7.

Before we discuss the logics for AG, we will present five variations on the above chocolate example (Section 8.5). These prototypical examples and their formal translation allow us to highlight important features of the logics in a very simple way.

In the two subsequent sections, we will present the logics $\mathbf{LA}^{\mathbf{r}}_{\forall}$ and $\mathbf{SILA}^{\mathbf{r}}$.[2] Both logics are based on a specific idea of how best to think of AG. The first one takes AG as a primitive inference pattern, and can be seen as the most straightforward of the two. The second logic provides a reconstruction of AG by a combination of inductive generalization and singular fact abduction, elaborating on an idea from [52] and [145]. As the reader will note, we will devote considerable space to the behavior of these systems for concrete examples, in order to argue for specific choices in the definition of the logics. In Section 8.8, we explain an important difference between $\mathbf{LA}^{\mathbf{r}}_{\forall}$ and $\mathbf{SILA}^{\mathbf{r}}$. We present a summary of our results and prospects for further research in Section 8.1.

[2] As the superscripts of the logics indicate, we will restrict the focus to adaptive logics that use the Reliability Strategy. This simplifies the discussion of specific examples. We briefly consider the Minimal Abnormality-variants of these logics in the concluding section.

**Preliminaries.** In the remainder, let $\mathcal{L}_f$ be the standard language of classical first order predicate logic (henceforth denoted by **CL**), obtained from the set of propositional letters $\mathcal{S} = \{p, q, r, \ldots\}$, a set of constants $\mathcal{C} = \{a, b, c, \ldots\}$, a set of variables $\mathcal{V} = \{x, y, z, \ldots\}$, a set of predicates $\mathcal{P} = \{P, Q, R, \ldots\}$, the connectives $\neg, \vee, \wedge, \supset, \equiv$ and the quantifiers $\forall$ and $\exists$.[3] $\mathcal{W}_f$ is the set of closed formulas in $\mathcal{L}_f$. An axiomatization of the first order fragment of **CL** without identity can be found in Appendix B.

We will use $A, B, C, \ldots$ as metavariables for members of $\mathcal{W}_f$ and $\alpha, \beta$ as metavariables for individual constants. Let $\mathcal{F}^\circ$ denote the set of *purely functional formulas*, i.e. formulas that do not contain individual constants, quantifiers, or sentential letters. For example, $Px \wedge (Qxy \vee Rx)$ is a purely functional formula, whereas $Pa \vee Qxy$ and $Px \wedge \exists y Qxy$ are not. Where $A \in \mathcal{F}^\circ$, let $\forall A$ abbreviate the universal quantification over every variable that is free in $A$. Finally, where $A, B \in \mathcal{F}^\circ$, we use $A \| B$ to denote the fact that $A$ and $B$ share no predicates.

## 8.2 Abduction of Generalizations

### 8.2.1 The phenomenon

Let us define *abduction of generalizations* informally, as every inference that fits the following pattern:

(P1) "All $A$ are $B$"
(P2) "Being $C$ explains being $B$"
(H) "Therefore, all $A$ are $C$"

In Section 8.3, we will explain how this definition can be operationalized in a first-order modal language. But first, let us point out some general characteristics of AG.

First of all, according to standard terminology, the first of the two premises in the above schema plays the role of the *explanandum* (that which is to be explained) in AG, whereas (H) is the *explanatory hypothesis.* The second premise has a somewhat special status – we will discuss it in Section 8.3.1.

In the remainder, we will restrict ourselves to the *Peircean* or *classical* notion of abduction, which is defined in a *deterministic* way: the truth of the explanatory hypothesis implies the *explanandum.*[4] For the above schema, this means that (H) should imply (P1). This is the case whenever $C$-hood *implies* $B$-hood, or in other words, when "all $C$ are $B$". In the remainder, we will start from the assumption that (P2) implies (but is *stronger* than) "all $C$ are $B$" – this will be further explained in Section 8.3.1.

It should be noted that the status of the explanandum is slightly different from its status in the classical notion of abduction, as defined e.g. by Peirce.

[3]Although we will only use unary predicates in concrete examples in this chapter, we do not exclude predicates of a rank $n \geq 2$. Also, for our present purposes, we need not include an identity symbol in the language schema.

[4]We do not suggest that this notion of abduction cannot be meaningfully extended to other accounts in which the motivation to adopt the abductive hypothesis is, for instance, probabilistic or comparative. However, we restrict ourselves in this chapter, as most of the literature on abduction does, to the classical case.

According to Peirce, the explanandum is a (surprising) *fact*, which is supposed to be *observed* – see [119, 5.189]. In the current definition, the explanandum is a *generalization*, which implies that it cannot be observed as such. It may e.g. be that this information was provided to us by an expert, or that we have learned that "all $A$ are $B$" by means of an inductive generalization on the basis of a large number of observations. However, what is crucial and common in both the classical account of abduction, and our definition of AG, is that the explanandum is something that is itself beyond doubt in the context in which we perform the abduction. In Section 8.8, we will consider the option that the induction is performed in the same context as AG.

Secondly, AG is distinct from what is called *singular fact abduction*, in which both the explanandum and the explanatory hypothesis are singular facts. Existing models for abduction usually limit themselves to singular fact abduction, as we will see in the next section.

Thirdly, AG is not a novel reasoning pattern. It has been known at least since Aristotle who treats something similar in his *Posterior Analytica* when he considers the "middle term" of a definition. This pattern is, according to his view, the essence of a good definition: it should not only say what the *definiendum* ($A$) is, it should also be an explanation ($C$) for its observed properties ($B$). As an example, he explains why horned animals ($A$) lack upper incisors ($B$) by defining horned animals as a subclass of animals that have inflected hard material from their mouth to their heads ($C$). According to Aristotle, this is a good definition of a class because it explains the properties of that class.[5] However, the reasoning pattern we are considering is much broader than what Aristotle had in mind. $A$, $B$ and $C$ can be any properties, and neither should $A$ be a definiendum, nor $C$ a definiens.

Fourthly, AG is frequently applied in human reasoning, often in combination with or following an instance of singular fact abduction. For instance, people do not only wonder why their heads hurt (they drank too much last night) or why there is a thunderstorm (it was very hot during the day). Not much of a reflective mind is needed to start asking questions such as why it is that every time one drinks a bit too much, one suffers from headaches, or why thunderstorms often follow hot days. In other words, people do not only wonder why certain facts are the case, they also wonder why certain regularities occur.

### 8.2.2 The Lack of Models for AG

Broadly speaking, two main currents in research on abduction can be discerned. On the one hand, research in AI and formal logic mostly focuses on a *syllogistic* interpretation of Peirce's work, in which abduction is introduced as part of a

[5] See [6, II.10] for Aristotle's distinction between two types of definitions and [6, II.12-14] for his view on the role of the middle term in a definition. A good treatment of the analogy between Aristotelian definitions and Peircean abduction can be found in [50]. In our opinion, Schurz refers in [131] to the wrong concept when he links AG (in his words: law abduction) to Aristotle. The concept "hitting upon the middle term" is only employed in the definition of quick wit [6, I.34], in which it is illustrated with an example of a singular fact abduction. In our view, a predecessor of AG can only be found in Aristotle's treatment of the role of the middle term in definitions.

tripod that is clarified with the following famous beans-example of Peirce [119, 2.623]:

> All the beans from this bag are white. (Rule)
> These beans are from this bag. (Case)
> These beans are white. (Result)

All reasoning deriving a result from a case and a rule is called *deductive*, all reasoning deriving a rule from a case and a result *inductive*, and all reasoning deriving a case from a rule and a result *abductive*. Having this schema in mind, researchers in AI or formal logic generally focus on instances of singular fact abduction, which are variations on the following inference rule:

$$B\alpha, \forall\beta(A\beta \rightarrow B\beta)/A\alpha$$

This rule is usually combined with a number of conditions, e.g. that the hypotheses it yields should be mutually consistent with our background knowledge, that they should be as parsimonious as possible (this term is explained below), etc. Aliseda even adds a further condition suggested by Peirce, i.e. that the observed fact should be surprising (in the sense that $B\alpha$ cannot be derived from the background theory alone) [4].

One notable exception to the exclusive focus on singular fact abduction is Thagard [145]. He obtains a similar pattern as AG, which he calls "rule abduction", by adding to his logic program PI the ability to generalize the results of singular fact abductions. Although his model does not abduce *from* generalizations, it has the same goal as AG, i.e. to derive an explanation for why all elements of a given class share a certain property.

On the other hand, research in philosophy of science usually departs from a *methodological* interpretation of Peirce. In his later writings Peirce distinguishes abduction, induction and deduction as different steps in a methodology of science [118, p. 212–218]. Abduction is the process of forming an explanatory hypothesis, from which deduction can draw predictions, which then can be tested by induction.[6] Research in this tradition, see e.g. [92, 131], considers abduction as a very broad concept including analogical reasoning, visual abduction, common cause reasoning, etc. Some still try to capture the concept of abduction under the single schema of *inference to the best explanation* (IBE).[7] However, these attempts to reduce the broadness of the considered concept prevent the discovery of interesting features of more specific patterns of abduction. Schurz explains this as follows [131, p. 205]:

> The majority of the recent literature on abduction has aimed at *one most general* schema of abduction (for example IBE) which matches every particular case. I do not think that good heuristic rules for

[6] It is generally acknowledged (see e.g. [52, p. 5–8]) that both interpretations can be found in Peirce's work, although they are not fully compatible. They represent an evolution in his thinking, as he hinted himself when he remarked that he "was too much taken up in considering syllogistic forms" [119, 2.102].

[7] See e.g. [76, 90, 48]. These scholars consider Peirce's remark that abduction should be as economical as possible [119, 7.220], as an essential and crucial condition.

> generating explanatory hypotheses can be found along this route, because these rules are dependent of the specific type of abduction scenario.

Schurz presents a taxonomy of distinct patterns of abduction, and argues that such a taxonomy is indispensable if one wants to understand the many uses of abduction. With Schurz' arguments in mind, we think that it is best to remain pluralistic on the logical form of abduction. We should maintain the rich concept of abduction as it is understood in the philosophy of science, but, in order to provide the formal rigor which is characteristic of the logic and AI community, we have to focus on each of the different specific forms of abduction separately – we will return to this point in the concluding section.

### 8.2.3 The Ubiquity of AG in Scientific Practice

At the end of Section 8.2.1, we mentioned several examples in which abduction of generalizations is triggered by a question concerning the result of a singular fact abduction. This question is brought up by a need for a deeper understanding of the observed relations. We can recognize this curious spirit in the endeavors of many scientists. For instance, Descartes was not satisfied with the folk explanation of the rainbow, i.e. that a rainbow appears because the sun breaks through shortly after a rain shower. He wanted to understand *why* rainbows appear whenever the sun shines while it rains. We will argue that AG is at least as important in scientific practice as singular fact abduction by considering two general characteristics of this practice.[8]

Firstly, in scientific practice one attempts to formulate theories, which have both a *universal* and *falsifiable* nature.[9] One does not want an explanation why, for instance, this particular person suffers from this disease. One wants to understand why and how this disease is transmitted in general. Formulating theories about particularities is seldom considered as good scientific practice; such theories are often labeled as *ad hoc*. Theories are thus mainly formulated for a whole class of objects and, by consequence, formulated in terms of generalizations. These generalizations allow us to derive singular fact predictions by means of which theories can be tested. Therefore, in the formation process of such theories, reasoning methods resulting in generalizations, such as inductive generalization or AG, are essential.

Secondly, augmented *unification* (as characterized, for instance, by [85]) is generally seen as an indicator of scientific progress.[10] Each application of AG is in essence a unification step, because it explains an observational generalization, e.g. "All $A$ are $B$", by characterizing its antecedent ($A$) as a subclass of a more general class ($C$) for which the observed properties ($B$) hold. Therefore, AG is

[8] This claim is about *scientific practice* and not about *scientific explanation*. In scientific explanation, a scientific theory is employed to explain a certain fact (which can be either a singular fact or a generalization). Scientific practice is the activity of forming such scientific theories and expanding current scientific knowledge.

[9] Universality should not be taken as an absolute notion, but as an achievable level of generality that is relative to the methods and scope of the specific field.

[10] Both the instrumentalist and realist view concerning the nature of scientific progress seem to agree on this point [115].

a key method to enhance unification in scientific practice. The most interesting examples in the history of science can be found when a new theory is proposed as a solution for some anomalies of an existing theory. In that case, the proponents of the new theory also need to show that most of the already known and well-tested observational laws, which are explained by the old theory, can be explained by the new theory. For instance, Newton could explain Huygens' pendulum law using his general laws of motion by pointing out how the different parameters of the pendulum law could be translated into his general mathematical framework. In the same way, Bohr could explain by means of his atomic model why the wave lengths of the visible emission spectrum of hydrogen can be calculated by the Balmer formula.[11]

## 8.3 Towards a Formal Logic for AG

### 8.3.1 The Explanatory Framework

The pattern presented in the definition of AG (see Section 8.2) can be formally explicated as follows:

(P1) $\forall x(Ax \supset Bx)$
(P2) $\forall x(Cx \supset Bx)$
(H) $\forall x(Ax \supset Cx)$

However, we must be careful here: the definition stipulates that $C$-hood *explains* $B$-hood, not just that everything that has the property $C$ also has the property $B$. In other words, where (P1) and (H) can be of any kind, the set of possible candidates for (P2) is restricted.[12] We call this set the *explanatory framework*. It consists of all generalizations of the form $\forall x(Fx \supset Gx)$ where being $F$ provides an explanation for being $G$. Whether or not a generalization belongs to the explanatory framework, depends on the context, i.e. on the phenomena we are reasoning about. All we assume is that it is clear for each generalization, given the abductive problem at hand, whether it is a member of the explanatory framework or not. In the latter case we call it a *mere generalization*.

With this new terminology, we are now able to characterize all the lines of the above schema: (P1) is the *explanandum*, (P2) is a generalization that is part of the explanatory framework for the current context; (H) is the explanatory hypothesis. An *explanation* or *explanans* for (P1) consists of an explanatory hypothesis together with one or more elements of the explanatory framework that connect the hypothesis to the explanandum.

Now what does it actually mean that $F$-hood explains $G$-hood? Needless to say, the philosophical literature abounds in theories of explanation. However, as we restrict ourselves to classical abduction (see Section 8.2.1), certain preconditions apply. First, if $F$-hood explains $G$-hood, then $F$-hood should also *imply*

[11] A philosophical introduction to the circumstances of these two major milestones in science can be found in [134], resp. [51].

[12] In our opinion, Schurz [131] puts too little emphasis on this point in his discussion of AG, or "law abduction" as he calls it. In his schema, (P2) is called a "background law", but as far as we see, no explicit definition or circumscription is provided.

$G$-hood. Second, as abduction is an inference, only argumentative accounts of explanation are relevant. Hence, the choices to explicate the notion of "explanation" in the definition of the *explanatory framework of a (classical) abductive problem* are limited to accounts of explanation that have the structure of a deductive argument such as a DN-argument (e.g. Hempel [78]), a causal argument (e.g. Hausman [77]) or an augmented unification argument (e.g. Kitcher [85]).[13]

In any of these accounts, (P2) has a specific status – it must be either lawlike, refer to an underlying causal mechanism, or be a more general argumentation scheme. We use the more abstract term *explanatory framework* to express this status of (P2). This specific status turns AG into a fundamentally *asymmetric* inference. It is not possible to derive $\forall x(Cx \supset Ax)$ from the same premises, since $A$-hood does not explain $B$-hood. Hence, if a logic explicates AG, it should be able to represent this asymmetry between (P1) and (P2) in its object language.

Before we explain how this can be done, let us briefly give an extra reason to motivate the distinction between the explanatory framework and mere generalizations as a valuable asset for any logic that models abductive processes in general. Mere generalizations are often used in abductions that involve knowledge about methods or procedures. Consider the following premises:

(P1) The Geiger counter produces audible clicks close to the object $a$.
(P2) If the Geiger counter produces audible clicks, $\beta$-radiation is present.
(P3) If an object contains C-14, $\beta$-radiation is emitted.

Without the distinction between the explanatory framework and mere generalizations, a logic for singular fact abduction treats (P2) and (P3) as having the same formal structure. But a physicist interested in explaining the presence of $\beta$-radiation is only interested in the hypothesis suggested by (P3), as the behavior of the Geiger counter provides no explanation. On the other hand, (P2) is needed to derive the fact that there is $\beta$-radiation in the first place (as it is not directly observable). Hence, (P2) cannot be omitted from this abductive reasoning context. Only a logic that is able to represent explanatory frameworks can handle this case properly.

### 8.3.2 A Modal Approach

The AG-logics presented below are all based on a predicative fragment of the well-known modal logic **T**. As explained in Section 2.1 of Chapter 2, we first have to define a "standard" logic $\mathbf{T_s}$, and afterwards extend it with the checked connectives to obtain the lower limit logic **T**.

Let $\mathcal{L}_t$ denote the extension of $\mathcal{L}_f$ with the necessity operator $\Box$. The set of formulas $\mathcal{W}_t$ is the smallest set for which the following holds:

For all $A \in \mathcal{W}_f$: $A, \Box A \in \mathcal{W}_t$
For all $A, B \in \mathcal{W}_t$: $\neg A, A \vee B, A \wedge B, A \supset B, A \equiv B \in \mathcal{W}_t$

[13]It is not implied that there are no other valuable accounts of explanation. We only claim that (classical) abductive hypotheses (the only ones that are our concern here) are part of a deductive argument that forms an explanation for the explanandum.

Hence, the modal operator $\Box$ does not occur in the scope of quantifiers, and we do not allow for the iteration of modal operators. We merely use the modal operator to distinguish between the explanatory framework and mere generalizations.[14] Hence, $\Box\forall x(Px \supset Qx)$ and $Pa \vee \Box\exists x(\neg Rx)$ are, for instance, members of $\mathcal{W}_t$, whereas $\Box\Box\forall x Px$ or $\forall x\Box Px$ are not. The language $\check{\mathcal{L}}_t$ is obtained by adding the checked connectives to $\mathcal{L}_t$; $\check{\mathcal{W}}_t$ is obtained by closing $\mathcal{W}_t$ under these connectives (see page 18).

An axiomatization for the logic $\mathbf{T_s}$ over the language $\mathcal{L}_t$ is obtained by taking the axioms of classical predicate logic (henceforth **CL**), adding the following axioms:

K $\quad \Box(A \supset B) \supset (\Box A \supset \Box B)$
T $\quad \Box A \supset A$

and closing them under the following two rules:

RN $\quad$ where $A \in \mathcal{W}_t$: from $\vdash A$, infer $\vdash \Box A$
MP $\quad$ from $A, A \supset B$, infer $B$

We will focus on the proof theoretic aspects of the adaptive logics for AG in this chapter, and therefore omit a definition of the $\mathbf{T_s}$-semantics. A standard semantics in terms of Kripke-models with varying domains and a reflexive accessibility relation $R$ can be obtained for $\mathbf{T_s}$, along the lines of [41]. We will return to the semantic counterpart of our approach in the concluding section.

The language $\mathcal{L}_t$ allows us to represent the premises involved in abductive reasoning processes with the expressive power of classical first-order logic, but gives us the extra operator $\Box$, which allows us to indicate at the object level that a certain generalization is in the explanatory framework. Recall that $\mathcal{F}^\circ$ denotes the set of *purely functional formulas*, i.e. formulas that do not contain individual constants, quantifiers, or sentential letters. Also, where $A \in \mathcal{F}^\circ$, $\forall A$ denotes the universal quantification over every variable that is free in $A$. The logic $\mathbf{LA}^{\mathbf{r}}_{\forall}$ treats any formula of the form $\Box\forall(A \supset B)$ with $A, B \in \mathcal{F}^\circ$ as an element of the explanatory framework.

The choice for $\mathbf{T_s}$ in order to model the explanatory framework has two important consequences. First of all, in view of the rule RN and the axiom K, classical logic consequences of the explanatory framework may themselves be used to generate explanatory hypotheses. For instance, if $\Box\forall x(Px \supset Qx)$ and $\Box\forall x(Qx \supset Rx)$ are premises of a particular abductive problem, not only these formulas but also $\Box\forall x(Px \supset Rx)$ will be part of the explanatory framework. Second, in view of axiom T, a generalization that is part of the explanatory framework is also assumed to be true as such. This is the formal expression of our restriction to the classical account of abduction, where "$A$ explains $B$" entails that "$A$ implies $B$".

[14] It might be possible to do without these restriction on the language, given a number of additional axioms such as the 4-axiom ($\Box A \equiv \Box\Box A$), the Barcan formula and/or the inverse Barcan formula. This would however severely complicate the logical apparatus, whereas the extended language would contain several expressions that have no sensible interpretation in the current context.

The logic **T** is obtained by adding the usual axioms for the checked connectives, and those that link the checked and non-checked connectives, to $\mathbf{T_s}$ (see Chapter 2, Section 2.4 where **K** is obtained from $\mathbf{K_s}$ in an analogous way). As we will explain below, the logics $\mathbf{LA}_\forall$ and $\mathbf{SILA^r}$ are non-monotonic extensions of **T**. Since **T** is itself a monotonic extension of **CL**, our logics only provide sensible consequences under the assumption that the explanatory framework and the set of known facts relevant to the abductive problem are mutually consistent – otherwise they yield a trivial consequence set.

Our formal expression of AG is in a sense minimal: iterations of boxes are excluded, and explanation is expressed by rather simple formal tools. It is a topic for further research whether our model can be meaningfully extended to include specific, more fine-grained accounts of explanation (e.g. adding asymmetric axioms to specify causal arguments in the sense of Hausman [77]).

## 8.4 Singular Fact Abduction: $\mathbf{LA}^{\mathbf{r}}_{\Box}$

In this section, we present a simple adaptive logic for singular fact abduction, called $\mathbf{LA}^{\mathbf{r}}_{\Box}$. As already indicated in the introduction, this has two reasons: (i) it allows us to show certain pitfalls for logics for abduction, and how the adaptive logics in this chapter evade these, and (ii) $\mathbf{LA}^{\mathbf{r}}_{\Box}$ is a constituent of the combined logic $\mathbf{SILA^r}$, which we present in Section 8.7.

$\mathbf{LA}^{\mathbf{r}}_{\Box}$ is obtained by a small variation on the logic $\mathbf{LA}^{\mathbf{r}}_{\mathbf{s}}$ from [107]. The latter system allows for the defeasible application of the rule $\forall(A \supset B), B\alpha/A\alpha$. In the terminology from the current chapter, it treats all generalizations as part of the explanatory framework. In $\mathbf{LA}^{\mathbf{r}}_{\Box}$, the abductive steps are restricted to those cases where $A$-hood explains $B$-hood, or formally, where we have derived $\Box\forall(A \supset B)$.

The logic $\mathbf{LA}^{\mathbf{r}}_{\Box}$ is defined by the triple $\langle \mathbf{T}, \Omega_{\mathbf{LA}^{\mathbf{r}}_{\Box}}, \mathbf{r}\rangle$, where the set of abnormalities of $\mathbf{LA}^{\mathbf{r}}_{\Box}$ is given by:[15]

$$\Omega_{\mathbf{LA}^{\mathbf{r}}_{\Box}} =_{\mathsf{df}} \{\Box\forall(A \supset B) \wedge (B\alpha \wedge \neg A\alpha) \mid A, B \in \mathcal{F}^{\circ} \text{ and } A\|B\}$$

We will abbreviate formulas of this kind by $(A \supset B)^{\alpha}$. The extra condition that $A$ and $B$ share no predicates implies that $A$ and $B$ are logically independent, whence any kind of self-explanation or circular explanation is avoided.[16] This also prevents certain counterintuitive or superfluous abductive steps, such as the following:

(i) $\Box\forall x((Px \wedge Qx) \supset Qx)$, $Qa$ / $Pa \wedge Qa$
(ii) $\Box\forall x(Px \supset Px), Pa$ / $Pa$
(iii) $\Box\forall x((Px \supset (Px \vee Qx)), Pa \vee Qa$ / $Pa$

There is also a technical reason for the restriction on the set of abnormalities.[17] Consider the following prototypical example: $\Gamma_{\mathsf{s1}} = \{\Box\forall x(Px \supset Qx), Qa\}$. Ob-

[15] Recall that, where $A, B \in \mathcal{F}^{\circ}$, we use $A\|B$ to denote the fact that $A$ and $B$ share no predicates.
[16] See also [108, p. 221-222].
[17] To the best of our knowledge, this reason was not spelled out before in the literature.

viously, we want to be able to finally derive the hypothesis $Pa$ from $\Gamma_{\mathsf{s1}}$, on the condition $(P \supset Q)^a$. But consider the following formula:

$$\Box\forall x(\neg Px \supset (\neg Px \vee Qx)) \wedge (\neg Pa \vee Qa) \wedge \neg\neg Pa \tag{8.1}$$

It can easily be verified that "(8.1) $\check{\vee}(P \supset Q)^a$" is a minimal disjunction that is **T**-derivable from $\Gamma_{\mathsf{s1}}$. Hence if (8.1) would be an abnormality, then (8.1) $\check{\vee}(P \supset Q)^a$ would be a minimal Dab-consequence of $\Gamma_{\mathsf{s1}}$, and hence $(P \supset Q)^a$ would be an unreliable abnormality in view of this premise set. In that case, we would not be able to finally derive $Pa$ in this very simple case, whence we could hardly speak of a logic that models singular fact abduction.

We will now illustrate the features of this specific logic by means of an example. Consider $\Gamma_{\mathsf{s2}} = \{Qa, Qb, \neg Ra, Rc, \Box\forall x(Px \supset Qx), \forall x(Px \supset Rx)\}$. We construct a proof from this premise set in which we abduce explanatory hypotheses for the observed phenomenon $Qa$:

| | | | |
|---|---|---|---|
| 1 | $\Box\forall x(Px \supset Qx)$ | PREM | $\emptyset$ |
| 2 | $Qa$ | PREM | $\emptyset$ |
| 3 | $Pa$ | 1,2;RC | $\{(P \supset Q)^a\}$ |

Recall that we can write line 3 because $\Gamma_{\mathsf{s2}} \vdash_{\mathbf{T}} Pa \,\check{\vee}(\Box\forall x(Px \supset Qx) \wedge (Qa \wedge \neg Pa))$. A key advantage of adaptive logics is that defeasible steps can be fully integrated with deductive steps. We can continue our proof, for instance, as follows:

| | | | |
|---|---|---|---|
| 4 | $\forall x(Px \supset Rx)$ | PREM | $\emptyset$ |
| 5 | $Ra$ | 3,4;RU | $\{(P \supset Q)^a\}$ |

The formula on line 5 is the result of a deductive step, but, as this step relies on a hypothesis, the condition of the hypothesis is transferred. In other words, $Ra$ can be derived on the same assumption as the hypothesis $Pa$ itself.

Note that, according to the terminology introduced in Section 8.3, the formula on line 4 is not part of the explanatory framework. Put differently, $P$-hood does not explain $R$-hood according to the premise set $\Gamma_{\mathsf{s2}}$. As a result, it is e.g. not possible to derive $Pc$ from the premises $Rc$ and $\forall x(Px \supset Rx)$.

At this point of the proof, it becomes clear that the hypothesis $Pa$ is not unproblematic. The condition of $Pa$ can be derived on the empty condition in the following way:

| | | | |
|---|---|---|---|
| 1 | $\Box\forall x(Px \supset Qx)$ | PREM | $\emptyset$ |
| 2 | $Qa$ | PREM | $\emptyset$ |
| 3 | $Pa$ | 1,2;RC | $\{(P \supset Q)^a\}$ ✓$^8$ |
| 4 | $\forall x(Px \supset Rx)$ | PREM | $\emptyset$ |
| 5 | $Ra$ | 3,4;RU | $\{(P \supset Q)^a\}$ ✓$^8$ |
| 6 | $\neg Ra$ | PREM | $\emptyset$ |
| 7 | $\neg Pa$ | 4,6;RU | $\emptyset$ |
| 8 | $(P \supset Q)^a$ | 1,2,7;RU | $\emptyset$ |

This implies that, at stage 8, the condition of line 3 is no longer reliable. Hence, line 3 and all lines that rely on it become marked at stage 8, as indicated by the $\checkmark^8$-sign. Because the formula on line 8 consists of a single abnormality, this abnormality will remain unreliable in every further extension of the proof, which implies that line 3 will remain marked in every such extension. Hence $Pa$ is not finally derived at line 3 of the above proof.

A hypothesis that is finally derivable from $\Gamma_{\mathsf{s2}}$ is, for instance, $Pb$:

| | | | |
|---|---|---|---|
| 9 | $Qb$ | PREM | $\emptyset$ |
| 10 | $Pb$ | 1,2;RC | $\{(P \supset Q)^b\}$ |

The above proof cannot be further extended in such a way that line 10 is marked. Hence, $Pb$ is an $\mathbf{LA}^{\mathbf{r}}_{\Box}$-consequence of $\Gamma_{\mathsf{s2}}$. This illustrates the fact that abduction, as modelled by $\mathbf{LA}^{\mathbf{r}}_{\Box}$, is case-sensitive: the failure of an abduction in one case does not imply that it also fails in similar cases.

The dynamic proof theory and the form of the abnormalities also ensure that certain unjustified hypotheses are not finally derivable. For instance, it is neither possible to finally derive random hypotheses from mere tautologies, nor to finally derive contradictory hypotheses from a given explanandum.[18] The following proof from $\Gamma_{\mathsf{s3}} = \{Sa\}$ illustrates how these mechanisms work:[19]

| | | | |
|---|---|---|---|
| 1 | $Pa \vee \neg Pa$ | -;RU | $\emptyset$ |
| 2 | $\Box\forall x(Rx \supset (Px \vee \neg Px))$ | -;RU | $\emptyset$ |
| 3 | $Ra$ | 1,2;RC | $\{(R \supset (P \vee \neg P))^a\}$ $\checkmark^{4-}$ |
| 4 | $(R \supset (P \vee \neg P))^a \check{\vee} (\neg R \supset (P \vee \neg P))^a$ | -;RU | $\emptyset$ |
| 5 | $Sa$ | PREM | $\emptyset$ |
| 6 | $\Box\forall x((Qx \wedge \neg Qx) \supset Sx)$ | -;RU | $\emptyset$ |
| 7 | $Qa \wedge \neg Qa$ | 5,6;RC | $\{((Q \wedge \neg Q) \supset S)^a\}$ $\checkmark^8$ |
| 8 | $((Q \wedge \neg Q) \supset S)^a$ | 5;RU | $\emptyset$ |

As the minimal *Dab*-formulas of the form $(B \supset (A \vee \neg A))^\alpha \check{\vee} (\neg B \supset (A \vee \neg A))^\alpha$ are theorems in $\mathbf{T}$, no explanation can ever be finally derived for a tautology of the form $A \vee \neg A$. Similar theorems can be found for other kinds of tautologies. Likewise, contradictory explanations $A$ can never be finally derived as an explanation for $B\alpha$, since for such $A$, $(A \supset B)^\alpha$ is always $\mathbf{T}$-derivable from $B\alpha$.

Another important property of $\mathbf{LA}^{\mathbf{r}}_{\Box}$ is that it provides the most *parsimonious* hypotheses. This requires some explanation. If "$Y$" suffices to explain "$X$", then we should not raise the explanatory hypothesis "$Y$ and $Z$". More generally, we want to derive only the logically weakest hypotheses that suffice to explain the explananda.[20]

[18] These properties were first shown in [107]; the reasoning there is analogous to the current one.

[19] The symbol $\checkmark^{4-}$ indicates that line 3 is marked from stage 4 onwards.

[20] As indicated by a referee, *logical* parsimony should be distinguished from *expressive* parsimony. For instance, if "$Y$ or $Z$" explains "$X$", than the explanatory hypothesis "$Y$" is expressively more parsimonious because it contains less different terms (assuming that $Y$ and $Z$ differ), but logically less parsimonious than the explanatory hypothesis "$Y$ or $Z$" because "$Y$" logically entails "$Y$ or $Z$".

Note that $\Box\forall x(Ax \supset Bx) \vdash_{\mathbf{T}} \Box\forall x((Ax \wedge Cx) \supset Bx)$ — this property is known under the name *Strengthening of the Antecedent*. However, if $A$-hood suffices to explain $B$-hood, then we should not derive $A\alpha \wedge C\alpha$ from a given premise $B\alpha$. The logic $\mathbf{LA}^{\mathbf{r}}_{\Box}$ handles this problem in the following way:

| | | | |
|---|---|---|---|
| 1 | $Pa$ | PREM | $\emptyset$ |
| 2 | $\Box\forall x(Qx \supset Px)$ | PREM | $\emptyset$ |
| 3 | $\Box\forall x((Qx \wedge Rx) \supset Px)$ | 2;RU | $\emptyset$ |
| 4 | $Qa \wedge Ra$ | 1,3;RC | $\{((Q \wedge R) \supset P)^a\}$ $\checkmark^5$ |
| 5 | $((Q \wedge R) \supset P)^a \,\check{\vee}((Q \wedge \neg R) \supset P)^a$ | 2;RU | $\emptyset$ |

More generally, minimal *Dab*-formulas of the form $((A \wedge C) \supset B)^\alpha \,\check{\vee}((A \wedge \neg C) \supset B)^\alpha$ prohibit that any random strengthening of a hypothesis can be derived.

The final feature which we will further exemplify is how this logic handles multiple explanatory hypotheses. Consider, for example, the premise set:

$$\Gamma_{\mathsf{s4}} = \{Ra, \Box\forall x(Px \supset Rx), \Box(\forall x)(Qx \supset Rx)\}$$

At first sight, both $Pa$ and $Qa$ can be derived as hypotheses for $Ra$. But, as shown in the proof below, these two hypotheses are not finally derivable from $\Gamma_{\mathsf{s4}}$, as their condition is unreliable in view of this premise set. Only the disjunction of $Pa$ and $Qa$ is finally derivable.

| | | | |
|---|---|---|---|
| 1 | $\Box(\forall x)(Px \supset Rx)$ | PREM | $\emptyset$ |
| 2 | $\Box(\forall x)(Qx \supset Rx)$ | PREM | $\emptyset$ |
| 3 | $Ra$ | PREM | $\emptyset$ |
| 4 | $Pa$ | 1,3;RC | $\{(P \supset R)^a\}$ $\checkmark^6$ |
| 5 | $Qa$ | 1,3;RC | $\{(Q \supset R)^a\}$ $\checkmark^7$ |
| 6 | $(P \supset R)^a \,\check{\vee}((Q \wedge \neg P) \supset R)^a$ | 1,2;RU | $\emptyset$ |
| 7 | $(Q \supset R)^a \,\check{\vee}((P \wedge \neg Q) \supset R)^a$ | 1,2;RU | $\emptyset$ |
| 8 | $\Box(\forall x)((Px \vee Qx) \supset Rx)$ | 1,2;RU | $\emptyset$ |
| 9 | $Pa \vee Qa$ | 3,8;RC | $\{((P \vee Q) \supset R)^a\}$ |

Hence, whenever two hypotheses $A\alpha$ and $B\alpha$ can explain a certain fact $C\alpha$, the two individual hypotheses will not be finally derivable, since the minimal *Dab*-formulas of the form $(A \supset C)^\alpha \,\check{\vee}((B \wedge \neg A) \supset C)^\alpha$ and $(B \supset C)^\alpha \,\check{\vee}((A \wedge \neg B) \supset C)^\alpha$ will be derivable. However, the abnormality $((A \vee B) \supset C)^\alpha$ will remain reliable, whence we can derive the explanation $A\alpha \vee B\alpha$. Similar results can easily be obtained for three or more available explanations: whenever there are multiple explanatory hypotheses, $\mathbf{LA}^{\mathbf{r}}_{\Box}$ only allows us to finally derive their disjunction. This again illustrates the fact that $\mathbf{LA}^{\mathbf{r}}_{\Box}$ only delivers the most parsimonious hypotheses.

## 8.5 Some Prototypical Examples

In this section we will discuss some prototypical premise sets, which represent variants of the chocolate example from the introduction. These will allow us to illustrate the differences and similarities between the logics for AG that are presented in the two subsequent sections.[21]

Let $P$ stand for "is made of chocolate", $Q$ for "tastes sweet", and $R$ for "contains sugar". Suppose we know that everything made of chocolate tastes sweet ($\forall x(Px \supset Qx)$), and that the fact that an object contains sugar explains why it tastes sweet ($\Box\forall x(Rx \supset Qx)$). It can be assumed that we know of at least one object, say $a$, that it is made of chocolate (and, hence, tastes sweet). Moreover, as we consider the sweetness of chocolate to be in need of explanation, we can safely assume that we know of at least one object, say $b$, that does not taste sweet. The formalization of this example gives us our first prototypical premise set:

$$\Gamma_{\mathsf{c1}} = \{\forall x(Px \supset Qx), \Box\forall x(Rx \supset Qx), Pa, \neg Qb\}.$$

Note that $\Gamma_{\mathsf{c1}} \vdash_{\mathbf{T}} \{Qa, \neg Pb, \neg Rb\}$. In this first case, each of the logics which we will present below allows us to finally derive $\forall x(Px \supset Rx)$, which is as expected.

Let us now consider some variations on the first prototypical example. First, if we come to know of an object $c$ that tastes sweet but contains no sugar, should we still be able to derive the hypothesis $\forall x(Px \supset Rx)$? And, second, should it make a difference if we would also know that this object $c$ is not made of chocolate? The following extensions of $\Gamma_{\mathsf{c1}}$ formalize these two cases:

$$\Gamma_{\mathsf{c2}} = \Gamma_{\mathsf{c1}} \cup \{Qc, \neg Rc\}$$

$$\Gamma_{\mathsf{c3}} = \Gamma_{\mathsf{c1}} \cup \{Qc, \neg Rc, \neg Pc\}$$

Now suppose that we learn about a specific brand of chocolate for diabetics, which does not contain sugar. This can be modeled by the following premise set:

$$\Gamma_{\mathsf{c4}} = \Gamma_{\mathsf{c3}} \cup \{Pd, \neg Rd\}$$

Obviously, in this case, we do not want to be able to finally derive the hypothesis that all chocolate contains sugar. In view of the syntactic reassurance of adaptive logics (see Part I of this thesis), we can immediately infer that none of the logics presented below yields such a (trivializing) result. But even in this case, one might wonder whether it should still be allowed that we derive the singular hypothesis $Ra$ as an explanation for $Qa$. As we will see, $\mathbf{LA}^{\mathbf{r}}_{\forall}$ does not allow us to derive $Qa$ from $\Gamma_{\mathsf{c4}}$, whereas $\mathbf{SILA}^{\mathbf{r}}$ does.

The previous examples are all extensions of $\Gamma_{\mathsf{c1}}$ with singular facts. However, we may also try to see what happens if other explanations come into play, or in the terminology from Section 8.3, if we add generalizations to the explanatory framework of our abductive problem. For instance, we might learn that if something contains asparthane (a synthetic sweetener), then it also tastes sweet. Hence, at this point, we know that both sugar and asparthane can explain the sweetness of chocolate. So where $S$ represents "contains asparthane", we have:

[21] In the premise sets $\Gamma_{\mathsf{c1}} - \Gamma_{\mathsf{c5}}$ presented here, singular facts are represented by means of named objects, e.g. $Pa, \neg Qb, Qc, \neg Rc$. Although we will not discuss them here, similar results were obtained for analogous premise sets that use existentially quantified formulae like e.g. $\exists x Px, \exists x \neg Qx, \exists x(Qx \wedge \neg Rx)$.

$\Gamma_{\mathsf{c5}} = \Gamma_{\mathsf{c1}} \cup \{\Box\forall x(Sx \supset Qx)\}$

As we will see below, in this case, both $\mathbf{LA}^{\mathbf{r}}_{\forall}$ and $\mathbf{SILA}^{\mathbf{r}}$ will lead to the conclusion that all chocolate contains either sugar or asparthane. Put differently, just as $\mathbf{LA}^{\mathbf{r}}_{\Box}$, $\mathbf{LA}^{\mathbf{r}}_{\forall}$ and $\mathbf{SILA}^{\mathbf{r}}$ only deliver the most parsimonious explanation, relative to a given explanandum and explanatory framework.

## 8.6 The Logic $\mathbf{LA}^{\mathbf{r}}_{\forall}$

**The General Idea** The first logic for abduction of generalizations is also the most straightforward one. It is called $\mathbf{LA}^{\mathbf{r}}_{\forall}$, where the subscript refers to the crucial role of universally quantified formulas in AG. The motor behind $\mathbf{LA}^{\mathbf{r}}_{\forall}$ is the conditional application of the inference schema of AG in its most obvious way, as presented in Section 8.2. In our modal translation, this gives us the following rule:

$$\text{RAG}: \text{ from } \forall x(Ax \supset Bx), \Box\forall x(Cx \supset Bx), \text{ infer } \forall x(Ax \supset Cx)$$

To allow for the defeasible application of RAG, we need to define our set of abnormalities accordingly. This requires some notational conventions. First, we introduce the following abbreviation:

$$A \not\rightharpoonup_C B =_{\mathsf{df}} \forall(A \supset B) \wedge \Box\forall(C \supset B) \wedge \neg\forall(A \supset C)$$

According to this definition, $A \not\rightharpoonup_C B$ can be read as: "although all $A$ are $B$, and although $C$-hood explains $B$-hood, it is not the case that all $A$ are $C$." Using this abbreviation, we define the set of abnormalities of $\mathbf{LA}^{\mathbf{r}}_{\forall}$ as follows:

$$\Omega_{\mathbf{LA}^{\mathbf{r}}_{\forall}} =_{\mathsf{df}} \{A \not\rightharpoonup_C B \mid A, B, C \in \mathcal{F}^{\circ}, A\|B \text{ and } B\|C\}$$

As in Section 8.4, the restrictions $A\|B$ and $B\|C$ are added to avoid self-explanation, and to avoid that irrelevant abnormalities block the final derivability of expected hypotheses – this will be explained in the next paragraph.[22]

The logic $\mathbf{LA}^{\mathbf{r}}_{\forall}$ is defined by the triple $\langle \mathbf{T}, \Omega_{\mathbf{LA}^{\mathbf{r}}_{\forall}}, \mathbf{r}\rangle$. To get an idea of how it works, consider the prototypical premise set $\Gamma_{\mathsf{c1}} = \{\forall x(Px \supset Qx), \Box\forall x(Rx \supset Qx), Pa, \neg Qb\}$. The adaptive logic allows us to derive the hypothesis $\forall x(Px \supset Rx)$, as happens in the following proof:

| | | | |
|---|---|---|---|
| 1 | $\forall x(Px \supset Qx)$ | PREM | $\emptyset$ |
| 2 | $\Box\forall x(Rx \supset Qx)$ | PREM | $\emptyset$ |
| 3 | $\forall x(Px \supset Rx)$ | 1,2;RC | $\{P \not\rightharpoonup_R Q\}$ |

[22] It need not be imposed that the subformulas $A$ and $C$ of an abnormality do not share any predicates. To see why, consider the garage-example from the introduction. Let $I$ stand for "is made of iron", $G$ for "is located in my garage", $C$ for "is corroded", and $W$ for "has been wet". Suppose we know, in addition, that there is at least one wooden object $c$ in our garage that has not been wet. Then this example can be modeled by $\Gamma_{\mathsf{g}} = \{\forall x((Ix \wedge Gx) \supset Cx), \Box\forall x((Ix \wedge Wx) \supset Cx), Ia, Ga, \neg Cb, Gc \wedge \neg Wc \wedge \neg Cc \wedge \neg Ic\}$. In this case, we need the abnormality $(I \wedge G) \not\rightharpoonup_{(I \wedge W)} C$ to finally derive the hypothesis that all *iron* objects in this garage have been wet.

At line 3 of the above proof, $\forall x(Px \supset Rx)$ is finally $\mathbf{LA}^{\mathbf{r}}_{\forall}$-derived from $\Gamma_{\mathsf{c1}}$. Hence $\mathbf{LA}^{\mathbf{r}}_{\forall}$ allows us to derive the hypothesis $\forall x(Px \supset Rx)$ in a very simple way, through a direct application of the reasoning pattern that is characteristic for AG.

**The Restrictions on the Abnormalities** Let us now explain why the abnormalities are defined in such a way that $A$ and $B$, but also $B$ and $C$ share no predicates. The following continuation of the proof illustrates the need for the first restriction:

| | | | |
|---|---|---|---|
| 4 | $Pa$ | PREM | $\emptyset$ |
| 5 | $\Box\forall x((Qx \wedge \neg Rx) \supset Qx)$ | -;RU | $\emptyset$ |
| 6 | $(Pa \wedge \neg Ra) \check{\vee} (Pa \wedge \neg(Qa \wedge \neg Ra))$ | 4;RU | $\emptyset$ |
| 7 | $\neg\forall x(Px \supset Rx) \check{\vee} \neg\forall x(Px \supset (Qx \wedge \neg Rx))$ | 6;RU | $\emptyset$ |
| 8 | $(P \not\rightarrow_R Q) \check{\vee} (P \not\rightarrow_{(\neg R \wedge Q)} Q)$ | 1,2,5,7;RU | $\emptyset$ |

Consider the formula on the last line of the above proof. If $P \not\rightarrow_{(\neg R \wedge Q)} Q$ would be an abnormality, then line 3 would be marked at stage 8 of this proof, and $\forall x(Px \supset Rx)$ would not be finally derivable from $\Gamma_{\mathsf{c1}}$. However, since $Q$ and $\neg R \wedge Q$ share a predicate, viz. $Q$, the formula $P \not\rightarrow_{(\neg R \wedge Q)} Q$ is not an abnormality, and hence the formula on line 8 is not a Dab-formula in the logic $\mathbf{LA}^{\mathbf{r}}_{\forall}$.

To explain the second restriction, i.e. $B \| C$ in the definition of $\Omega_{\mathbf{LA}^{\mathbf{r}}_{\forall}}$, consider $\Gamma_{\mathsf{c2}} = \{\forall x(Px \supset Qx), \Box\forall x(Rx \supset Qx), Pa, \neg Qb, Qc, \neg Rc\}$. Note that although $R$-hood explains $Q$-hood according to these premises, it is nevertheless the case that $c$ has property $Q$ but not property $R$. There are two possible ways to deal with this situation: either one considers this as a sufficient argument to withdraw the hypothesis $\forall x(Px \supset Rx)$, or one sticks to this hypothesis and concludes that $\neg Pc$ is the case. The logic $\mathbf{LA}^{\mathbf{r}}_{\forall}$ takes the second option, but it can only do so given the second restriction on the abnormalities.

As before, we illustrate this fact by means of an object-level proof. Since $\Gamma_{\mathsf{c1}} \subset \Gamma_{\mathsf{c2}}$, we can continue the above proof and add the premises $Qc, \neg Rc$:

| | | | |
|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 9 | $Qc$ | PREM | $\emptyset$ |
| 10 | $\neg Rc$ | PREM | $\emptyset$ |
| 11 | $Pc \check{\vee} \neg Pc$ | -;RU | $\emptyset$ |
| 12 | $\exists x(Px \wedge \neg Rx) \check{\vee} \exists x(\neg Px \wedge Qx \wedge \neg Rc)$ | 9,10,11;RU | $\emptyset$ |
| 13 | $\forall x((\neg Px \wedge Qx) \supset Qx)$ | -;RU | $\emptyset$ |
| 14 | $(P \not\rightarrow_R Q) \check{\vee} ((\neg P \wedge Q) \not\rightarrow_R Q)$ | 2,12,13;RU | $\emptyset$ |
| 15 | $\neg Pc$ | 3,10;RU | $\{P \not\rightarrow_R Q\}$ |

It can easily be verified that the disjunction on line 14 is minimal, i.e. that none of its disjuncts can be $\mathbf{T}$-derived from $\Gamma_{\mathsf{c2}}$. However, given the above definition of $\Omega_{\mathbf{LA}^{\mathbf{r}}_{\forall}}$, the second disjunct of the formula on line 8 is not an abnormality, whence $(P \not\rightarrow_R Q)$ is a reliable abnormality at stage 15. Hence, tautologies such as $\forall x((\neg Px \wedge Qx) \supset Qx)$ (line 13) do not allow us to block the derivation of the explanatory hypothesis $\forall x(Px \supset Rx)$ from $\Gamma_{\mathsf{c2}}$.

Note that $\neg Pc$ is finally derived on line 15 of the above proof, and hence $\Gamma_{\mathsf{c2}} \vdash_{\mathbf{LA}^{\mathbf{r}}_{\forall}} \neg Pc$. As a result, the sets $\Gamma_{\mathsf{c2}}$ and $\Gamma_{\mathsf{c3}}$ are $\mathbf{LA}^{\mathbf{r}}_{\forall}$-equivalent.[23]

**Dynamics of the Proof Theory** The dynamic aspect of the $\mathbf{LA}^{\mathbf{r}}_{\forall}$-proof theory can be illustrated by means of the premise set $\Gamma_{\mathsf{c4}}$. Since this premise set is an extension of $\Gamma_{\mathsf{c2}}$, we can again continue the same proof and add the premises $Pd, \neg Rd$:

| | | | |
|---|---|---|---|
| 1 | $\forall x((Ix \wedge Gx) \supset Cx)$ | PREM | $\emptyset$ |
| 2 | $\Box\forall x((Ix \wedge Wx) \supset Cx)$ | PREM | $\emptyset$ |
| 3 | $\forall x(Px \supset Rx)$ | 2;RC | $\{P \nrightarrow_R Q\}$ ✓[19] |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 16 | $Pd$ | PREM | $\emptyset$ |
| 17 | $\neg Rd$ | PREM | $\emptyset$ |
| 18 | $\neg\forall x(Px \supset Rx)$ | 16,17;RU | $\emptyset$ |
| 19 | $P \nrightarrow_R Q$ | 1,2,18;RU | $\emptyset$ |

We have derived the Dab-formula $P \nrightarrow_R Q$ at line 19, which renders line 3 marked. Since the Dab-formula at line 19 only contains one disjunct, line 3 will remain marked in every extension of the proof.

**Multiple Explanatory Hypotheses** As announced, $\mathbf{LA}^{\mathbf{r}}_{\forall}$ only allows us to derive a "disjunctive" general hypothesis, if two or more explanations can be found for a certain property that is shared by all members of a class. This can be illustrated by means of the example $\Gamma_{\mathsf{c5}} = \{\forall x(Px \supset Qx), \Box\forall x(Rx \supset Qx), \Box\forall x(Sx \supset Qx), Pa, \neg Qb\}$.

At first sight, both the hypotheses $\forall x(Px \supset Rx)$ and $\forall x(Px \supset Sx)$ can be derived from $\Gamma_{\mathsf{c5}}$. But, as shown in the proof below, these two formulas are not finally derivable. The hypothesis $\forall x(Px \supset (Rx \vee Sx))$ is, however, finally derivable from $\Gamma_{\mathsf{c5}}$.[24]

| | | | |
|---|---|---|---|
| 1 | $\forall x(Px \supset Qx)$ | PREM | $\emptyset$ |
| 2 | $\Box\forall x(Rx \supset Qx)$ | PREM | $\emptyset$ |
| 3 | $\Box\forall x(Sx \supset Qx)$ | PREM | $\emptyset$ |
| 4 | $\forall x(Px \supset Rx)$ | 1,2;RC | $\{P \nrightarrow_R Q\}$ ✓[9] |
| 5 | $\forall x(Px \supset Sx)$ | 1,3;RC | $\{P \nrightarrow_S Q\}$ ✓[10] |
| 6 | $Pa$ | PREM | $\emptyset$ |
| 7 | $\neg\forall x(Px \supset Rx) \vee \neg\forall x(Px \supset (Sx \wedge \neg Rx))$ | 6;RU | $\emptyset$ |
| 8 | $\Box\forall x((Sx \wedge \neg Rx) \supset Qx)$ | 3;RU | $\emptyset$ |
| 9 | $(P \nrightarrow_R Q) \check{\vee} (P \nrightarrow_{S\wedge\neg R} Q)$ | 1,2,7,8;RU | $\emptyset$ |
| 10 | $(P \nrightarrow_S Q) \check{\vee} (P \nrightarrow_{R\wedge\neg S} Q)$ | 1,2,3,6;RU | $\emptyset$ |
| 11 | $\Box\forall x((Rx \vee Sx) \supset Qx)$ | 2,3;RU | $\emptyset$ |
| 12 | $\forall x(Px \supset (Rx \vee Sx))$ | 1,11;RC | $\{P \nrightarrow_{R\vee S} Q\}$ |

[23] As shown in Chapter 2, two premise sets $\Gamma$ and $\Gamma'$ are $\mathbf{AL}$-equivalent whenever $\Gamma \vdash_{\mathbf{AL}} \Gamma'$ and $\Gamma' \vdash_{\mathbf{AL}} \Gamma$. In the current case, we have $\Gamma_{\mathsf{c3}} \vdash_{\mathbf{LA}^{\mathbf{r}}_{\forall}} \Gamma_{\mathsf{c2}}$ by the reflexivity of $\mathbf{LA}^{\mathbf{r}}_{\forall}$, and $\Gamma_{\mathsf{c2}} \vdash_{\mathbf{LA}^{\mathbf{r}}_{\forall}} \Gamma_{\mathsf{c3}}$, since $\Gamma_{\mathsf{c2}} \vdash_{\mathbf{LA}^{\mathbf{r}}_{\forall}} \neg Pc$ and $\Gamma_{\mathsf{c3}} - \Gamma_{\mathsf{c2}} = \{\neg Pc\}$.

[24] Line 10 is derived in a way analogous to the derivation of line 9 – note that the predicates $R$ and $S$ are symmetric on $\Gamma_{\mathsf{c5}}$.

Although we omit illustrations for reasons of space, it can easily be shown that $\mathbf{LA}^{\mathbf{r}}_{\forall}$ also does not allow us to finally derive self-contradictory hypotheses, or random hypotheses from a tautological explanandum such as $\forall x(Px \supset Px)$. Moreover, $\mathbf{LA}^{\mathbf{r}}_{\forall}$ avoids the problem of strengthening the antecedent in a way similar to $\mathbf{LA}^{\mathbf{r}}_{\forall}$. For instance,

$$\Gamma_{\mathsf{c1}} \vdash_{\mathbf{T}} (P \not\rightarrow_{R\wedge S} Q) \check{\vee} (P \not\rightarrow_{R\wedge\neg S} Q)$$

Hence, we cannot finally $\mathbf{LA}^{\mathbf{r}}_{\forall}$-derive either $\forall x(Px \supset (Rx \wedge Sx))$ or $\forall x(Px \supset (Rx \wedge \neg Sx))$ from $\Gamma_{\mathsf{c2}}$.

## 8.7 The Logic SILA$^{\mathbf{r}}$

### 8.7.1 The General Idea

As mentioned in Section 8.2, Thagard incorporates a pattern that is similar to AG in his logic program PI, by allowing for the inductive generalization of abduced singular hypotheses – see [145, pp. 58-60]. The same approach to AG is proposed by Flach and Kakas in their [52, Chapter 1]. Consider again the chocolate-example as modeled by the premise set $\Gamma_{\mathsf{c1}}$. Flach and Kakas' suggestion is spelled out in Table 8.1.

| | |
|---|---|
| (P1) $(\forall x)(Px \supset Qx)$ | All chocolate tastes sweet. |
| (P2) $\Box(\forall x)(Rx \supset Qx)$ | Everything that contains sugar, tastes sweet. |
| (P3) $Pa$ | $a$ is a made of chocolate. |
| $Qa$ | $a$ tastes sweet. (by Classical Logic) |
| $Ra$ | $a$ contains sugar. (by Singular Fact Abduction) |
| $(\forall x)(Px \supset Rx)$ | All chocolate contains sugar. (by Inductive Generalization) |

Table 8.1: The Flach&Kakas-Reconstruction of AG.

AG is thus reduced to a combination of singular fact abduction and inductive generalization. More specifically, the conclusion of the AG is obtained via the assumption that the explanation for $a$'s being $Q$ can be generalized to the whole class of objects that are $P$.

According to Schurz [131, p. 212], the decomposition of AG into singular fact abduction and inductive generalization is "somewhat artificial":[25]

> Law-abductions are usually performed in one single conjectural step. We don't form the abductive hypothesis of containment of sugar for each observed pineapple, one after the other, and then generalize it, but we form the law-conjecture "pineapples contain sugar" at once.

[25] In Schurz' example, the explanandum is the fact that all pineapples taste sweet.

Nevertheless, it is useful to see whether the logical combination of singular fact abduction and inductive generalization results in interesting observations, and where it differs from the more direct approach to AG as constituted by $\mathbf{LA}^{\mathbf{r}}_{\forall}$. Moreover, as will be explained below, the way we characterize the inductive step in the reconstruction of AG slightly differs from what Schurz suggests: we need not perform a whole series of singular fact abductions in order to be able to generalize these. In our model, a single instance of a singular fact abduction may already provide the basis for an induction, which gives us the conclusion of an AG. Still, this does not lead to all an excessive amount of finally derivable generalizations, as we will explain below.

In the remainder of this section, we will first introduce the adaptive logic $\mathbf{IM}^{\mathbf{r}}$ for the inductive generalization of observed facts (Section 8.7.2). Next, we will argue that in order to obtain a sensible logic for AG, one has to combine $\mathbf{LA}^{\mathbf{r}}_{\Box}$ and $\mathbf{IM}^{\mathbf{r}}$ in a prioritized way (Section 8.7.3). We will briefly discuss the proof theory of the resulting system (Section 8.7.4), and finally show how it handles the prototypical examples (Section 8.7.5).

## 8.7.2 Adaptive Induction

Adaptive logics of inductive generalization are a well-studied branch of adaptive logics – see [17, 29, 24, 13] and [25, Chapter 3]. In fact, a wide range of such logics has been developed; we will only present a variation on one of the existing systems. As all other ALs for induction, this logic models *qualitative* inductive generalization – what this means will be specified below.

$\mathbf{IM}^{\mathbf{r}}$ is an adaptive logic in standard format, characterized by the triple $\langle \mathbf{T}, \Omega_{\mathbf{IM}^{\mathbf{r}}}, \mathbf{r}\rangle$, where[26]

$$\Omega_{\mathbf{IM}^{\mathbf{r}}} =_{\mathsf{df}} \{\exists A\alpha \wedge \exists\neg A\alpha \mid A \in \mathcal{F}^{\circ}\}$$

To lighten notation, these abnormalities will be abbreviated as follows: $!A = \exists A \wedge \exists\neg A$.[27]

To see how $\mathbf{IM}^{\mathbf{r}}$ allows us to obtain universal generalizations from singular facts, note that $\exists\alpha A\alpha \vdash_{\mathbf{CL}} \forall\alpha A\alpha \vee (\exists\alpha A\alpha \wedge \exists\alpha\neg A\alpha)$. Hence whenever $\Gamma$ entails an instance of the (possibly complex) formula $A$, we can derive the hypothesis $\forall x Ax$ in an $\mathbf{IM}^{\mathbf{r}}$-proof from $\Gamma$, on the condition $\{!A\}$.

Consider the following proof from $\Gamma_{\mathsf{i}} = \{Pa, Pa \supset Qa, \neg Pb, \neg Qb \wedge Rc, \neg Qc\}$:

| | | | |
|---|---|---|---|
| 1 | $Pa$ | PREM | $\emptyset$ |
| 2 | $Pa \supset Qa$ | PREM | $\emptyset$ |
| 3 | $\neg Pb$ | PREM | $\emptyset$ |
| 4 | $\neg Qb \wedge Rc$ | PREM | $\emptyset$ |
| 5 | $\neg Qc$ | PREM | $\emptyset$ |

From $Pa \supset Qa$, we may can $\mathbf{IM}^{\mathbf{r}}$-derive the inductive hypothesis that all $P$ are $Q$ as follows:

[26] For the motivation of the requirement that $A \in \mathcal{F}^{\circ}$, see [17, p. 11-12].

[27] $\mathbf{IM}^{\mathbf{r}}$ is a variant of the logic $\mathbf{IL}^{\mathbf{r}}$ from [29], which has the same set of abnormalities but $\mathbf{CL}$ as its lower limit logic. Hence $\mathbf{IM}^{\mathbf{r}}$ is a conservative extension of $\mathbf{IL}^{\mathbf{r}}$ to the modal framework, in that for non-modal premise sets $\Gamma$, $Cn_{\mathbf{IL}^{\mathbf{r}}}(\Gamma) = Cn_{\mathbf{IM}^{\mathbf{r}}}(\Gamma)$.

| | | | |
|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 6 | $\exists x(Px \supset Qx)$ | 2;RU | $\emptyset$ |
| 7 | $\forall x(Px \supset Qx) \check{\vee} !(P \supset Q)$ | 6;RU | $\emptyset$ |
| 8 | $\forall x(Px \supset Qx)$ | 7;RC | $\{!(P \supset Q)\}$ |

One might think that, since $\Gamma_{\mathsf{i}}$ does not entail a counterinstance of $\forall x(Px \supset Qx)$, this generalization will be finally derivable. However, we can derive a Dab-formula from this premise set that blocks the above derivation:

| | | | |
|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 8 | $\forall x(Px \supset Qx)$ | 7;RC | $\{!(P \supset Q)\}$ $\checkmark^{11}$ |
| 9 | $\neg Pc \check{\vee} Pc$ | -;RU | $\emptyset$ |
| 10 | $(\neg Pc \wedge Rc) \check{\vee} (Pc \wedge \neg Qc)$ | 4,5,9;RU | $\emptyset$ |
| 11 | $!(P \vee \neg R) \check{\vee} !(P \supset Q)$ | 1,2;10;RU | $\emptyset$ |

As indicated, line 8 is marked at stage 11. Since the formula on line 11 is a minimal Dab-consequence of $\Gamma_{\mathsf{i}}$, line 8 will remain marked in every extension of the proof.

This points to a very basic aspect of all adaptive logics of induction: all these yield a set of generalizations that are *mutually* compatible with the premise set. As the above example suggests, and as explained in much more detail in [24], this results in a rather small set of finally derivable generalizations.

At the beginning of this section, we said that $\mathbf{IM^r}$ models qualitative inductive generalization. This can most easily be understood in view of a concrete example. Consider $\Gamma'_{\mathsf{i}} = \Gamma_{\mathsf{i}} \cup \{Pd, Qd\}$, $\Gamma''_{\mathsf{i}} = \Gamma_{\mathsf{i}} \cup \{Pd, Qd, Pe, Qe\}$, .... $\mathbf{IM^r}$ yields the same generalizations for all these premise sets as those that are finally derivable from $\Gamma_{\mathsf{i}}$. Hence, $\mathbf{IM^r}$ does not take into account the number of instances of a specific formula $A \in \mathcal{F}_\circ$ – all that matters is which combinations of predicates occur together in the **CL**-consequence set of the premises.

This has a rather important consequence for the application to AG: unlike what is suggested by Schurz' quote from the preceding section, one successful singular fact abduction suffices to conditionally derive, by $\mathbf{IM^r}$, the generalization that explains the observed regularity. So we may restrict ourselves to a singular fact abduction about one "prototypical" piece of chocolate, and immediately generalize the result of this abduction to everything that is made of chocolate.

### 8.7.3 The Need to Prioritize

As explained in Part I of this thesis, there are several ways to combine adaptive logics in standard format. In the current section, we will only consider two of them: (i) the combination of two ALs that results in a new flat AL, and (ii) the superposition of two ALs. More specifically, we will explain why (i) does not lead to an adequate logic for AG, and next show that (ii) provides an interesting account of AG. Notice that both (i) and (ii) require that the ALs to be combined share the same lower limit logic and strategy. This immediately explains why we extended the logic $\mathbf{IL^r}$ to the modal framework, obtaining $\mathbf{IM^r}$, which has the same lower limit logic as $\mathbf{LA^r_\Box}$.

The idea behind a flat combination of two logics $\mathbf{AL_1} = \langle \mathbf{LLL}, \Omega_1, \mathbf{r}\rangle$ and $\mathbf{AL_2} = \langle \mathbf{LLL}, \Omega_2, \mathbf{r}\rangle$ was already mentioned in Chapter 2: simply define the combination as a flat adaptive logic $\mathbf{AL} = \langle \mathbf{LLL}, \Omega_1 \cup \Omega_2, \mathbf{r}\rangle$. The resulting system is an AL in standard format, whence all the generic definitions, proof theoretic rules and metatheory can be transferred to **AL**.

Let us apply this technique to $\mathbf{LA^r_\Box}$ and $\mathbf{IM^r}$. We then get a logic which we dub $\mathbf{ILA^r_\Box}$, and which takes as its set of abnormalities $\Omega_\cup = \Omega_{\mathbf{LA^r_\Box}} \cup \Omega_{\mathbf{IM^r}}$. Thus, whenever we derive $A \vee Dab(\Delta)$ in an $\mathbf{ILA^r_\Box}$-proof from $\Gamma$, for a $\Delta \subseteq \Omega_\cup$, then we may push $\Delta$ to the condition on the next line of this proof. Similarly, $Dab(\Theta)$ is a Dab-formula in a proof iff $\Theta \subseteq \Omega_\cup$.

However, such an approach does not allow us to model (interesting cases of) AG, as we will now show. Let $\Gamma_{\mathsf{p}} = \{\forall x(Px \supset Qx), \Box\forall x(Rx \supset Qx), Pa, \neg Qb, \neg Pc \wedge Rc\}$. In line with the preceding, we can interpret this premise set as follows:

- "All chocolate tastes sweet"
- "Whatever contains sugar, tastes sweet"
- "$a$ is a piece of chocolate"
- "$b$ does not taste sweet"
- "$c$ is not a piece of chocolate, but $c$ contains sugar"

This premise set seems to correspond to a realistic situation. The knowledge that at least some things do not taste sweet seems a prerequisite to even start wondering why chocolate tastes sweet. Also, the fact that there other things than chocolate which contain sugar merely underlines the idea that $R$ is not just an accidental property, and that it allows us to explain the behavior of certain (classes of) objects.

Note that each of the following are **T**-consequences of $\Gamma_{\mathsf{p}}$: $Qa, \neg Pb, \neg Rb, Qc$. This means that for all objects and predicates mentioned in $\Gamma_{\mathsf{p}}$, only the value of $Ra$ remains undecided: $\Gamma_{\mathsf{p}} \nvdash_{\mathbf{T}} Ra$ and $\Gamma_{\mathsf{p}} \nvdash_{\mathbf{T}} \neg Ra$. But this means that we can derive a minimal Dab-consequence from this premise set, as follows:

| | | | |
|---|---|---|---|
| 1 | $\forall x(Px \supset Qx)$ | PREM | $\emptyset$ |
| 2 | $\Box\forall x(Rx \supset Qx)$ | PREM | $\emptyset$ |
| 3 | $Pa$ | PREM | $\emptyset$ |
| 4 | $\neg Qb$ | PREM | $\emptyset$ |
| 5 | $\neg Pc \wedge Rc$ | PREM | $\emptyset$ |
| 6 | $Ra \,\check{\vee}\, \neg Ra$ | -;RU | $\emptyset$ |
| 7 | $(Ra \wedge Pa) \,\check{\vee} (Qa \wedge \neg Ra)$ | 3,6;RU | $\emptyset$ |
| 8 | $((Rc \supset \neg Pc) \wedge \neg(Ra \supset \neg Pa)) \,\check{\vee} (\Box\forall x(Rx \supset Qx) \wedge Qa \wedge \neg Ra)$ | 2,5,7;RU | $\emptyset$ |
| 9 | $!(R \supset \neg P) \,\check{\vee} (R \supset Q)^a$ | 8;RU | $\emptyset$ |

In a similar vein, we can derive the following minimal disjunction of $\mathbf{IM^r}$-abnormalities from $\Gamma_{\mathsf{p}}$:

$$!(R \supset \neg P) \,\check{\vee} !(P \supset R) \tag{8.2}$$

In view of the formula on line 9 of the above proof, we cannot make the singular fact abduction that leads to $Ra$. But even if there was a way to circumvent this problem, then we would still not be able to generalize $Pa \wedge Ra$ and hence finally derive $\forall x(Px \supset Rx)$, in view of (8.2).

The same problem remains when we replace $\mathbf{IM^r}$ by another logic from the class discussed in [24]. For reasons of space, we cannot present a complete argument for this claim here, but let us briefly point out why this is the case. From the viewpoint of a logic for inductive generalization, $\Gamma_{\mathsf{p}}$ does not give us any more reasons to believe that $Ra$ is the case, than to believe that $\neg Ra$ is the case. That is, if $Ra$ is the case, then this contradicts the hypothesis $\forall x(Rx \supset \neg Px)$ – note that this hypothesis is not contradicted by the premise set, and that there is a positive instance for it. However, if we believe that $\neg Ra$ is the case, then we have the $\mathbf{LA}^{\mathbf{r}}_{\Box}$-abnormality $(R \supset Q)^a$. So at the end of the day, since abduction and induction are put on a par, the logic does not allow us to choose between either.

One might argue that this is mainly due to the incompleteness of the premise set $\Gamma_{\mathsf{p}}$. That is, in real life, we know of at least some object $d$ that is made of chocolate and contains sugar. Indeed, if we add e.g. $Pd, Rd$ to $\Gamma_{\mathsf{p}}$, then the problem will disappear, since then $!(R \supset \neg P)$ becomes derivable in itself. However, this addition begs the question whether abduction should play a role at all in the whole process, since then we get the information $\exists x(Px \supset Rx)$ for free. In other words, if we want to model AG as a process in which both abduction and induction play an active role, we should be able to handle cases such as $\Gamma_{\mathsf{p}}$.

A solution for this predicament is obtained by giving abduction priority over induction, in the reconstruction of AG. This is done by superposing $\mathbf{IM^r}$ on $\mathbf{LA}^{\mathbf{r}}_{\Box}$, whence the resulting consequence relation can be defined as follows:

$$Cn_{\mathbf{SILA^r}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{IM^r}}(Cn_{\mathbf{LA}^{\mathbf{r}}_{\Box}}(\Gamma))$$

Some readers might wonder why we use a superposition of logics, and not e.g. a hierarchic combination (see Chapter 4) or a lexicographic AL (see Chapter 5). Our motivation is twofold. On the one hand, using a hierarchic AL, we would be able to derive the hypothesis $Pa$ from $\Gamma_{\mathsf{p}}$, but we would not be able to generalize it, since $!(R \supset \neg P)$ would be in the set of unreliable formulas $U^{\star}(\Gamma)$ (in view of the minimal Dab-consequence 8.2). Put differently, the format $\mathbf{HAL^r}$ is still too weak to model AG in cases like $\Gamma_{\mathsf{p}}$. On the other hand, it is likely that a logic in the formats $\mathbf{SAL}^{\mathbf{r}}_{(\mathbf{I})}$ and $\mathbf{AL}^{\mathbf{r}}_{\sqsubset}$ would yield the expected consequences for this example. We chose the format $\mathbf{SAL^r}$ for the sake of simplicity, since this way, the logic can be immediately defined from $\mathbf{LA}^{\mathbf{r}}_{\Box}$ and $\mathbf{IM^r}$.

In the remainder of this section, we will show that $\mathbf{SILA^r}$ gives the intended outcome in the case of $\Gamma_{\mathsf{p}}$, and that it is also well-behaved in each of the prototypical examples $\Gamma_{\mathsf{c1}} - \Gamma_{\mathsf{c6}}$.

### 8.7.4 A Proof Theory for SILA$^{\mathbf{r}}$

To model reasoning with $\mathbf{SILA^r}$, we will use the second proof theory for superpositions of adaptive logics, which was proposed in Chapter 3, Section 3.5. Note that $\Omega_{\mathbf{LA}^{\mathbf{r}}_{\Box}} \cap \Omega_{\mathbf{IM^r}} = \emptyset$, whence we need not use sequences for conditions, but can simply use sets of abnormalities $\Delta \subset \Omega_{\mathbf{LA}^{\mathbf{r}}_{\Box}} \cup \Omega_{\mathbf{IM^r}}$. As explained in Chapter 3, this means that $\mathbf{SILA^r}$-proofs have the same outlook as $\mathbf{ILA}^{\mathbf{r}}_{\Box}$-proofs. The generic rules are such that conditions in a proof can consist of both abductive and inductive abnormalities, and that we can apply both singular fact abduction and inductive generalization at any point in the proof. The only difference consists in the marking of lines at every stage of the proof.

We first use this proof theory to illustrate that $\mathbf{SILA^r}$ gives the intended outcome in the case of $\Gamma_{\mathsf{p}} = \{\forall x(Px \supset Qx), \Box\forall x(Rx \supset Qx), Pa, \neg Qb, \neg Pc \wedge Rc\}$. The following $\mathbf{SILA^r}$-proof illustrates how we can derive the hypothesis $\forall x(Px \supset Rx)$ from these premises:

| | | | |
|---|---|---|---|
| 1 | $\forall x(Px \supset Qx)$ | PREM | $\emptyset$ |
| 2 | $\Box\forall x(Rx \supset Qx)$ | PREM | $\emptyset$ |
| 3 | $Pa$ | PREM | $\emptyset$ |
| 4 | $Qa$ | 1,3;RU | $\emptyset$ |
| 5 | $Ra$ | 4,2;RC | $\{(R \supset Q)^a\}$ |
| 6 | $\exists x(Px \supset Rx)$ | 5;RU | $\{(R \supset Q)^a\}$ |
| 7 | $\forall x(Px \supset Rx)$ | 6;RC | $\{(R \supset Q)^a, !(P \supset R)\}$ |

The priorities come into play where the marking is concerned. In line with the definitions from Chapter 3, we define Dab$_1$-formulas at stage $s$ as disjunctions of the members of $\Omega_{\mathbf{LA}^{\mathbf{r}}_{\Box}}$, derived on the empty condition. $Dab(\Delta)$ is a *minimal* Dab$_1$-formula at stage $s$ iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is a Dab$_1$-formula at stage $s$. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab$_1$-formulas at stage $s$, let $U^1_s =_{\mathsf{df}} \Delta_1 \cup \Delta_2 \cup \ldots$. The first step in the marking procedure is given by the following definition:

**Definition 8.1** *A line with condition* $\Delta$ *is* 1*-marked at stage* $s$ *iff* $\Delta \cap U^1_s(\Gamma) \neq \emptyset$.

We say that $Dab(\Delta)$ is a Dab$_2$-formula at stage $s$ iff $Dab(\Delta)$ is derived on the condition $\Theta \subset \Omega_{\mathbf{LA}^{\mathbf{r}}_{\Box}}$, on a line that is not 1-marked at stage $s$. We define the minimal Dab$_2$-formulas at stage $s$ and $U^2_s(\Gamma)$ accordingly. Next, we have:

**Definition 8.2** *A line with condition* $\Delta$ *is* 2*-marked at stage* $s$ *iff* $\Delta \cap U^2_s(\Gamma) \neq \emptyset$.

We say that a line is marked at stage $s$ iff it is either 1- or 2-marked at this stage. Derivability at a stage and final derivability are defined as in the standard format – see Definitions 3.8 and 3.9.

Let us continue the preceding proof to illustrate how the prioritized marking ensures that we can finally derive $\forall x(Px \supset Rx)$ from $\Gamma_{\mathsf{p}}$. First of all, we can derive two disjunctions of abnormalities, in view of the preceding subsection:

| | | | |
|---|---|---|---|
| 1 | $\forall x(Px \supset Qx)$ | PREM | $\emptyset$ |
| 2 | $\Box\forall x(Rx \supset Qx)$ | PREM | $\emptyset$ |
| 3 | $Pa$ | PREM | $\emptyset$ |
| 4 | $Qa$ | 1,3;RU | $\emptyset$ |
| 5 | $Ra$ | 4,2;RC | $\{(R \supset Q)^a\}$ |
| 6 | $\exists x(Px \supset Rx)$ | 5;RU | $\{(R \supset Q)^a\}$ |
| 7 | $\forall x(Px \supset Rx)$ | 6;RC | $\{(R \supset Q)^a, !(P \supset R)\}\checkmark^{11}_{2}$ |
| 8 | $Ra \vee \neg Ra$ | -;RU | $\emptyset$ |
| 9 | $\neg Pc \wedge Rc$ | PREM | $\emptyset$ |
| 10 | $!(R \supset \neg P) \check{\vee} (R \supset Q)^a$ | 2,3,8,9;RU | $\emptyset$ |
| 11 | $!(R \supset \neg P) \check{\vee} !(P \supset R)$ | 3,8,9;RU | $\emptyset$ |

Note however that only the formula on line 11 is a $\mathrm{Dab}_2$-formula; the formula on line 10 contains both kinds of abnormalities. This means that, on the one hand, $U^1_{11}(\Gamma_{\mathsf{p}}) = \emptyset$, whence line 6 remains unmarked. On the other hand, $U^2_{11}(\Gamma_{\mathsf{p}}) = \{!(R \supset \neg P), !(P \supset R)\}$. As a result, line 7 is 2-marked at stage 11. This problem is overcome by showing that the first member of the Dab-formula on line 11 can be derived in itself, using the abnormalities from $\Omega_{\mathbf{LA}^{\mathbf{r}}_{\Box}}$:

| | | | |
|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 7 | $\forall x(Px \supset Rx)$ | 6;RC | $\{(R \supset Q)^a, !(P \supset R)\}$ |
| 8 | $Ra \vee \neg Ra$ | -;RU | $\emptyset$ |
| 9 | $\neg Pc \wedge Rc$ | PREM | $\emptyset$ |
| 10 | $!(R \supset \neg P) \check{\vee} (R \supset Q)^a$ | 2,3,8,9;RU | $\emptyset$ |
| 11 | $!(R \supset \neg P) \check{\vee} !(P \supset R)$ | 3,8,9;RU | $\emptyset$ |
| 12 | $!(R \supset \neg P)$ | 10;RC | $\{(R \supset Q)^a$ |

Note that the formula on line 12 is a $\mathrm{Dab}_2$-formula at stage 12 (it is a disjunction of members of $\Omega_{\mathbf{LA}^{\mathbf{r}}_{\Box}}$, and it is derived on a line that is not 1-marked). Hence the formula on line 11 is not a minimal $\mathrm{Dab}_2$–formula at stage 12. As a result, the abnormality $!(P \supset R)$ is reliable at stage 12, whence line 7 is not 2-marked at this stage.

### 8.7.5 The Prototypical Examples

**The standard case** Recall the first prototypical premise set $\Gamma_{\mathsf{c1}} = \{\forall x(Px \supset Qx), \Box\forall x(Rx \supset Qx), Pa, \neg Qb\}$. As could be expected, $\Gamma_{\mathsf{c1}} \vdash_{\mathbf{SILA}^{\mathbf{r}}} \forall x(Px \supset Rx)$. The following proof illustrates how $\mathbf{SILA}^{\mathbf{r}}$ models the reduction of AG to a combination of singular fact abduction and inductive generalization, as proposed by Flach & Kakas:

| | | | |
|---|---|---|---|
| 1 | $\forall x(Px \supset Qx)$ | PREM | $\emptyset$ |
| 2 | $\Box\forall x(Rx \supset Qx)$ | PREM | $\emptyset$ |
| 3 | $Pa$ | PREM | $\emptyset$ |
| 4 | $\neg Qb$ | PREM | $\emptyset$ |
| 5 | $Qa$ | 1,3;RU | $\emptyset$ |
| 6 | $Ra$ | 2,5:RC | $\{(R \supset Q)^a\}$ |
| 7 | $\forall x(Px \supset Rx)$ | 3,6;RC | $\{(R \supset Q)^a, !(P \supset R)\}$ |

At stage 7 of the above proof, $\forall x(Px \supset Rx)$ is finally derived from $\Gamma_{\mathsf{c1}}$, which is as expected.

**Extensions of $\Gamma_{\mathsf{c1}}$** Let us now consider the cases modeled by $\Gamma_{\mathsf{c2}} = \Gamma_{\mathsf{c1}} \cup \{Qc, \neg Rc\}$. We can continue the above proof as follows:

| | | | |
|---|---|---|---|
| 7 | $\forall x(Px \supset Rx)$ | 3,6;RC | $\{(R \supset Q)^a, !(P \supset R)\}$ |
| 8 | $Qc$ | PREM | $\emptyset$ |
| 9 | $\neg Rc$ | PREM | $\emptyset$ |
| 10 | $\exists x(Px \supset Rx)$ | 3,7;RU | $\emptyset$ |
| 11 | $\exists x((\neg Px \wedge Qx) \supset Rx)$ | 6;RU | $\{(R \supset Q)^a\}$ |
| 12 | $\neg\forall x(Px \supset Rx) \check{\vee} \neg\forall x((\neg Px \wedge Qx) \supset Rx)$ | 8,9;RU | $\emptyset$ |

| | | | |
|---|---|---|---|
| 13 | $!(P \supset R) \check{\vee} !((\neg P \wedge Q) \supset R)$ | 10,11,12;RU | $\{(R \supset Q)^a\}$ |

At stage 13 of the above proof, line 7 becomes marked in view of the Dab$_2$-formula on line 13. Moreover, there is no way to render line 7 unmarked in an extension of the proof, and hence $\forall x(Px \supset Rx)$ cannot be finally $\mathbf{SILA^r}$-derived from $\Gamma_{c2}$.[28] How should we interpret this result?

From the viewpoint of the inductive logic, there is no more support for the generalization $\forall x(Px \supset Rx)$ than for $\forall x((\neg Px \wedge Qx) \supset Rx)$. If the former is true, than $\neg Pc$ holds, whence we can obtain a counterinstance to the latter. If the latter is true, then $Pc$ holds, which means that we have a negative instance of the former.

Nevertheless, if we add the premise $\neg Pc$ to $\Gamma_{c2}$, we *can* finally derive $\forall x(Px \supset Rx)$. The reason is that in this case the second disjunct of the Dab-formula on line 13 is derivable in itself. Hence, $\Gamma_{c3} \vdash_{\mathbf{SILA^r}} \forall x(Px \supset Rx)$. So another way to interpret the difference between $\mathbf{LA^r_\forall}$ and $\mathbf{SILA^r}$ with respect to $\Gamma_{c2}$, is that the latter system directs our search for additional data: it tells us to find out whether or not $c$ has property $P$. If it turns out that $Pc$ is the case, then the hypothesis $\forall x(Px \supset Rx)$ is falsified. If, however, $\neg Pc$ is the case, then we can again derive $\forall x(Px \supset Rx)$.

Another point to note is the fact that, even where AG fails, $\mathbf{SILA^r}$ still allows us to draw singular fact abductions. So for instance in the case of $\Gamma_{c4}$, we can still uphold the hypothesis $Ra$, even though the general hypothesis $\forall x(Px \supset Rx)$ has to be retracted:

| | | | |
|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ |
| 6 | $Ra$ | 2,5:RC | $\{(R \supset Q)^a\}$ |
| 7 | $\forall x(Px \supset Rx)$ | 3,6;RC | $\{(R \supset Q)^a, !(P \supset R)\}\checkmark^{17}$ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 14 | $Pd$ | PREM | $\emptyset$ |
| 15 | $\neg Rd$ | PREM | $\emptyset$ |
| 16 | $\exists x \neg(Px \supset Rx)$ | 14,15;RU | $\emptyset$ |
| 17 | $!(P \supset R)$ | 10;RU | $\emptyset$ |

**Multiple Explanatory Hypotheses** The pitfalls of abduction (as spelled out in Section 8.4) are avoided by $\mathbf{SILA^r}$ in exactly the same way as by $\mathbf{LA^r_\Box}$ (see Section 8.4), since the abductive step in $\mathbf{SILA^r}$ is modeled by the same set of abnormalities. For reasons of space, we will only illustrate this for the case of multiple explanatory hypotheses – the other cases are completely analogous.

Consider $\Gamma_{c5} = \{\forall x(Px \supset Qx), \Box\forall x(Rx \supset Qx), \Box\forall x(Sx \supset Qx), Pa, \neg Qb\}$. In the following $\mathbf{SILA^r}$-proof from $\Gamma_{c5}$, we first derive the explanatory hypothesis

[28]One way to save the hypothesis $\forall x(Px \supset Rx)$ in the case of $\Gamma_{c2}$ is by replacing $\mathbf{IM^r}$ with a more refined logic of induction, i.e. one that prioritizes stronger generalizations over weaker ones. As explained in [24], these logics first minimize abnormalities $!A$ where $A$ only contains one predicate; next they minimize abnormalities $!B$ with two predicates, and so on. As a result, the generalization $\forall x(Px \supset Rx)$ would be finally derivable, and the generalization $\forall x((\neg Px \wedge Qx) \supset Rx)$ would be defeated. However, even though this seems to solve the problem with the current case, this would be merely an ad hoc solution with no real justification.

$Ra$, and generalize it to $\forall x(Px \supset Rx)$ (see lines 6, resp. 7). After that, a Dab-formula is derived that blocks these derivations (line 11). Finally, the weaker hypothesis $Ra \vee Sa$ is derived (line 13), which allows us to finally derive the generalization $\forall x(Px \supset (Rx \vee Sx))$ (line 14).

| | | | |
|---|---|---|---|
| 1 | $\forall x(Px \supset Qx)$ | PREM | $\emptyset$ |
| 2 | $\Box\forall x(Rx \supset Qx)$ | PREM | $\emptyset$ |
| 3 | $\Box\forall x(Sx \supset Qx)$ | PREM | $\emptyset$ |
| 4 | $Pa$ | PREM | $\emptyset$ |
| 5 | $\neg Qb$ | PREM | $\emptyset$ |
| 5 | $Qa$ | 1,4;RU | $\emptyset$ |
| 6 | $Ra$ | 2,5;RC | $\{(R \supset Q)^a\}$ $\checkmark^{11-}_{1}$ |
| 7 | $\forall x(Px \supset Rx)$ | 4,6;RC | $\{(R \supset Q)^a, !(P \supset R)\}$ $\checkmark^{11-}_{1}$ |
| 8 | $\neg Ra \,\check{\vee}\, \neg(Sa \wedge \neg Ra)$ | -;RU | $\emptyset$ |
| 9 | $(Qa \wedge \neg Ra) \,\check{\vee} (Qa \wedge \neg(Sa \wedge \neg Ra))$ | 5,8;RU | $\emptyset$ |
| 10 | $\Box\forall x((Sx \wedge \neg Rx) \supset Qx)$ | 3;RU | $\emptyset$ |
| 11 | $(R \supset Q)^a \,\check{\vee} ((S \wedge \neg R) \supset Q)^a$ | 2,9,10;RU | $\emptyset$ |
| 12 | $\Box\forall x((Rx \vee Sx) \supset Qx)$ | 2,3;RU | $\emptyset$ |
| 13 | $Ra \vee Sa$ | 5,12;RC | $\{((R \vee S) \supset Q)^a\}$ |
| 14 | $\forall x(Px \supset (Rx \vee Sx))$ | 4,13;RC | $\{((R \vee S) \supset Q)^a, !(P \supset (R \vee S))\}$ |

As we can see in the above proof, the derivation of the stronger hypothesis $\forall x(Px \supset Rx)$ is blocked because it relies on the hypothesis $Ra$ – note that lines 6 and 7 are 1-marked from stage 11 on. By a similar reasoning, the hypotheses $Sa$ and $\forall x(Px \supset Sx)$ are not finally $\mathbf{SILA^r}$-derivable from $\Gamma_{\mathsf{c5}}$. In other words, already on the level of abduction, the logic enforces that, whenever two explanations are available, we can only finally derive their disjunction.

## 8.8 Induction And AG In One Context?

Recall that in our definition of AG, we assumed that the generalization "all $A$ are $B$" in the pattern of AG is given beforehand. Under this assumption, it turned out that $\mathbf{LA^r_\forall}$ and $\mathbf{SILA^r}$ yield very similar (though not identical) consequences – we refer to our discussion of the examples $\Gamma_{\mathsf{c1}} - \Gamma_{\mathsf{c5}}$ to illustrate this fact.

There is however one specific feature of $\mathbf{SILA^r}$ which heretofore remained largely unnoticed, precisely because of the assumption that underlies our definition of AG. Consider the following premise set:

$$\Gamma_{\mathsf{d}} = \{Pa_1, Qa_1, \ldots, Pa_9, Qa_9, \Box\forall x(Rx \supset Qx)\}$$

$\mathbf{SILA^r}$ allows us to derive $Ra_1, \ldots, Ra_9$ and $\forall x(Px \supset Rx)$ from $\Gamma_{\mathsf{d}}$. That is, by $\mathbf{LA^r_\Box}$, we can finally derive $Ra_i$ for all $i \in \{1, \ldots, 9\}$. From $Pa_1, Ra_1$ we can finally derive $\forall x(Px \supset Rx)$ by $\mathbf{IM^r}$. Hence, thanks to the inductive behavior of $\mathbf{SILA^r}$, it allows us to derive the explanatory hypothesis of an AG in cases like $\Gamma_{\mathsf{d}}$.

At this point, one may ask: is this behavior of $\mathbf{SILA^r}$ justified? In the remainder, we will argue that the answer depends i.a. on (i) the alleged variability of the kind of objects we are reasoning about, and (ii) pragmatic factors, such as the effort that is needed to find out whether a plausible generalization is indeed true or highly likely, and the risk involved in getting it wrong. We will not spell out a whole theory of these factors – doing so would require yet another thesis –, but merely illustrate them to show that in some cases, the inductive power of $\mathbf{SILA^r}$ can be justified.

Claim (i) is but a specific instance of a more general theory propagated by Thagard and Nisbett [146]. They argue on the basis of psychological evidence, examples from the history of science and thought experiments that background knowledge concerning the variability in kinds is crucial to understand and justify any sort of inductive generalization.[29] We give a brief description of the notion of variability from [146] below. However, let us first give two more realistic examples that illustrate the distinction between what we call variable and invariable kinds or classes of objects, in the context of AG.

**Example 8.1** *Professor Schmitt is teaching a course for the students of psychology and biology. Before he starts his classes, Schmitt checks the list with names of students who subscribed to this course. There are about 20 students from biology, and about 30 from psychology on the list. When Schmitt has gone through half of his list, he notes that all 13 students from the psychology group which he proclaimed so far are absent. There are also 2 absent biology students, but at least 7 of them are there. At that point, Schmitt remembers that one of the psychology student's father has died two days ago, and that his funeral takes place today. Two students in psychology, Nick and Jess, have notified him that they would be attending the funeral, but probably more of them actually attended it. Also, some students may be absent for more usual reasons, such as illness.*

**Example 8.2** *When walking in New York's Central Park, Winston notes that some oaks in the park have holes in some of their leaves. These holes are not very easy to spot, but every oak that he has inspected carefully, turns out to have them. This makes Winston – an amateur biologist – curious. After consulting the internet, he finds out that there are two possible explanations for such holes: either the oaks are inhabited by a parasitic plant that affects their immunity against sour rain, or they are infected by a species of caterpillars that eat bits of their leaves.*

In the first example, it seems more plausible that Schmitt first goes through the whole list, before he raises the hypothesis that each of his psychology students (from this group) is either ill, or went to the funeral. In the second example, it is much more likely that Winston concludes right away that all oaks in the park have holes in their leaves, and that each oak in the park is either inhabited by a parasitic plant, or infected by caterpillars.[30] In Schmitt's case, it is clear that a very variable class of objects, viz. students, is considered. Especially with regards to their social behavior, students do all kinds of things, and it would be very surprising that they are all absent. On the other hand, if a number of oaks

[29]The more general idea that background knowledge is crucial to understand the applicability of certain inductive methods, can be traced back to Mill [112]. More recently, it has also been advocated i.a. by Goodman [65], Davies [45] and Norton [116].

[30]In fact, it seems plausible that Winston draws an even stronger conclusion, viz. that either all oaks in the part are inhabited by a parasitic plant, or that they are all infected by caterpillars – I return to this point in the concluding section.

from a park have holes in its leaves, then it seems highly unlikely that this is so by accident, and that other oaks in the same park do not share this deformity. The bottom line is that, whether or not we are allowed to reason inductively, depends in part on our background knowledge concerning the variability of the kinds or classes we reason about.

Some qualifications are needed here, since the concept of variability is open for many interpretations. First of all, it needs to be relativized to a *target property or a group of target properties* of the reasoning process. For instance, if we are considering their chemical behavior, then the class of all objects that are solely made of iron is a very homogeneous class. If we are considering the way these objects are used, then there are very few relevant features that are shared by these objects. Likewise, the biological properties of psychology students provides a much more stable basis for inductive inferences than their social lives.

Second, even if we relativize it to the reasoning process at hand, the distinction between variable and invariable kinds is not absolute. To some extent, even the social behavior of students is homogeneous – if only for the fact that they interact, follow certain trends in fashion, and follow the same courses. Nevertheless, in a significant number of contexts – cf. Schmitt's reasoning versus that of Winston – it is clear on which side of the distinction we are. We refer to [146] for more examples and an elaborate discussion of the role of variability in inductive methods.

We also claimed that induction can be justified in terms of pragmatic factors. When it is very difficult to find out whether indeed, all members of a specific class share a certain property, one might simply take this for granted and continue reasoning until and unless problems arise. Also, if there is much risk involved, the reasoner may be less eager to raise a hypothesis that is crucial for his further actions.

These two factors can again be illustrated by means of the above two examples. In the case of Schmitt, it is plausible that he will first go over the entire list of students, since this was after all what he planned to do – it does not take any additional effort. In principle, Winston could also check whether all trees of the park have the same problem. However, since he is merely interested in these trees for the sake of curiosity, Winston will not bother doing this. If Winston were a biologist working for the New York Community, he would have needed strong evidence that indeed, all or most of the trees in the park have holes in their leaves, before he could act on the alleged cause of this problem.

Much more can be said about the question whether, and in which contexts, a given inductive method is justified – see [165] for a gentle introduction to the literature on this subject. However, we merely wanted to illustrate the fact that the distinctions from that literature can be applied in the context of AG as well, and that from this perspective, one can justify the behavior of $\mathbf{SILA^r}$ in cases like $\Gamma_{\mathsf{d}}$.

## 8.9 Summary and Outlook

As argued in this chapter, abduction of generalizations (AG) is ubiquitous in everyday and scientific reasoning. We provided a first general analysis of this pattern, and argued that the notion of an explanatory framework should be embodied in any formal model for AG. This idea was implemented in two adaptive logics. The following points summarize our main results:

- Given a few preliminary adjustments (cf. the restrictions on the set of abnormalities of $\mathbf{LA}^{\mathbf{r}}_{\forall}$), we can obtain a very sensible and powerful logic for AG, by letting this inference pattern figure as a kind of default rule in the logic $\mathbf{LA}^{\mathbf{r}}_{\forall}$.

- The Flach & Kakas reconstruction of AG in terms of a combination of singular fact abduction and inductive generalization can only be implemented in an adaptive logic if we prioritize abduction over induction. The resulting logic, $\mathbf{SILA}^{\mathbf{r}}$, leads to overall very similar consequences as $\mathbf{LA}^{\mathbf{r}}_{\forall}$, when applied to prototypical examples of AG.

- Due to its inductive power, $\mathbf{SILA}^{\mathbf{r}}$ allows us to derive the explanatory hypothesis of an AG in the absence of any mere generalization. Whether this is justified, depends on background knowledge about the variability of kinds we reason about, and pragmatic factors such as the effort it takes to check an inductive hypothesis, and the risk involved in getting it wrong.

One interesting question, which we left untouched so far, is whether the Minimal Abnormality-variants of $\mathbf{LA}^{\mathbf{r}}_{\forall}$ and $\mathbf{SILA}^{\mathbf{r}}$ would display interesting differences with the two logics considered here. For instance, it seems that in the Winston example from the previous section, the flat adaptive logic $\mathbf{LA}^{\mathbf{m}}_{\forall}$, defined by the triple $\langle \mathbf{T}, \Omega_{\mathbf{LA}^{\mathbf{r}}_{\forall}}, \mathbf{m} \rangle$ allows us to finally derive the conclusion that "either all oaks in the park are inhabited by a parasitic plant, or they are all infected by caterpillars." Note that this conclusion is stronger than what $\mathbf{LA}^{\mathbf{r}}_{\forall}$ allows us to derive, viz. "each oak in the park is either inhabited by a parasitic plant, or infected by caterpillars." More generally, consider the following situation:

(P1) $\forall(A \supset B)$ (all $A$ are $B$, a mere generalization)
(P2$_1$) $\Box\forall(C_1 \supset B)$ (that something is $C_1$, explains why it is $B$)
$\vdots$
(P2$_n$) $\Box\forall(C_n \supset B)$ (that something is $C_n$, explains why it is $B$)

In this case, there seem to be two options: either we infer $\forall(A \supset (C_1 \vee \ldots \vee C_n))$, or we infer the stronger hypothesis $\forall(A \supset C_1) \vee \ldots \vee \forall(A \supset C_n)$. The Reliability-variants take the first option, whereas the Minimal Abnormality-variants seem to take the second. More work is needed however, to see whether this is indeed the case, and to find out which of the two strategies is preferable in a given context.

There are various other topics for further research as well. Several enrichments of our formal model can be studied, in order to deal with e.g. probabilistic information (see Section 8.2.1), causal arguments (see Section 8.3.1), and abductive anomalies.[31]

[31] In Aliseda's terminology [4], an anomaly is a fact, the negation of which follows from our background theory.

Case studies of some of the historical examples mentioned in Section 8.2 may shed new light on the relation between AG, unification and other patterns of abduction. Such case studies may help us to get more grip on the difference between the abductive methods characterized by (variations on) the logics $\mathbf{LA}^{\mathbf{r}}_{\forall}$ and $\mathbf{SILA}^{\mathbf{r}}$.

Finally, we may look at other patterns of abduction as put forward by Schurz, such as common cause abduction and second-order existential abduction [131]. If we are able to characterize each of these methods by means of (flat or prioritized) adaptive logics, this further substantiates the claim that ALs provide a unifying framework for the study of defeasible reasoning in general, and abduction in particular.

# Chapter 9

# Logics for Relevant Belief Revision

*This chapter is based on the papers "'The Dynamics of Relevance: Adaptive Belief Revision" (Synthese, conditionally accepted), co-authored by Peter Verdée, and "Prime Implicates and Relevant Belief Revision" (Journal of Logic and Computation 2011; doi: 10.1093/logcom/exr040). We thank two anonymous referees for their comments on the first of these two papers. I thank Audun Stolpe, David Makinson, Meghyn Bienvenu, Giuseppe Primiero, two anonymous referees and the anonymous handling editor for their fruitful comments on the second paper.*

## 9.1 Introduction

Belief revision has been a subject of intensive research since the middle of the 1980s. The starting point of what is often called "the logic of belief revision" [74] is the following question: given a set of initial beliefs $\Upsilon$ formulated in a propositional language, and some piece of new information $A$ that possibly contradicts $\Upsilon$, how are we to revise $\Upsilon$ such that $A$ can be incorporated? This is typically done by defining a revision operation $\oplus$, which is a function that maps every couple $\langle \Upsilon, A \rangle$ to a set of formulas $\Upsilon \oplus A$, called the revision set of $\Upsilon$ by $A$.

An important distinction in this domain is that between *theories*, i.e. **CL**-closed sets of formulas (also called *belief sets*) and *belief bases.* A belief base $\Upsilon$ can be any set of propositional formulas. Hence, theories are a border case of belief bases. Similarly, one distinguishes between theory-based revision operations, and revisions of a belief base. We will consider belief revision from the more general perspective of belief bases, although we will treat different belief bases that are **CL**-equivalent in the same way – this will be explained below.

In the standard approach, initiated by Alchourron, Gärdenfors and Makinson, belief revision is reformulated as a combination of belief *contraction* and belief *expansion*, via the so-called Levi identity (after Isaac Levi). To contract $\Upsilon$ by $B$ means to select a $\Upsilon' \subseteq \Upsilon$ (or a $\Upsilon' \subseteq Cn_{\mathbf{CL}}(\Upsilon)$) which maximally approximates $\Upsilon$, but such that $\Upsilon' \nvdash_{\mathbf{CL}} B$. To expand $\Upsilon$ by $C$ simply means to add $C$ to $\Upsilon$ —

for theory-based expansion, the resulting set is closed under **CL**. The revision of $\Upsilon$ by $A$ is then reformulated as follows: we first contract $\Upsilon$ by $\neg A$ – this gives us the contraction set $\Upsilon \ominus \neg A$ — and next we expand the latter set by $A$.

In the current and next chapter, we will focus on revision as an independent operation. In Chapter F of the appendix, the operation of contraction is also considered.

One way to understand the logic of belief revision, is as a two-sided endeavor: on the one hand, one formulates postulates (also called axioms) that any operation $\oplus$ should obey and, on the other hand, one gives generic definitions of revision operations. An example of a postulate is the *Success* postulate, which requires that for all sets of beliefs $\Upsilon$ and all formulas $A$, $A \in \Upsilon \oplus A$. An example of a "generically defined" revision operation is partial meet contraction – see below. The formal challenge for the logic of belief revision is then to prove representation theorems, which link these two characterizations of revision operations to each other. In addition, several scholars study the relations between various ways to define revision operations – e.g. revision operations based on entrenchment levels [57], those based on kernel contraction [72], partial meet revisions [3], model-based revision operations [67], etc.

The current chapter concerns a specific axiom that was formulated by Rohit Parikh, viz. the *Axiom of Relevance*, spelled out in terms of the so-called "finest splitting" of a set of beliefs. We will give the definition of the finest splitting and the axiom of relevance in Section 9.2. Intuitively, this axiom states that whenever a proposition is in the set of initial beliefs, and the new information you receive is not related to this proposition, then you should hold on to this proposition – even if some of your other beliefs have to be revised. That two propositions are related is specified in terms of a relation between the schematic letters that occur in them and the set of initial beliefs.

Suppose you initially believe $p \wedge q$. Now if you learn that actually $p$ is not the case, then the relevance axiom states that this should not alter your belief in $q$. So although you have to abandon the belief in $p$ and hence also in $p \wedge q$, you will stick to $q$. As we will see below, there are much more subtle and complex cases, all of which can be decently handled by the relevance axiom.

Since the publication of [117], Parikh's definition of the finest splitting of a set of beliefs and the related axiom of relevance have received quite some attention in the literature on belief revision. In their [87], Kourousias and Makinson extended Parikh's splitting result to the infinite case and showed how the AGM partial meet revision operations can be adapted in such a way that the relevance axiom is obeyed. Stolpe applied this result to overcome a triviality problem in input-output logic [135]. Makinson discussed the relation of propositional relevance modulo the finest splitting to propositional relevance modulo a canonical form, and proved the two notions are equivalent [94]. Investigations into the computational aspects of finest splittings took a start with [167].

Although the relevance axiom has by far not the same status as the AGM postulates for belief revision, many authors find it useful to prove that the revision operations they define obey this additional axiom – see e.g. [38, 120, 43, 168]. The current chapter takes a step further in this direction: it presents a series of logics for belief revision that not only warrant relevance, but also provide a more dynamic account of the concept of relevance itself.

A key feature of these logics is that their proof theory models revision as a process in which beliefs are analyzed only as soon as they turn out to be relevant to the new information. It will be argued that this gives relevant belief revision a natural flavor. Moreover, it will be shown that relevance-obeying belief revision can take on several distinct forms, each of which are nicely captured within the unifying framework of adaptive logics.

The logics we will present each characterize a revision operation in a direct way, i.e. without the usual detour via a contraction. They are formulated in a modal framework, which makes it possible to draw a distinction between initial beliefs and the revision set in the object language: initial beliefs are preceded by a $\Box$, whereas the revision set consists of all the non-modal formulas that are derivable from the premise set by the logic under consideration. It is possible to define similar systems that characterize relevant contraction and to define revision operations from it, using a bi-modal language, but this would complicate the formal framework without adding much in return – in the end, revision is what matters for the logic of belief revision.

The revision operations can be applied to both belief bases and theories, but they always result in a **CL**-closed revision set. When applied to theories, they behave exactly like the revision operations in the sense of the AGM framework. Furthermore, they will be shown to satisfy all Gärdenfors' postulates for belief revision, as stated in [56].

The remainder of this chapter is structured as follows. We will recapitulate the most salient results in the context of relevant belief revision in Section 9.2. Then we will introduce two conceptual distinctions: one between global and local analysis of a set of beliefs, and one between the external and internal dynamics of belief revision (Section 9.3). These distinctions will help us to clarify the main goal of this chapter, i.e. to present the eight adaptive logics for belief revision. These logics model relevant belief revision by local analysis, and capture the internal dynamics of (relevant) belief revision.

In Section 9.4, the logics $\mathbf{AR_1^r}$ and $\mathbf{AR_1^m}$ are presented. We will present some examples in Section 9.5 to illustrate the basic features of the $\mathbf{AR_1^r}$-proof theory, and highlight its relation to the concepts we introduced in Section 9.3. After that, we will show how small variations in the definition of $\mathbf{AR_1^r}$, resp. $\mathbf{AR_1^m}$ lead to three other distinct couples of logics, which each determine their own distinct revision operations (Section 9.6). We will illustrate the differences between these logics with the aid of examples. Section 9.8 lists the most important meta-theoretic properties of the logics we discussed. We make some concluding remarks in Section 9.9.

## 9.2 Relevant Belief Revision

### 9.2.1 The Axiom of Relevance

Recall that $\mathcal{W}_c$ refers to the set of well-formed formulas of propositional classical logic, $\mathcal{S}$ to the set of sentential letters, and that $\mathcal{W}_c^l = \mathcal{S} \cup \{\neg A \mid A \in \mathcal{S}\}$. In the remainder, we use $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$ as metavariables for sets of sets of formulas.[1] Where $A \in \mathcal{W}_c$, resp. $\Delta \subseteq \mathcal{W}_c$, let $E(A)$, $E(\Delta)$ denote the set of sentential letters that occur in $A$, resp. $\Delta$. We use $\bigvee \Delta$ ($\bigwedge \Delta$) to denote the classical disjunction (resp. conjunction) of the members of $\Delta$, where in this notation, $\Delta$ is always assumed to be finite (if $\Delta = \emptyset$, then $\bigvee \Delta$ and $\bigwedge \Delta$ denote the empty string; if $\Delta = \{A\}$, then $\bigvee \Delta = \bigwedge \Delta = A$). We use $\bigvee_{i \in I} A_i$ ($\bigwedge_{i \in I} A_i$) to abbreviate $\bigvee \{A_i \mid i \in I\}$ (resp. $\bigwedge \{A_i \mid i \in I\}$). Finally, let $\Delta^{\neg} =_{\mathsf{df}} \{\neg A \mid A \in \Delta\}$.

In the remainder, we will generalize the notion of revision in such a way that the new information is also conceived of as a set, denoted by (notational variants of) $\Psi$.[2] Hence revision becomes an operation $\wp(\mathcal{W}_c) \times \wp(\mathcal{W}_c) \to \wp(\mathcal{W}_c)$. This generalization will be very useful in Chapter 10, where we consider a specific kind of iterated revisions. The classical revision operations in the AGM sense – and those which we are mostly concerned with in the current chapter – are thus of the form $\Upsilon \oplus \{A\}$. However, to simplify the reading, we will slightly abuse notation and denote them by $\Upsilon \oplus A$ when we consider concrete examples.

**The Rationality Postulates.** In his [55] and [56], Gärdenfors formulates postulates that every operation for belief revision should fulfill. These postulates can be stated as follows:[3]

**G1** *Closure:* $\Upsilon \oplus \Psi = Cn_{\mathbf{CL}}(\Upsilon \oplus \Psi)$
**G2** *Success:* $\Psi \subseteq \Upsilon \oplus \Psi$
**G3** *Inclusion:* $\Upsilon \oplus \Psi \subseteq Cn_{\mathbf{CL}}(\Upsilon \cup \Psi)$
**G4** *Vacuity:* If $\Upsilon \cup \Psi$ is consistent, then $\Upsilon \oplus \Psi = Cn_{\mathbf{CL}}(\Upsilon \cup \Psi)$
**G5** *Consistency:* If $\Psi$ is consistent, then $\Upsilon \oplus \Psi$ is consistent
**G6** *Extensionality:* If $\Psi \dashv\vdash_{\mathbf{CL}} \Psi'$ , then $\Upsilon \oplus \Psi = \Upsilon \oplus \Psi'$

There are two supplementary postulates, i.e. *Superexpansion* and *Subexpansion*, which are often cited in the belief revision literature. Following the suggestion of a referee, these are spelled out in Appendix E, where their relation to the logics presented in this chapter is discussed. However, for our present purposes, it suffices to focus on the above six postulates, often called the *basic* rationality postulates.

As Parikh remarks in [117], the basic rationality postulates are still too weak, in that they allow for the "trivial update" (henceforth $\oplus_{\mathbf{T}}$). This operation is defined as follows: if $\Upsilon \cup \Psi$ is consistent, then $\Upsilon \oplus_{\mathbf{T}} \Psi =_{\mathsf{df}} Cn_{\mathbf{CL}}(\Upsilon \cup \Psi)$; otherwise, $\Upsilon \oplus_{\mathbf{T}} \Psi =_{\mathsf{df}} Cn_{\mathbf{CL}}(\Psi)$ . As Parikh notes, "this is unsatisfactory, because we would like to keep as much of the old information as possible [even when it contradicts the new information]. Hence the above list [= the list of postulates] needs to be supplemented to rule out the trivial update" [117, p. 3].

As Kourousias and Makinson explain in [87], this problem can easily be generalized to revisions of all kinds of belief *bases*. For example, when revising the

[1] For some specific sets, such as e.g. $\Phi(\Gamma)$, this convention is violated in order to stay in line with the notational conventions from the Ghent group of logicians.

[2] Fuhrmann and Hansson consider contraction and revision operations in view of sets of propositions in their [53]. As they show, many results from the traditional account – in which the new information is a single formula – can be generalized to this setting.

[3] This list is based on the one from [74], but generalized to be applicable to revisions by sets. Also, the *Vacuity* postulate is generalized in order to include the case where $\Upsilon \neq Cn_{\mathbf{CL}}(\Upsilon)$.

base $\Upsilon_1 = \{p \wedge q\}$ by $\neg p$, there are "rational" (in the sense of [73]) belief operations that yield $\{\neg p\}$ as the only resulting belief, hence removing both the implicit beliefs $p$ and $q$.

**The Axiom of Relevance.** Parikh's positive contribution consists in the formulation of an additional postulate, i.e. the axiom of relevance **P**. To spell out this axiom, he defines the finest splitting of a set of formulas. This requires some notational preparation. A *partition* $\mathbb{A} = \{\Lambda_i\}_{i \in I}$ of a set $\Delta$ is a set of non-empty, pairwise disjoint sets such that $\bigcup_{i \in I}\{\Lambda_i\} = \Delta$. In this notation, the sets $\Lambda_i$ are called the *cells* of $\mathbb{A}$.

**Definition 9.1 ([94]: Def. 3.1)** *Let $\mathbb{E} = \{\Lambda_i\}_{i \in I}$ be a partition of $\mathcal{S}$. We say that $\mathbb{E}$ is a* splitting *of $\Gamma$ iff there is a $\Delta = \bigcup_{i \in I} \Delta_i$ such that each $E(\Delta_i) \subseteq \Lambda_i$ and $\Delta \dashv\vdash_{\mathbf{CL}} \Gamma$.*[4]

Remark that a splitting of $\Upsilon$ is not a partition of $\Upsilon$ itself, but of the letter set $\mathcal{S}$. Intuitively, a splitting tells us how we may cut $\mathcal{S}$ into distinct subsets $\Lambda_i$, in such a way that we can express $\Gamma$ by distinct sets of formulae $\Delta_i$, each expressed by means of letters that only belong to one set $\Lambda_i$.

**Example 9.1** *Let $\Upsilon_2 = \{(p \vee q) \wedge r, \neg r \vee s, q \vee t, r \vee u\}$. Note that this set is* **CL***-equivalent to $\Upsilon_2' = \{p \vee q, q \vee t, r, s\}$. From the latter, we may generate the following splittings of $\Upsilon_2$:*

$\mathbb{E}_1(\Upsilon_2) = \{\mathcal{S}\}$
$\mathbb{E}_2(\Upsilon_2) = \{\{p, q, t\}, \{r, s\}\} \cup \{\{A\} \mid A \in \mathcal{S} - \{p, q, r, s, t\}\}$
$\mathbb{E}_3(\Upsilon_2) = \{\{p, q, t\}, \{r\}, \{s\}\} \cup \{\{A\} \mid A \in \mathcal{S} - \{p, q, r, s, t\}\}$

$\mathbb{E}$ is *at least as fine as* $\mathbb{E}'$ iff every cell of $\mathbb{E}'$ is the union of cells of $\mathbb{E}$; $\mathbb{E}$ is finer than $\mathbb{E}$ iff it $\mathbb{E}$ is at least as fine as $\mathbb{E}'$ but the converse fails. Note that if $\mathbb{E}$ is a splitting of $\Gamma$, and $\mathbb{E}$ is finer than the partition $\mathbb{E}'$ of $\mathcal{S}$, it immediately follows that $\mathbb{E}'$ is also a splitting of $\Gamma$ (see [117, pp. 4-5]). We say that $\mathbb{E}$ is a *finest splitting* of $\Gamma$ iff there is no splitting $\mathbb{E}'$ of $\Gamma$ that is finer than $\mathbb{E}$.

**Example 9.2** *Take $\Upsilon_2$ from Example 9.1. Note that $\mathbb{E}_2(\Upsilon_2)$ is finer then $\mathbb{E}_1(\Upsilon_2)$, and $\mathbb{E}_3(\Upsilon_2)$ is finer then $\mathbb{E}_2(\Upsilon_2)$. Provably, $\mathbb{E}_3(\Upsilon_2)$ is a finest splitting of $\Upsilon_2$.*

Note that if $\Upsilon \dashv\vdash_{\mathbf{CL}} \Upsilon'$, and $A \in \mathcal{S} - E(\Upsilon')$, then $\{A\}$ is a cell of a splitting of $\Upsilon$ – see Example 9.1: $\{u\}$ is a cell in $\mathbb{E}_3(\Upsilon_2)$. To avoid clutter, we will henceforth only mention the letters that are non-redundant in $\Upsilon$ when we represent splittings of $\Upsilon$. E.g. $\mathbb{E}_3(\Upsilon_2)$ will be represented as $\{\{p, q, t\}, \{r\}, \{s\}\}$.

As we will explain below, the following technical result is crucial for Parikh's notion of relevance:

**Theorem 9.1 ([117] for the finite case; [87] for the general case)** *Every $\Gamma \subseteq \mathcal{W}_c$ has a unique finest splitting.*

[4] The idea of a splitting originates in [117]. I use Makinson's definition because it includes the case where $\Gamma$ is infinite.

Parikh uses the finest splitting to define his notion of relevance in the context of belief revision. However, in Parikh's initial formulation, relevance still depends heavily on the exact way we formulate the new information $\Psi$. For instance, according to Parikh, $q$ is not relevant to the revision of $\{p \wedge q\}$ by $\neg p$, although it is relevant to the revision of $\{p \wedge q\}$ by $\neg p \wedge (q \vee \neg q)$, which is equivalent to $\neg p$. This small inconvenience is overcome by the notion of a least letter-set representation from [93, 94]:

**Definition 9.2** *$\Psi^*$ is a least letter-set representation of $\Psi$ iff (i) $\Psi^* \dashv\vdash_{\mathbf{CL}} \Psi$ and (ii) for every $\Delta$ such that $\Delta \dashv\vdash_{\mathbf{CL}} \Psi$, $E(\Delta) \subseteq E(\Psi^*)$.*

As shown in [93], every set $\Psi$ has a least letter-set representation $\Psi^*$. Let $E^*(\Psi) = E(\Psi^*)$, where $\Psi^*$ is an arbitrary least letter-set representation of $\Psi$.[5] We call $E^*(\Psi)$ the least-letter set of $\Psi$. Given the uniqueness of $E^*(\Psi)$, and the uniqueness of the finest splitting of $\Upsilon$, we can define relevance to $\Upsilon \oplus \Psi$ ($\Upsilon \ominus \Psi$) as follows:[6]

**Definition 9.3** *Let $\mathbb{E}$ be the finest splitting of $\Upsilon$. We say that a formula $B$ is irrelevant to the revision (contraction) of $\Upsilon$ by $\Psi$ iff for every cell $\Lambda_i \in \mathbb{E}$: $\Lambda_i \cap E^*(\Psi) = \emptyset$ or $\Lambda_i \cap E(B) = \emptyset$.*

Note that relevance to the revision of $\Upsilon$ by $A$ is equivalent to relevance to the contraction of $\Upsilon$ by $A$. Also, note that where $\Upsilon \dashv\vdash_{\mathbf{CL}} \Upsilon'$ and $\Psi \dashv\vdash_{\mathbf{CL}} \Psi'$, relevance to the revision (contraction) of $\Upsilon$ by $\Psi$ is equivalent to relevance to the revision (contraction) of $\Upsilon'$ by $\Psi'$. Hence relevance is a syntax-independent notion, i.e. it is independent of the way we represent the information embodied by $\Upsilon$ and $\Psi$.

The preceding definitions finally allow us to state Parikh's axiom of relevance. His original formulation of this axiom is the following:

**P** *Relevance:* If $B \in \Upsilon$ is irrelevant to the revision (contraction) of $\Upsilon$ by $\Psi$, then $B \in \Upsilon \oplus \Psi$ ($B \in \Upsilon \ominus \Psi$)

However, Parikh only intends to apply this axiom to **CL**-theories – for bases, it would not solve the above problem. Consider again the example $\Upsilon_1$: the formula $p \wedge q \in \Upsilon$ is *relevant* to the revision of $\Upsilon$ by $\neg p$. Hence if we take **P** literally, there is no problem in dropping $p \wedge q$, which is the only belief in $\Upsilon$. In order to deal with both belief bases and **CL**-theories, we may generalize the axiom as follows:[7]

$\mathbf{P}_g$ *(Generalized) Relevance:* If $B \in Cn_{\mathbf{CL}}(\Upsilon)$ is irrelevant to the revision (contraction) of $\Upsilon$ by $\Psi$, then $B \in Cn_{\mathbf{CL}}(\Upsilon \oplus \Psi)$ ($B \in Cn_{\mathbf{CL}}(\Upsilon \ominus \Psi)$)

**Example 9.3** *Consider the contraction of $\Upsilon_2$ by $r$. If $\mathbf{P}_g$ is obeyed, then this implies that $p \vee q$, $p \vee t$ and $s$ are in the contraction set of $\Upsilon_2$ by $r$.*

[5] Every formula $A$, resp. set of formulas $\Gamma$ has infinitely many least letter-set representations. E.g. $p, p \wedge p, p \wedge p \wedge p, \ldots$ are all least letter-set representations of $p$.

[6] In [94], Makinson also uses the least letter-set representation of $\Psi$ to define relevance.

[7] Whenever $\Upsilon$, $\Upsilon \oplus \Psi$ and $\Upsilon \ominus \Psi$ are closed under **CL**, as in the traditional AGM-approach, this formulation reduces to Parikh's original axiom.

Henceforth, I will take this more general relevance-axiom as the point of reference. As explained in Appendix F, axiom $\mathbf{P}_g$ only makes sense if we assume that $\Upsilon$ is consistent; otherwise it yields a revision set that is arguably nonsensical. To solve this problem, and hence to apply the relevance axiom to inconsistent theories in a sensible way, a notion of subclassical relevance is developed in Chapter F.

**Normalized Partial Meet Revision.** The best-known syntax-based class of revision functions are partial meet revisions, as defined in [3]. Where the new information is expressed by a single formula $A$, every partial meet revision operation can be defined as follows:

**Definition 9.4** *(i) let $\Upsilon \dot{\perp} A$ be the set of all maximal subsets $\Delta$ of $\Upsilon$ that do not imply $A$.*

*(ii) let $\gamma$ be a selection function, such that for every $\Upsilon, A$: if $\Upsilon \dot{\perp} A \neq \emptyset$, then $\gamma(\Upsilon \dot{\perp} A)$ is a non-empty subset of $\Upsilon \dot{\perp} A$; otherwise, $\gamma(\Upsilon \dot{\perp} A) = \{\Upsilon\}$.*

*(iii) let $\Upsilon \oplus_\gamma A = Cn_{\mathbf{CL}}(\bigcap \gamma\{\Upsilon \dot{\perp} \neg A\} \cup \{A\})$.*

It was shown in [3] that partial meet revision obeys all the postulates G1 up to G6 when applied to **CL**-closed sets. Also, whenever a revision operation on a **CL**-closed set obeys G1-G6, then it is equivalent to a partial meet contraction – this result is the most famous representation theorem of the AGM framework. However, Parikh established the following fact:

**Fact 9.1** *There are partial meet revisions that do not obey* **P***.*

We refer to [117, 87, 86] for examples and discussions of this fact. In their [87], Kourousias and Makinson argue in favor of a contextual approach to this fact: sometimes, so they claim, **P** seems too strong in the sense that it neglects the specific formulation of the initial beliefs; however, in other cases, one might want to ensure that **P** is guaranteed. In the latter case, it is possible to tweak partial meet revision in such a way that relevance is obeyed. This requires that we first define the set of canonical forms of $\Upsilon$:

**Definition 9.5** *Where $\mathbb{E} = \{\Lambda_i\}_{i \in I}$ is the finest splitting of $\Upsilon$: $\mathbb{C}_\Upsilon = \{\Delta = \bigcup_{i \in I}\{\Delta_i\} \mid \Delta \dashv\vdash_{\mathbf{CL}} \Upsilon$ and for every $i \in I : E(\Delta_i) \subseteq \Lambda_i\}$*

In the literature, $\mathbb{C}_\mathcal{K}$ has also been called the set of "canonical forms" or "normal forms" of $\Upsilon$ – in the remainder, I use the two terms as synonyms. In the next section, it is pointed out that $\mathbb{C}_\Upsilon$ need not be, and often is not a singleton – we refer to that section for examples. There we will also show that the so-called set of prime implicates of $\Upsilon$ – see below – is a canonical form of $\Upsilon$, and can be used to determine the finest splitting of $\Upsilon$.

The following is proven in [87]:

**Theorem 9.2 ([87], Th. 4.1)** *For every consistent $\Upsilon$: partial meet revision (contraction) with respect to a $\Delta \in \mathbb{C}_\Upsilon$ obeys the relevance axiom.*

Two small warnings are in place here. First of all, Kourousias and Makinson only consider the revision (contraction) of belief bases in view of single formulas, not in view of sets of formulas as we do it here. Second, they do not refer to the least letter-set representation of the new information $\Psi$ in their definition of relevance.[8] In Section F.5.4 from Appendix F, it is shown how their result can be rephrased in the current framework.

However, if we abstract from the minor technical differences, a crucial fact remains. In view of Theorem 9.2, it is possible to obey **P** if we first massage $\Upsilon$ into one of its canonical forms and only afterwards apply a partial meet revision to this set. In the remainder, we call such revision operations "normalized revisions".

### 9.2.2 Prime Implicates and Canonical Forms

*This Section summarizes some results published in "Prime Implicates and Relevant Belief Revision", Journal of Logic and Computation 2011; doi: 10.1093/logcom/exr040. I thank Peter Verdée for spotting several shortcomings in the next to last version of this section.*

**Not One but Many Canonical Forms.** As indicated in the previous section, there is often not just *one* normal form of $\Upsilon$, and hence "normalized revision (contraction) of $\Upsilon$ in view of $\Psi$" is usually not a unique operation. Many scholars write "*the* finest splitting of $\Upsilon$" to refer to sets of formulas in normal form – see e.g. [87, 94, 135, 167]. A simple example shows that the definite article is not in place here.

Take $\Upsilon_3 = \{p \vee q, q \vee r, r \vee s\}$. The finest splitting of $\Upsilon_3$ is $\mathbb{E} = \{\{p, q, r, s\}\}$. Hence both $\Theta = \Upsilon_3$ and $\Theta' = \{(p \vee q) \wedge (q \vee r) \wedge (r \vee s)\}$ are normal forms of $\Upsilon_3$. This underdetermination of "the" normal form of $\Upsilon$ obviously carries over to the normalized contraction (and hence also revision) operations. To see why, consider a full meet contraction of the normal forms of $\Upsilon_3$ by $p \vee q$. This operation is defined as follows (cf. [3]):

**Definition 9.6** *Let $\gamma_0$ be a selection function, such that for every $\Upsilon, A$: if $\Upsilon \dot{\perp} A \neq \emptyset$, then $\gamma_0(\Upsilon \dot{\perp} A) =_{\mathsf{df}} \Upsilon \dot{\perp} A$; otherwise, $\gamma_0(\Upsilon \dot{\perp} A) =_{\mathsf{df}} \{\Upsilon\}$. Then $\Upsilon \ominus_{\gamma_0} A =_{\mathsf{df}} \bigcap \gamma_0(\Upsilon \dot{\perp} A)$.*

Note that $\Theta \dot{\perp} p \vee q = \{\{q \vee r, r \vee s\}\}$. Hence, the full meet contraction of $\Theta$ by $p \vee q$ gives us the set $\{q \vee r, r \vee s\}$. However, $\Theta' \dot{\perp} p \vee q = \{\emptyset\}$, whence the full meet contraction of $\Theta'$ gives us $\emptyset$.

To summarize, although every $\Upsilon$ has a unique finest splitting, it may have several different normal forms, and the differences between them has a great impact on the contraction and revision sets obtained from them. So if we want to implement the result of Kourousias and Makinson (as stated by Theorem 9.2), we have to further specify which normalized contraction or revision operation we are using.[9] The question then becomes: which $\Theta \in \mathbb{C}_\Upsilon$ should we use to define a

[8] In contrast, in his [94], Makinson does refer to the least letter-set representation of $A$ when he defines relevance to the revision of $\Upsilon$ by $A$.

[9] Stolpe e.g. uses Theorem 9.2 when applying input-output logic to codes of laws, see [135].

normalized contraction or revision operation? In the remainder, I will show that one well-known set can be used for this purpose.

**Prime Implicates.** The set $\Pi(\Gamma)$ of prime implicates of a set of formulas $\Gamma$ can be defined as follows:

**Definition 9.7** *Where* $\Gamma \subseteq \mathcal{W}_c$, $\Pi(\Gamma) =_{\mathsf{df}} \{\bigvee\Theta \mid (i)\ \Theta \subset \mathcal{W}^l_c, (ii)\ \nvdash_{\mathbf{CL}} \bigvee\Theta,$ $(iii)\ \Gamma \vdash_{\mathbf{CL}} \bigvee\Theta$ *and* $(iv)$ *for no* $\Theta' \subset \Theta : \Gamma \vdash_{\mathbf{CL}} \bigvee\Theta'\}$.

Note that by clause (ii), there are no tautologies in $\Pi(\Gamma)$. The underlying idea behind the definition of $\Pi(\Gamma)$ can be stated as follows: $\Pi(\Gamma)$ is the set that "breaks down" the information in $\Pi(\Gamma)$ as much as possible, i.e. into minimal disjunctions of literals. The following lemma will be helpful in the remainder:

**Lemma 9.1** *For all* $\Gamma \subseteq \mathcal{W}_c$: $\Pi(\Gamma) \dashv\vdash_{\mathbf{CL}} \Gamma$.

*Proof.* The right-left direction is immediate in view of Definition 9.7. For the left-right direction, suppose $A \in \Gamma$. Let $\bigwedge_{i\in I}\bigvee\Theta_i$ be a conjunctive normal form of $A$. Note that for all $i \in I$, $\Gamma \vdash_{\mathbf{CL}} \bigvee\Theta_i$. For every $i \in I$, define $\Theta'_i$ as follows:

if $\vdash_{\mathbf{CL}} \Theta_i$, then $\Theta'_i = \emptyset$
otherwise, let $\Theta'_i \subseteq \Theta_i$ be minimal such that $\Gamma \vdash_{\mathbf{CL}} \Theta'_i$

Note that in view of this construction, (†) $\bigwedge_{i\in I}\bigvee\Theta'_i \vdash_{\mathbf{CL}} \bigwedge_{i\in I}\bigvee\Theta_i$, and hence $\bigwedge_{i\in I}\bigvee\Theta'_i \vdash_{\mathbf{CL}} A$.[10] Note also that for every $i \in I$, if $\Theta'_i \neq \emptyset$, then $\bigvee\Theta'_i$ fulfills all the clauses (i)-(iv) in Definition 9.7. Hence for all $i \in I$ such that $\Theta'_i \neq \emptyset$, $\bigvee\Theta'_i \in \Pi(\Gamma)$. It follows that $\Pi(\Gamma) \vdash_{\mathbf{CL}} \bigwedge_{i\in I}\bigvee\Theta'_i$, and hence by (†) and the transitivity of **CL**, $\Pi(\Gamma) \vdash_{\mathbf{CL}} A$. ■

In the remainder, I will prove the following theorem:

**Theorem 9.3** *For every belief base* $\Upsilon$: $\Pi(\Upsilon) \in \mathbb{C}_\Upsilon$.

A proof of essentially this result was offered in [167]. However, it contains a flaw, which I analyze after giving my own. The crucial motor behind the proof I will present, is the relation $\sim_\Delta$ of path-relevance modulo a set $\Delta$, which is borrowed from [94] and is also applied in Chapter F. This relation is defined as follows:

**Definition 9.8** *Let* $\Delta \subseteq \mathcal{W}_c$ *and* $A, B \in \mathcal{W}_c$. *A is path-relevant to B modulo* $\Delta$ *(*$A \sim_\Delta B$*) iff there are* $C_1, \ldots, C_n \in \Delta$ *such that* $E(A) \cap E(C_1) \neq \emptyset$, $E(C_1) \cap E(C_2) \neq \emptyset$, $E(C_2) \cap E(C_3) \neq \emptyset$, …, *and* $E(C_n) \cap E(B) \neq \emptyset$.

As shown below, $\sim_{\Pi(\Upsilon)}$ constitutes an equivalence relation on $\Pi(\Upsilon)$, whence it can be used to obtain a partition of $\Pi(\Upsilon)$ into $\sim_{\Pi(\Upsilon)}$-connected subsets $\Upsilon_1, \Upsilon_2, \ldots$. From this, we can obtain a partition of $\mathcal{S}$: $\mathbb{E}_\Pi(\Upsilon) = \{E(\Upsilon_1), E(\Upsilon_2), \ldots\}$. Finally, it is proven that $\mathbb{E}_\Pi(\Upsilon)$ is the finest splitting of $\Upsilon$. This implies that $\Pi(\Upsilon)$ can be used to determine the finest splitting of $\Upsilon$.

[10] Recall the convention that, where $\Delta = \emptyset$, $\bigvee\Delta$ and $\bigwedge\Delta$ denote the empty string.

The set $\Pi(\Upsilon)$ as defined above is used for a specific kind of model-based revision in [97] – the authors link their paper to Parikh's and Makinson's work on the notion of relevant belief change, but do not explicitly discuss the relation between $\Pi(\Upsilon)$ and $\mathbb{C}_\Upsilon$. In [120], the same authors propose a solution to the problem of relevance in belief revision in terms of preferences over prime *implicants*, minimal conjunctions of literals that entail $\Upsilon$. The idea of defining revision of $\Upsilon$ in view of its prime implicate set was put forward in [38], where it is conjectured that this revision obeys relevance. However, so far no one seems to have made the distinction between $\Pi(\Upsilon)$ and other sets in $\mathbb{C}_\Upsilon$.

**Proof of Theorem 9.3.** Note that $A \sim_\Delta B$ does not necessarily imply that $A$ and $B$ are members of $\Delta$, only that there is a path from $A$ to $B$ *through* $\Delta$. It will however also be convenient to rely on the following properties specific to $\sim_\Delta$ defined only over the members of $\Delta$:

**Fact 9.2** *$\sim_\Delta$ is a transitive, reflexive and symmetric on $\Delta$, whence $\sim_\Delta$ is an equivalence relation on $\Delta$.*

**Fact 9.3** *If $A \sim_\Delta B$, then $A \sim_{\Delta\cup\Delta'} B$ for every $\Delta'$.*

I now define a multiset $\mathbb{M}_\Pi(\Upsilon)$ from $\Pi(\Upsilon)$ and the equivalence relation $\sim_{\Pi(\Upsilon)}$, and a partition of $\mathcal{S}$ on the basis of this multiset:

**Definition 9.9** *$\mathbb{M}_\Pi(\Upsilon)$ is the quotient set of $\Pi(\Upsilon)$ by $\sim_{\Pi(\Upsilon)}$.*[11] *Where $\mathbb{M}_\Pi(\Upsilon) = \{\Delta_i\}_{i\in I}$, $\mathbb{E}_\Pi(\Upsilon) = \{E(\Delta_i)\}_{i\in I} \cup \{\{A\} \mid A \in \mathcal{S} - E(\Pi(\Upsilon))\}$.*

Since $\sim_{\Pi(\Upsilon)}$ is an equivalence relation on $\Pi(\Upsilon)$, $\mathbb{M}_\Pi(\Upsilon)$ is a partition of $\Pi(\Upsilon)$. Also, note that for no $\Delta_i \in \mathbb{M}_\Pi(\Upsilon) : \Delta_i = \emptyset$, whence also for no $E_i \in \mathbb{E}_\Pi(\Upsilon)$: $E_i = \emptyset$. In the remainder, I prove that $\mathbb{E}_\Pi(\Upsilon)$ is the finest splitting of $\Upsilon$.

Let me first show that $\mathbb{E}_\Pi(\Upsilon)$ is a partition of $\mathcal{S}$. This follows immediately from (1) the fact that every $E_i$ is non-empty, (2) the fact that $\bigcup \mathbb{E}_\Pi(\Upsilon) = \mathcal{S}$, and the following lemma:

**Lemma 9.2** *For every $E_i, E_j \in \mathbb{E}_\Pi(\Upsilon)$: $E_i \neq E_j$ iff $E_i \cap E_j = \emptyset$.*

*Proof.* Let $E_i, E_j \in \mathbb{E}_\Pi(\Upsilon)$. The right-left direction is obvious since no $E_i \in \mathbb{E}_\Pi(\Upsilon)$ is empty. For the left-right direction, suppose that for $E_i, E_j \in \mathbb{E}_\Pi(\Upsilon)$, $E_i \cap E_j \neq \emptyset$. I only consider the case where $E_i = E(\Delta_i)$ and $E_j = E(\Delta_j)$ for $\Delta_i, \Delta_j \in \mathbb{M}_\Pi(\Upsilon)$ – in the other case, it follows immediately that $E_i \cap E_j = \emptyset$. Suppose that $E(\Delta_i) \cap E(\Delta_j) \neq \emptyset$. This implies that there are $A \in \Delta_i, B \in \Delta_j$: $E(A) \cap E(B) \neq \emptyset$, whence $A \sim_{\Pi(\Upsilon)} B$. It follows that $A$ and $B$ are in the same equivalence class. As a result, $\Delta_i = \Delta_j$, whence $E_i = E_j$. ■

To see why $\mathbb{E}_\Pi(\Upsilon)$ is a *splitting* of $\Upsilon$, note that each of the following holds:

(i) $\bigcup \mathbb{M}_\Pi(\Upsilon) = \Pi(\Upsilon)$, and hence by Lemma 9.1, $\bigcup \mathbb{M}_\Pi(\Upsilon) \dashv\vdash_{\mathbf{CL}} \Upsilon$.

[11] This is the set of all equivalence classes of $\Pi(\Upsilon)$, given the equivalence relation $\sim_{\Pi(\Upsilon)}$ on $\Pi(\Upsilon)$.

(ii) For every $\Delta_i \in \bigcup \mathbb{M}_\Pi(\Upsilon)$, there is a cell $E_j \in \mathbb{E}_\Pi(\Upsilon)$ such that $E(\Delta_i) \subseteq E_j$, viz. the cell $E_i$.

In view of (i), (ii) and Definition 9.1, $\mathbb{E}_\Pi(\Upsilon)$ is a splitting of $\Upsilon$. To prove that $\mathbb{E}_\Pi(\Upsilon)$ is also the *finest* splitting of $\Upsilon$, I need two lemmas:

**Lemma 9.3 ([87], Theorem 1.1.)** *Let $\Delta = \bigcup_{i\in I}\{\Delta_i\}$ where the letter sets $E(\Delta_i)$ are pairwise disjoint, and suppose $\Delta \vdash_{\mathbf{CL}} A$. Then there are formulas $B_i$ such that (1) each $E(B_i) \subseteq E(\Delta_i) \cap E(A)$, (2) each $\Delta_i \vdash_{\mathbf{CL}} B_i$, and (3) $\bigcup_{i\in I}\{B_i\} \vdash_{\mathbf{CL}} A$. (Parallel Interpolation)*

**Lemma 9.4** *If (1) $\{A, B\} \vdash_{\mathbf{CL}} C \vee D$, (2) $E(A) \subseteq E(C)$, (3) $E(B) \subseteq E(D)$, and (4) $E(C) \cap E(D) = \emptyset$, then $\{A\} \vdash_{\mathbf{CL}} C$ or $\{B\} \vdash_{\mathbf{CL}} D$.*

*Proof.* Suppose (1)-(4) holds, but $\{A\} \nvdash C$ and $\{B\} \nvdash D$. In that case, $A \wedge \neg C$ and $B \wedge \neg D$ are both **CL**-satisfiable. In view of (2), (3) and (4), $E(A \wedge \neg C) \cap E(B \wedge \neg D) = \emptyset$, whence $(A \wedge \neg C) \wedge (B \wedge \neg D)$ is **CL**-satisfiable.[12] This implies that $\{A, B\} \nvdash C \vee D$, which contradicts (1). ■

**Theorem 9.4** $\mathbb{E}_\Pi(\Upsilon)$ *is the finest splitting of* $\Upsilon$.

*Proof.* Assume that there is a splitting $\mathbb{E} = \{E_i\}_{i\in I}$ of $\Upsilon$, such that $\mathbb{E}$ is finer than $\mathbb{E}_\Pi(\Upsilon)$. Hence for an $E \in \mathbb{E}_\Pi(\Upsilon)$, there is an $i \in I$: $\emptyset \subset E_i \subset E$. This means that $E$ cannot be a singleton, whence we can derive that $E = E(\Pi(\Upsilon^j))$ for an $\Upsilon^j \in \mathbb{M}_\Pi(\Upsilon)$. So we have:

(†) For an $\Upsilon^j \in \mathbb{M}_\Pi(\Upsilon)$, there is an $i \in I$: $\emptyset \subset E_i \subset E(\Upsilon^j)$

Let $A \in \Upsilon^j$ be such that $E(A) \not\subseteq E_i$ and let $B\Upsilon^j$ be such that $E(B) \cap E_i \neq \emptyset$. It can easily be verified that $A$ and $B$ exist.[13] Since $A, B \in \Upsilon^j$, $A \sim_{\Pi(\Upsilon)} B$. Hence there are $C_1, \ldots, C_n \in \Pi(\Upsilon)$ such that $E(A) \cap E(C_1) \neq \emptyset$, $E(C_1) \cap E(C_2) \neq \emptyset$, $E(C_2) \cap E(C_3) \neq \emptyset, \ldots$, and $E(C_n) \cap E(B) \neq \emptyset$. Let $A = C_0$ and $B = C_{n+1}$.

Assume now that (‡) for every $k$ with $0 \leq k \leq n+1$, either $E(C_k) \cap E_i = \emptyset$ or $E(C_k) \subseteq E_i$. Then it can be shown by mathematical induction that

($\star$) for every $k$ with $0 \leq k \leq n+1$, $E(C_k) \cap E_i = \emptyset$.

The base case ($k = 0$) is immediate, in view of (‡) and the fact that $E(A) \not\subseteq E_i$. For the induction step, suppose that $E(C_k) \cap E_i = \emptyset$. Since $E(C_k) \cap E(C_{k+1}) \neq \emptyset$, it follows that $E(C_{k+1}) \not\subseteq E_i$. But then, in view of (‡), $E(C_{k+1}) \cap E_i = \emptyset$.

From ($\star$) and the fact that $B = C_{n+1}$, we can derive that $E(B) \cap E_i = \emptyset$ — a contradiction. So assumption (‡) must be false: there is a $k$ with $0 \leq k \leq n+1$, such that $E(C_k) \cap E_i \neq \emptyset$ and $E(C_k) \not\subseteq E_i$. Let $l$ be such that $E(C_l) \cap E_i \neq \emptyset$ and $E(C_l) \not\subseteq E_i$, and let $D = C_l$. Note that since $D \in \Pi(\Upsilon)$, $D$ is a disjunction of literals.

[12] I rely on the fact that if $A$ and $B$ are **CL**-satisfiable and share no elementary letters, then $A \wedge B$ is **CL**-satisfiable.

[13] To see why $A$ exists, assume that for every $A' \in \Upsilon^j$, $E(A') \subseteq E_i$. In that case, $E(\Upsilon^j) \subseteq E_i$, which contradicts (†). To see why $B$ exists, assume that for every $B \in \Upsilon^j$, $E(B) \cap E_i = \emptyset$. In that case, $E(\Upsilon^j) \cap E_i = \emptyset$, which again contradicts (†).

Since $\mathbb{E}$ is a splitting of $\Upsilon$, $\mathbb{E}' = \{E_i, \bigcup\mathbb{E} - E_i\}$ is also a splitting of $\Upsilon$. Since $\Upsilon \vdash_{\mathbf{CL}} D$, by Lemma 9.3, there are two formulae $F_i$ and $F$ such that (1) $E(F_i) \subseteq E_i \cap E(D)$ and (2) $E(F) \subseteq (\bigcup\mathbb{E} - E_i) \cap E(D)$, $\Upsilon \vdash_{\mathbf{CL}} F_i$, $\Upsilon \vdash_{\mathbf{CL}} F$ and $\{F_i, F\} \vdash_{\mathbf{CL}} D$.

Let $D = G_i \vee G$, where $G_i$ and $G$ are disjunctions of literals such that $\emptyset \subset E(G_i) \subseteq E_i, \emptyset \subset E(G) \subseteq (\bigcup\mathbb{E} - E_i)$, hence also $E(G) \cap E(G_i) = \emptyset$. By (1) and (2), we obtain: (1') $E(F_i) \subseteq E(G_i)$ and (2') $E(F) \subseteq E(G)$. By Lemma 9.4, this implies that either $\{F_i\} \vdash_{\mathbf{CL}} G_i$ or $\{F\} \vdash_{\mathbf{CL}} G$. Since $\Upsilon \vdash_{\mathbf{CL}} F_i$ and $\Upsilon \vdash_{\mathbf{CL}} F$, also $\Upsilon \vdash_{\mathbf{CL}} G_i$ or $\Upsilon \vdash_{\mathbf{CL}} G$ by the transitivity of **CL**. This however implies that $D$ is not a minimal disjunction of literals that is **CL**-derivable from $\Upsilon$, whence $D \notin \Pi(\Upsilon)$ — a contradiction. ■

**Further Comments on the Proof.** Note that every set $\Upsilon$ is associated with a unique set $\Pi(\Upsilon)$, and from this $\Pi(\Upsilon)$, we can uniquely obtain the set $\mathbb{E}_\Pi(\Upsilon)$. Hence Theorem 9.4 implies that every set $\Upsilon$ has a unique finest splitting. Just as in [87], I needed the Parallel Interpolation theorem to arrive at this result.

In [167], Wu and Zhang also attempted to prove that the finest splitting of $\Upsilon$ can be obtained from $\Pi(\Upsilon)$ – their set $C(K) \setminus K^{\#}$ is almost identical to what I defined as $\Pi(\Upsilon)$.[14] The way they obtain the finest splitting from $\Pi(\Upsilon)$ is also highly similar to the way I did it here: they define an equivalence relation $R^\star$ on $\Pi(\Upsilon)$, which is equivalent to my $\sim_{\Pi(\Upsilon)}$, and obtain a set of $R^\star$-connected subsets $\Upsilon^1, \Upsilon^2, \ldots$ of $\Pi(\Upsilon)$. Finally, they claim that the set $\mathbb{E} = \{E(\Upsilon^1), E(\Upsilon^2), \ldots\}$ is the finest splitting of $\Upsilon$.

However, Wu and Zhang's actual *proof* for this claim is mistaken. To see why, let me recapitulate their notion of indivisibility:

**Definition 9.10** *$\Delta$ is* indivisible *iff for every partition $\mathbb{E} = \{E_1, E_2\}$ of $E(\Delta)$, there is an $A \in \Delta$ such that: $E(A) \cap E_1 \neq \emptyset$ and $E(A) \cap E_2 \neq \emptyset$*

Wu and Zhang seem to assume that the following holds, for every multiset $\mathbb{D} \in \wp(\mathcal{W}_c)$:

(†) Where $\bigcup\mathbb{D} \dashv\vdash_{\mathbf{CL}} \Upsilon$: $\bigcup\mathbb{D}$ is a canonical form of $\Upsilon$ iff every cell of $\mathbb{D}$ is indivisible.

The authors prove that every $\sim_{\Pi(\Upsilon)}$-connected subset of $\Pi(\Upsilon)$ is indivisible. By the left-right direction of (†), they infer that $\Pi(\Upsilon)$ is "*the* finest splitting set of $\Upsilon$ [my emphasis]". In view of the preceding, this claim is slightly confusing. So I take it that they actually mean that $\Pi(\Upsilon)$ is one of the normal forms of $\Upsilon$, and that $\mathbb{E}_\Pi(\Upsilon) = \{E(\Upsilon^1), E(\Upsilon^2), \ldots\}$ is the finest splitting of $\Upsilon$. The crucial problem concerns the left-right direction of (†).

Let $\Upsilon_4 = \{p, q\}$. Note that every cell of $\mathbb{D} = \{\{p, q, p \supset q\}\}$ is indivisible. The finest splitting of $\Upsilon_4$ is obviously $\mathbb{E} = \{\{p\}, \{q\}\}$. Hence $\mathbb{E}_\Delta = \{\{p, q\}\}$ is not the finest splitting of $\Upsilon_4$. More generally, it can be proven that for any $\Upsilon \subseteq \mathcal{W}_c$, $\{Cn(\Upsilon)\}$ is indivisible, whereas $Cn(\Upsilon)$ is obviously not a canonical form of $\Upsilon$. As a result, the left-right direction of (†) fails. Put differently, the property of

[14] The only difference is that tautologies are allowed for in $C(K) \setminus K^{\#}$, whereas I omitted them. This restriction does not make a difference with respect to the argument here or the proof in the preceding section.

indivisibility has little bearing on the question whether a set of formulas $\Delta$ is a canonical form of $\Upsilon$.

## 9.3 Two Crucial Distinctions

### 9.3.1 Global versus Local Analysis

It seems that if one wants to obey the axiom of relevance using a syntax-based revision, one first has to translate $\Upsilon$ into one of its canonical forms. This translation can be seen as a kind of analysis: we cut $\Upsilon$ into as many small pieces of knowledge as possible, before we revise it. Analytic steps can consist in the simplification of single formulas (e.g. infer $A$ from $A \wedge B$), but may also rely on several formulas taken together (e.g. infer $B$ from $A, \neg A \vee B$). In most realistic cases, many analytic steps have to be combined in order to obtain a canonical form of $\Upsilon$. Also, to obtain the canonical form of $\Upsilon$, one has to perform what we shall call a *global analysis*: the whole of $\Upsilon$ has to undergo the analytic procedure.[15]

Let us clarify this a bit further. In Section 9.2.2, it was proven that the set of prime implicates or prime clauses of $\Upsilon$ is a canonical form of $\Upsilon$. As far as we know, this is the only concrete circumscription of a canonical form of $\Upsilon$. Note however that the computation of $\Pi(\Upsilon)$ is exponential in $\Upsilon$.[16]

So although it eventually leads to the right outcome, the strategy of global analysis seems quite unnatural: most often, intelligent agents only analyze *some* of their initial beliefs *to some extent*, i.e. when this turns out useful. It requires a great effort to perform a global analysis, and agents usually prefer to rely on the analytic steps they have taken so far, until and unless they have sufficient reasons to take the analysis to the next level. Even if it is in principle possible to obtain absolute certainty, agents often lack the means to perform an extensive search, and hence are forced to act on their present best insights.

Let us clarify this by a simple example. Suppose your initial belief base is $\Upsilon_5 = \{p \wedge q, q \supset (r \vee s), \neg t \wedge u\}$, and you learn that $\neg p$ is the case. Since $\Upsilon_5 \vdash_{\mathbf{CL}} p$, you have to revise the set $\Upsilon_5$ somehow. According to Parikh's definition, the belief $\neg t \wedge u$ is not relevant to the revision of $\Upsilon_5$ by $\neg p$, whence we may simply keep it as it is. Nevertheless, if we would compute a canonical form of $\Upsilon_5$, we would cut $\neg t \wedge u$ into $\neg t$ and $u$. This means we perform an analysis that is, strictly speaking, not necessary for the revision operation under consideration.

Note however that a relevant revision of $\Upsilon_5$ by $\neg p$ can only be ensured if we *do* analyze $p \wedge q$ into $p$ and $q$, such that the latter belief can be retained. This analysis is also necessary to get $r \vee s$ in the revision set. More generally, which logical steps are to be taken in order to guarantee relevance when revising $\Upsilon$ by

[15] In [43], Chopra and Parikh propose that we view $\Upsilon$ not as a single set of formulas, but as a "structured" base, i.e. a set of sub-theories $\Upsilon_1, \Upsilon_2, \ldots$, where each sub-theory deals with its own subject matter. If we take this as our starting point, it is much easier to warrant (a certain degree of) relevance for a revision in view of $A$: we only revise those sets $\Upsilon_i$ for which $E(\Upsilon_i) \cap E(A) \neq \emptyset$ and leave the other sets unchanged. Although the overall framework Chopra and Parikh present is worthy of pursuit, it seems equally interesting to see whether a *logic* can sieve out which beliefs are relevant to the new information, instead of just presupposing that our beliefs are structured in such a way that relevance can easily be obeyed.

[16] See [83] for a comparative study of three procedures to compute $\Pi(\Upsilon)$ from $\Upsilon$.

$\Psi$, depends on the specific $\Upsilon$ and $\Psi$ we are dealing with. In the end, it seems that we still need a global analysis in order to receive absolute certainty about the necessity of any analytic step.

We will model a process in which an agent first reasons about his or her initial and upheld beliefs in order to find out which analytic steps (s)he should take. This results in a tentative process: unless (s)he turns to a global analysis, the agent will never be entirely sure that his or her current conclusions are warranted. In more formal terms: in order to prove that a derived formula really follows by the logic, we have to make a reasoning at the meta-level. Nevertheless, the more inferences we draw, the more insights we gain with respect to which beliefs are relevant to the revision. Moreover, the fact that conclusions are defeasible seems to be perfectly in line with the general idea behind the study of belief change, i.e. that our knowledge is always open for revision. Finally, once we take first order predicate logic as the frame of reference, decidability is no longer in general attainable even in principle. In that case, the most we can get is a reasonable estimate of which beliefs can be upheld.

### 9.3.2 Internal versus External Dynamics

It will be useful to introduce yet another distinction in order to get a better picture of the logics we will present below. This distinction is the one between internal and external dynamics of a reasoning process.[17] Although it is fairly well-known in the field of adaptive logics (see e.g. [14, 16, 21]), and was already illustrated and discussed in Chapter 1, it seems worthwhile to explain its meaning in the context of belief revision.

In a sense, all belief revision operations are "dynamic": they all try to deal with the fact that we receive new information and update our beliefs accordingly. Doing so requires that we not only add the new information and derive what follows from the initial beliefs together with this new information, but also that we retract some initial beliefs and/or some of their consequences. In other words, gaining information from the outside leads to a dynamic process in which we revise our initial beliefs. In the literature on defeasible reasoning, this feature is referred to as the *external dynamics* of a reasoning method.

There is, however, also a dynamics which is internal to revision as a process, not as a relation between input (initial beliefs plus new information) and output (revision set). Sven Ove Hansson seems to refer to such processes when he writes that "[a]ctual subjects change their minds as a result of deliberations that are not induced by new inputs." [73, p. 8] These deliberations contribute to what is often called the *internal dynamics* of a reasoning method. It is the dynamics of drawing inferences from a body of knowledge (resp. a set of premises), relying on a certain standard of normality or condition. Whenever it turns out – in view of the same and/or other inferences – that in some particular case, the standard of normality or condition cannot be relied upon, we retract some of these inferences. Although we may eventually arrive at a stable outcome, we only do so by passing through a number of different epistemic stages. At every such stage, we use the

[17] Pollock dubs the external dynamics the synchronic defeasibility, and the internal dynamics the diachronic defeasibility of inferences – see [121].

previous inferences and rely on them while the reasoning continues.

From the viewpoint of the internal dynamics, partial meet revision, and many other revision operations (including model-based revisions in terms of distances, see e.g. [97]) are fairly static. Put differently, the internal dynamics is thought of as a problem for computing: (how) can we produce the right output given our definition of the revision operation. Little effort is made to capture formally how a rational agent could reason from his or her initial beliefs, retract some of his or her previous conclusions, and finally arrive at a stable revision set.[18]

Consider for example $\Upsilon_6 = \{p \vee q, \neg r \wedge \neg q, (t \vee s) \supset r\}$ and the revision of this set by $\neg p \vee s$. Just to see that the new information contradicts the initial beliefs, one already has to make a number inferences of the following kind:

(1) $\neg r \wedge \neg q \ / \ \neg q$
(2) $\neg q, p \vee q \ / \ p$
(3) $\neg r \wedge \neg q \ / \ \neg r$
(4) $\neg r, (t \vee s) \supset r \ / \ \neg(t \vee s)$
(5) $\neg(t \vee s) \ / \ \neg s$
(6) $p, \neg s, \neg p \vee s \ / \ \bot$

Once we are at this point, we might even need some more inferences in order to see whether e.g. the belief $t$ can be upheld. It seems worthwhile to try to model these processes themselves, and not solely their outcome, at the object-level.

The distinction between internal and external dynamics may equally well be applied to the concept of relevance. On the one hand, there is a clear and fixed criterion that determines whether or not a formula is relevant to a given revision operation, as spelled out by Definition 9.3. On the other hand, finding out which formulas are so, is a matter of actual reasoning (or actual computing), whence it may well be the case that we change our mind about the relevance of a particular formula throughout the reasoning process.[19] Moreover, the internal dynamics of revision and relevance are strongly interrelated: only when we learn that some old beliefs have to be revised, we can infer that those beliefs were relevant to the revision operation.

## 9.4 The Adaptive Logics $\mathbf{AR_1^r}$ and $\mathbf{AR_1^m}$

In this section, we present the logics $\mathbf{AR_1^r}$ and $\mathbf{AR_1^m}$. These systems not only define a revision set for every set of initial beliefs $\Upsilon$ in view of a piece of new information $A$. They also provide a proof theory that shows how you can derive the former from the latter. The revision operations based on them are obtained through a translation of the set of initial beliefs in a modal language.[20]

[18] See e.g. [73, p. 8]: "The idealized belief states of belief dynamics only change as a direct result of new inputs."

[19] We are indebted to Dunja Šešelja for pointing this out to us in a discussion.

[20] Adaptive logics have already been successfully applied to (prioritized) belief bases in the past – see e.g. [164, 163, 32]. Nevertheless, this is the first application of that framework to the problem of *relevant* belief revision.

**The Modal Language for Belief Revision.** The language $\mathcal{L}_r$ is obtained by adding $\Box$ to $\mathcal{L}_c$. The associated set of formulas, is the smallest set $\mathcal{W}_r$ such that

(i) $\mathcal{W}_c \subset \mathcal{W}_r$
(ii) Where $A \in \mathcal{W}_c$, $\Box A \in \mathcal{W}_r$
(iii) Where $A, B \in \mathcal{W}_r$, $\neg A, A \wedge B, A \vee B, A \supset B, A \equiv B \in \mathcal{W}_r$

The language $\mathcal{L}_r$ will be used to express the different components of a revision operation. Where $\Upsilon$ is the set of initial beliefs, let $\Upsilon^\Box = \{\Box A \mid A \in \Upsilon\}$. Where $\Psi$ is the new information, $\Upsilon^\Box \cup \Psi$ is the premise set we feed into the adaptive logic. For example, where $\Upsilon = \{p \wedge q\}$ and the new information is $\{\neg p\}$, the premise set will be $\{\Box(p \wedge q), \neg p\}$.

The adaptive logic $\mathbf{AR^r_1}$, which is defined below, yields a revision set for every $\Upsilon, \Psi$ in the following way:

**Definition 9.11** $\Upsilon \oplus_{\mathbf{AR^r_1}} \Psi = \{B \in \mathcal{W}_c \mid \Upsilon^\Box \cup \Psi \vdash_{\mathbf{AR^r_1}} B\}$.

In words, the revision set of $\Upsilon$ by $\Psi$ is the set of all non-modal formulas that can be derived from $\Upsilon^\Box \cup \Psi$ by the logic $\mathbf{AR^r_1}$. Note that, although the notion of adaptive revision relates to a certain kind of "translation", this translation differs substantially from the translation into a normal form that was criticized in Section 9.3.1. More specifically, the translation needed for adaptive revision is very straightforward: simply put a $\Box$ in front of each initial belief.

For the remainder of this chapter, it is important to keep in mind that where $\mathbf{AR^r_1}$ is a function that maps sets of formulas in $\mathcal{L}_r$ to other sets of formulas in the same language, $\oplus_{\mathbf{AR^r_1}}$ maps a couple to a set, or more formally, $\oplus_{\mathbf{AR^r_1}}$ : $\wp(\mathcal{W}_c) \times \wp(\mathcal{W}_c) \rightarrow \wp(\mathcal{W}_c)$.

**The lower limit logic.** The first adaptive logic for belief revision we will present, $\mathbf{AR^r_1}$, is based on the very simple modal logic $\mathbf{Kt_s}$, which stands for "$\mathbf{K_s}$ with only **t**wo possible worlds". A $\mathbf{Kt_s}$-model $M$ is a quadruple $\langle W, R, v, w_0\rangle$, where $W = \{w_0, w_K\}$ is a set of possible worlds, $R$ an accessibility relation on $W$, $v : \mathcal{S} \times W \rightarrow \{0, 1\}$ an assignment function and $w_0$ the actual world. The valuation $v_M : \mathcal{W}_r \rightarrow \{0, 1\}$ defined by the model $M$ is characterized by:

C1 where $A \in \mathcal{S}$, $v_M(A, w) = v(A, w)$
C2 $v_M(\neg A, w) = 1$ iff $v_M(A, w) = 0$
C3 $v_M(A \vee B, w) = 1$ iff $v_M(A, w) = 1$ or $v_M(B, w) = 1$
C4 $v_M(A \wedge B, w) = 1$ iff $v_M(A, w) = 1$ and $v_M(B, w) = 1$
C5 $v_M(A \supset B, w) = 1$ iff $v_M(A, w) = 0$ or $v_M(B, w) = 1$
C6 $v_M(\Box A, w) = 1$ iff, $v_M(A, w') = 1$ for all $w'$ such that $Rww'$

We define $M \Vdash A$ iff $v_M(A, w_0) = 1$. Where $M \in \mathcal{M}_{\mathbf{Kt_s}}$, we say that $M$ is a $\mathbf{Kt_s}$-model of $\Gamma$ iff $M \Vdash A$ for all $A \in \Gamma$. Finally, $\Gamma \models_{\mathbf{Kt_s}} A$ iff $A$ is true in all $\mathbf{Kt_s}$-models of $\Gamma$.

The following is obvious in view of the definition of the semantics:

**Lemma 9.5** $\{\Box(A_1 \vee \ldots \vee A_n)\} \models_{\mathbf{Kt_s}} (\Box A_1 \vee \ldots \vee \Box A_n)$

A syntax for $\mathbf{Kt_s}$ is obtained as follows. We extend an axiomatization of **CL** with the following axioms (where $A, B \in \mathcal{W}_c$):

A1 $\quad \Box(A \supset B) \supset (\Box A \supset \Box B)$
A2 $\quad \Box A \vee \Box\neg A$

and close it under modus ponens (MP) and the following rule (where $A, B \in \mathcal{W}_c$):

RN $\quad$ if $\vdash A$, then $\vdash \Box A$

Where $\vdash_{\mathbf{Kt_s}}$ indicates membership in the set of $\mathbf{Kt_s}$-axioms, we define $\Gamma \vdash_{\mathbf{Kt_s}} A$ iff there are $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_{\mathbf{Kt_s}} (B_1 \wedge \ldots \wedge B_n) \supset A$. Note that according to these definitions, $\mathbf{Kt_s}$ is a compact Tarski-logic. The proof of the following is safely left to the reader:

**Theorem 9.5** *$\Gamma \vdash_{\mathbf{Kt_s}} A$ iff $\Gamma \models_{\mathbf{Kt_s}} A$.*

As usual, we need to enrich the language, syntax and semantics of $\mathbf{Kt_s}$ in order to deal with the checked connectives. Again, this can be done in a very straightforward way, since $\mathbf{Kt_s}$ is a supraclassical logic – see Chapter 2, Section 2.4 where **K** was obtained from $\mathbf{K_s}$ in the same way. We use $\check{\mathcal{L}}_r$ to refer to the extended language, $\check{\mathcal{W}}_r$ to the associated set of formulas, and **Kt** for the logic obtained by extending $\mathbf{Kt_s}$ with the appropriate axioms for the checked connectives.

**The set of abnormalities.** The general idea behind adaptive revision is the following: where $B \in Cn_{\mathbf{CL}}(\Upsilon)$, and $B$ is not relevant to $\Upsilon \oplus A$, we want to be able to infer that $B$ is in the revision set. So we have to be able to express that $B$ is relevant to $\Upsilon \oplus A$ in the object language. Once that is done, we may assume that a belief is not relevant to $\Upsilon \oplus A$, and hence can be upheld, until and unless proven otherwise.

The most straightforward idea for a set of abnormalities is to treat any formula of the form $\Box B \wedge \neg B$, where $B \in \mathcal{W}_c$, as an abnormality. That is, the assumption that formulas of the form $\Box B \wedge \neg B$ are false boils down to the assumption that if $B$ is an initial belief, then it can be upheld. We will henceforth abbreviate $\Box B \wedge \neg B$ by $!B$. In this context, we may say that "the formula $B$ behaves abnormally" if $B$ is an initial belief, but it is contradicted by the new information. However, where does relevance come in?

In view of Definition 9.3, relevance is not a function of the logical form of a formula, but only of the elementary letters that occur in it. So the crucial shift we have to make, in order to model relevance as a kind of abnormality, is to go from the abnormality of formulas to the abnormality of letters. One way this can be done is the following. Where $B \in \mathcal{S}$, define the abnormality of the letter $B$: $\rho(B) =!B \vee !\neg B$. In words, the letter $B$ behaves abnormally if and only if either the formula $B$ or the formula $\neg B$ behaves abnormally. This gives us the set of abnormalities, which we dub $\Omega_1^{\mathsf{r}}$ in the current chapter:

$$\Omega_1^{\mathsf{r}} =_{\mathsf{df}} \{\rho(B) \mid B \in \mathcal{S}\}$$

One may also read $\rho(B)$ as follows: "every formula that contains the letter $B$, is relevant to the new information". As a matter of fact, it holds that if $\Upsilon^{\Box} \cup \Psi \vdash_{\mathbf{Kt}} \rho(B)$, then $B$ is relevant to $\Upsilon \oplus \Psi$ – this is a corollary of Theorem 9.2 from Section 9.8.3.[21]

The logic $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$ is an adaptive logic in standard format, defined by the triple $\langle \mathbf{Kt}, \Omega^{\mathsf{r}}_1, \mathbf{r}\rangle$. In the remainder, we will recapitulate some definitions from Chapter 2 for this specific logic, and highlight some important features of it.

**The Semantics of $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$.** To define the $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$-semantics, we first need to introduce a few technical concepts from the standard format. In the remainder of this chapter, let a Dab$_1$-formula $Dab_1(\Delta)$ be the disjunction of the members of a finite $\Delta \subseteq \Omega^{\mathsf{r}}_1$. We define the set of unreliable formulas in view of $\Gamma$, $U^1(\Gamma)$, in the standard way: where $Dab_1(\Delta_1), Dab_1(\Delta_2), \ldots$ are the minimal Dab$_1$-consequences of $\Gamma$, $U^1(\Gamma) =_{\mathsf{df}} \Delta_1 \cup \Delta_2 \cup \ldots$. Let $Ab_1(M) = \{A \in \Omega^{\mathsf{r}}_1 \mid M \Vdash A\}$. Then we can define the set of reliable models in view of $\Gamma$, for the logic $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$, as follows:

**Definition 9.12** *$M \in \mathcal{M}_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{Kt}}(\Gamma)$ and $Ab_1(M) \subseteq U^1(\Gamma)$.*

**Definition 9.13** *$\Gamma \models_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}} A$ iff for every $M \in \mathcal{M}_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}}(\Gamma)$, $M \Vdash A$.*

To see how the adaptive logic works, consider the following theorem:[22]

**Theorem 9.6** *If $\Upsilon \models_{\mathbf{CL}} B$, then $\Upsilon^{\Box} \models_{\mathbf{Kt}} B \check{\vee} \bigvee\{\rho(C) \mid C \in E(B)\}$.*

*Proof.* Suppose $\Upsilon \models_{\mathbf{CL}} B$. Hence $\Upsilon^{\Box} \models_{\mathbf{Kt}} \Box B$. Let $D = \bigwedge_{i\in I}\bigvee_{j\in J_i} D_j$ be a conjunctive normal form of $B$, such that $E(D) \subseteq E(B)$. Note that $\Upsilon^{\Box} \models_{\mathbf{Kt}} B \check{\vee}(\Box B \wedge \neg B)$, whence also:

$$\Upsilon^{\Box} \models_{\mathbf{Kt}} B \check{\vee}(\Box \bigwedge_{i\in I}\bigvee_{j\in J_i} D_j \wedge \neg \bigwedge_{i\in I}\bigvee_{j\in J_i} D_j) \tag{9.1}$$

Since $\Box(A_1 \wedge \ldots \wedge A_n) \vdash_{\mathbf{Kt}} \Box A_1 \wedge \ldots \wedge \Box A_n$ and by **CL**-properties:

$$\Upsilon^{\Box} \models_{\mathbf{Kt}} B \check{\vee} \bigvee_{i\in I}(\Box \bigvee_{j\in J_i} D_j \wedge \neg \bigvee_{j\in J_i} D_j) \tag{9.2}$$

By Lemma 9.5 and **CL**-properties:

$$\Upsilon^{\Box} \models_{\mathbf{Kt}} B \check{\vee} \bigvee_{i\in I}(\bigvee_{j\in J_i}(\Box D_j \wedge \neg D_j)) \tag{9.3}$$

Or shorter, $\Upsilon^{\Box} \models_{\mathbf{Kt}} B \check{\vee} \bigvee_{i\in I}\bigvee_{j\in J_i} !D_j$. Note that $!D_j \models_{\mathbf{Kt}} \rho(E(D_j))$. Hence we may derive:

$$\Upsilon^{\Box} \models_{\mathbf{Kt}} B \check{\vee} \bigvee_{i\in I}\bigvee_{j\in J_i} \rho(E(D_j)) \tag{9.4}$$

[21] In Section 9.6, we will show that there are yet other ways to express relevance in terms of the abnormality of formulas.

[22] This theorem is analogous to Lemma 7.3.1 in Chapter.

Since each $E(D_j) \in E(B)$, we have:

$$\Upsilon^{\Box} \models_{\mathbf{Kt}} B \check{\vee} \bigvee\{\rho(C) \mid C \in E(B)\} \tag{9.5}$$

■

Theorem 9.6 implies that if $B \in Cn_{\mathbf{CL}}(\Upsilon)$, then we may infer that $B$ can be upheld, unless one of the elementary letters in $B$ behaves abnormally. Moreover, as shown in Section 9.8.3, if $B$ is not relevant to $\Upsilon \oplus \Psi$, then none of the elementary letters in $B$ will behave abnormally. Altogether, this explains why $\oplus_{\mathbf{AR_1^r}}$ obeys the axiom **P**.

**The Proof Theory of $\mathbf{AR_1^r}$.** Note that in view of Theorem 9.6 and the soundness and completeness of **Kt**, the rule RC allows us to derive $A$ from $\Box A$ in an $\mathbf{AR_1^r}$-proof from $\Gamma$. This means we may apply the derived rule RD$_1$:

$$\text{RD}_1 \qquad \begin{array}{ll} \Box A & \Delta \\ \hline A & \Delta \cup \{\rho(B) \mid B \in E(A)\} \end{array}$$

So we obtain a very simple way to turn an initial belief into a revised belief. Moreover, a straightforward way to derive Dab-formulas is by the application of the rule RD (see page 22), which can be reformulated as follows in the presence of $\perp$:

$$\text{RD}_2 \quad \text{If } A_1, \ldots, A_n \vdash_{\mathbf{Kt}} \perp: \quad \begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline Dab_1(\Delta_1 \cup \ldots \cup \Delta_n) & \emptyset \end{array}$$

When applied to $\mathbf{AR_1^r}$, the marking definition of $\mathbf{AL^r}$ reflects the internal dynamics of revision: we may take some belief to be in the revision set at some point, but only later on learn that this belief contradicts the new information. At that point, we have to retract some of our earlier conclusions. In Section 9.5, we will further clarify this mechanism by an example of an $\mathbf{AR_1^r}$-proof.

**The Minimal Abnormality-variant.** Like every adaptive logic that is defined in the standard format, $\mathbf{AR_1^r}$ also has a Minimal-Abnormality-variant, i.e. $\mathbf{AR_1^m}$, characterized by the triple $\langle \mathbf{Kt}, \Omega_1^{\mathsf{r}}, \mathbf{m}\rangle$. In accordance with the notation of the current chapter, we have:

**Definition 9.14** *$M \in \mathcal{M}_{\mathbf{AR_1^m}}(\Gamma)$ iff $M \in \mathcal{M}_{\mathbf{Kt}}(\Gamma)$ and there is no $M' \in \mathcal{M}_{\mathbf{Kt}}(\Gamma)$ such that $Ab_1(M') \subset Ab_1(M)$.*

**Definition 9.15** *$\Gamma \models_{\mathbf{AR_1^m}} A$ iff for every $M \in \mathcal{M}_{\mathbf{AR_1^m}}(\Gamma)$, $M \Vdash A$.*

As expected, the logic $\mathbf{AR_1^m}$ is slightly stronger than $\mathbf{AR_1^r}$ – we will present an example in Section 9.6. Hence for every $\Upsilon, \Psi$, it holds that $\Upsilon \oplus_{\mathbf{AR_1^r}} \Psi \subseteq \Upsilon \oplus_{\mathbf{AR_1^m}} \Psi$, where in some cases this set inclusion is proper.

All previous observations about the proof theory of $\mathbf{AR_1^r}$ of course apply to $\mathbf{AR_1^m}$ as well: we can easily derive $A$ from $\Box A$ on the condition $\{\rho(B) \mid B \in E(A)\}$, and we can apply the rule RD$_2$ in order to derive Dab-formulas.

## 9.5 Example of an $\mathbf{AR^r_1}$-proof

In this section, we will illustrate what is meant by (i) the internal dynamics of belief revision in general, and relevance in particular and (ii) the fact that $\mathbf{AR^r_1}$ models local analysis of the initial beliefs in order to obey relevance. We will use one example of a revision operation and explain how the proof theory deals with it.

Let $\Upsilon_7 = \{p \wedge q, p \supset (s \wedge \neg r), q \equiv t\}$ and consider the revision of $\Upsilon_7$ by $A_1 = \neg(p \wedge \neg r)$. Note that each of the following holds:

(1) $\Upsilon_7 \vdash_{\mathbf{CL}} p, q, s, \neg r, t$.
(2) Hence the finest splitting[23] of $\Upsilon_7$ is $\mathbb{E}_7 = \{\{p\}, \{q\}, \{r\}, \{s\}, \{t\}\}$.
(3) Hence $q, s, t$ are not relevant to $\Upsilon_7 \oplus A_1$.
(4) Hence if a revision operation $\oplus_{\mathbf{X}}$ obeys $\mathbf{P}$, then $q, s, t \in \Upsilon_7 \oplus_{\mathbf{X}} A_1$.

We start an $\mathbf{AR^r_1}$-proof from $\Upsilon_7^{\Box} \cup \{A_1\}$ as follows:

| | | | |
|---|---|---|---|
| 1 | $\Box(p \wedge q)$ | PREM | $\emptyset$ |
| 2 | $\Box(p \supset (s \wedge \neg r))$ | PREM | $\emptyset$ |
| 3 | $\Box(q \equiv t)$ | PREM | $\emptyset$ |
| 4 | $\neg(p \wedge \neg r)$ | PREM | $\emptyset$ |

Since the new information $A_1$ does not contradict the initial belief $p \wedge q$, we might want to derive $p \wedge q$ from $\Box(p \wedge q)$, using the rule $\mathrm{RD}_1$:

| | | | |
|---|---|---|---|
| 5 | $p \wedge q$ | 1; $\mathrm{RD}_1$ | $\{\rho(p), \rho(q)\}$ |

Likewise, we may derive that the second initial belief can be upheld:

| | | | |
|---|---|---|---|
| 6 | $p \supset (s \wedge \neg r)$ | 2; $\mathrm{RD}_1$ | $\{\rho(p), \rho(s), \rho(r)\}$ |

Note that the application of $\mathrm{RD}_1$ is extremely easy in both cases: just skip the $\Box$ and put the abnormality of the elementary letters of the formula in the condition. We may now continue our proof by relying on the previous inferences, and derive more facts about the revision set. For example, we can derive that if the formulas on line 5 and line 6 can be upheld, then we may also infer that $s$ and $\neg r$ are in the revision set:

| | | | |
|---|---|---|---|
| 7 | $p$ | 5; RU | $\{\rho(p), \rho(q)\}$ |
| 8 | $s \wedge \neg r$ | 6,7; RU | $\{\rho(p), \rho(s), \rho(r), \rho(q)\}$ |
| 9 | $s$ | 8; RU | $\{\rho(p), \rho(s), \rho(r), \rho(q)\}$ |
| 10 | $\neg r$ | 8; RU | $\{\rho(p), \rho(s), \rho(r), \rho(q)\}$ |

Note that, as prescribed by the unconditional rule RU, the formulas on lines 9 and 10 take over the condition of line 8, which is itself the union of the conditions of line 6 and 7.

At this point, we face a problem: the formulas on line 4, line 7 and line 10 lead to a contradiction: $\neg(p \wedge \neg r), p, \neg r \vdash_{\mathbf{CL}} \bot$, whence also $\neg(p \wedge \neg r), p, \neg r \vdash_{\mathbf{K}} \bot$. This means we can derive a disjunction of abnormalities, through the application of the second derived rule:

[23] As in previous examples, we skip all letters that do not occur in $\Upsilon$.

| | | | |
|---|---|---|---|
| 11 | $\rho(p) \check{\vee} \rho(s) \check{\vee} \rho(r) \check{\vee} \rho(q)$ | 4,7,10; $\mathrm{RD}_2$ | $\emptyset$ |

At stage 11, the set of unreliable abnormalities is

$$U^1_{11}(\Upsilon_1 \cup \{A_1\}) = \{\rho(p), \rho(s), \rho(r), \rho(q)\}$$

As a result, all lines with a condition that contains any of these abnormalities are marked at stage 11:

| | | | |
|---|---|---|---|
| ⋮ | ⋮ | ⋮ | ⋮ |
| 5 | $p \wedge q$ | 1; $\mathrm{RD}_1$ | $\{\rho(p), \rho(q)\}$✓ |
| 6 | $p \supset (s \wedge \neg r)$ | 2; $\mathrm{RD}_1$ | $\{\rho(p), \rho(s), \rho(r)\}$✓ |
| 7 | $p$ | 5; RU | $\{\rho(p), \rho(q)\}$✓ |
| 8 | $s \wedge \neg r$ | 6,7; RU | $\{\rho(p), \rho(s), \rho(r), \rho(q)\}$✓ |
| 9 | $s$ | 8; RU | $\{\rho(p), \rho(s), \rho(r), \rho(q)\}$✓ |
| 10 | $\neg r$ | 8; RU | $\{\rho(p), \rho(s), \rho(r), \rho(q)\}$✓ |
| 11 | $\rho(p) \check{\vee} \rho(s) \check{\vee} \rho(r) \check{\vee} \rho(q)$ | 4,7,10; $\mathrm{RD}_2$ | $\emptyset$ |

Hence, if we want to retain at least some parts of the first two initial beliefs, we will have to take some more analytic action. Roughly speaking, there are two ways in which analysis may help us to rescue a belief: it may show us that the belief can be derived on a different condition, and it may show us that a certain $\mathrm{Dab}_1$-formula is not a minimal $\mathrm{Dab}_1$-consequence of the premises. In the current case, both aspects of the adaptive logic will be necessary to rescue $q$ and $s$, as the following continuation of the proof shows:

| | | | |
|---|---|---|---|
| 12 | $\Box q$ | 1; RU | $\emptyset$ |
| 13 | $q$ | 12; $\mathrm{RD}_1$ | $\{\rho(q)\}$ |
| 14 | $\Box p$ | 1; RU | $\emptyset$ |
| 15 | $\Box(s \wedge \neg r)$ | 14,2; RU | $\emptyset$ |
| 16 | $\Box s$ | 15; RU | $\emptyset$ |
| 17 | $s$ | 16; $\mathrm{RD}_1$ | $\{\rho(s)\}$ |
| 18 | $\Box\neg r$ | 15;RU | $\emptyset$ |
| 19 | $\rho(p) \check{\vee} \rho(r)$ | 4,14,18;RU | $\emptyset$ |

Lines 11-17 show us that both $q$ and $s$ can be derived on a condition that is a proper subset of the conditions on lines 5 and 9. Hence we need only rely on the normal behavior of $\rho(q)$ and $\rho(s)$ in order to derive $q$, resp. $s$. Line 19 shows that the formula on line 11 is not a minimal $\mathrm{Dab}_1$-consequence of $\Upsilon_7^{\Box} \cup \{A_1\}$, and that the real problematic letters are $p$ and $r$. Hence at the current stage of the proof, lines 13 and 17 are unmarked.

As it turns out, $q$ and $s$ are finally derived in this $\mathbf{AR_1^r}$-proof. That is, there is no minimal $\mathrm{Dab}_1$-consequence $Dab_1(\Delta)$ of $\Upsilon_7^{\Box} \cup \{A_1\}$ such that either $\rho(q) \in \Delta$ or $\rho(s) \in \Delta$.[24] Hence even if we would extend the proof and derive a Dab-consequence $Dab_1(\Theta)$ such that e.g. $\rho(q) \in \Theta$, then we can further extend it such that $Dab_1(\Theta')$ is derived with $\Theta' \subseteq \Theta - \{\rho(q)\}$.

[24] This follows immediately in view of Theorem 9.1 from Section 9.8.3 below, and the fact that $q$ and $s$ are not relevant to $\Upsilon_7 \oplus A_1$.

So how about the third initial belief? Note that since the initial beliefs entail $q$, $q \equiv t$ could be analyzed further into $t$. However, this is not necessary. We may equally well derive $q \equiv t$ without any analysis, and only afterwards apply disjunctive syllogism to derive $t$:

| | | | |
|---|---|---|---|
| 3 | $\Box(q \equiv t)$ | PREM | $\emptyset$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 20 | $q \equiv t$ | 3; $\mathrm{RD}_1$ | $\{\rho(q), \rho(t)\}$ |
| 21 | $t$ | 13,18; RU | $\{\rho(q), \rho(t)\}$ |

## 9.6 Six Alternatives

In this section we introduce a number of alternative adaptive logics for belief revision. Each of these logics defines a unique revision operation, i.e. a function that maps every couple $\langle \Upsilon, \Psi \rangle$ to a revision set $\Upsilon \oplus \Psi$. The differences between each of these shed new light on the many ways one might achieve a belief revision that obeys postulate **P**. After providing their definition, we will first present an overview of all the differences between the eight resulting consequence relations, resp. revision operations (Section 9.6.1). In Section 9.7, we prove that some of the revision operations from this chapter are conservative extensions of others.

### 9.6.1 The Alternative Systems and Some Examples

**The Logics.** Recall that every adaptive logic in standard format is defined by a triple: a lower limit logic, a set of abnormalities and a strategy. As we will show, varying each of these elements leads to six new logics for belief revision.

The variation on the lower limit logic is very straightforward: instead of **Kt**, we can also use **K** – the latter system was defined on in Section 2.4.2 of Chapter 2. However, this requires that the set of abnormalities is adjusted, as we will do below. For some definitions and metaproofs, it will be convenient to use the names $\mathbf{LLL_1}$, ..., $\mathbf{LLL_4}$ for the lower limit logics of our adaptive logics for belief revision. We have:

$\mathbf{LLL_1} =_{\mathsf{df}} \mathbf{LLL_2} =_{\mathsf{df}} \mathbf{Kt}$
$\mathbf{LLL_3} =_{\mathsf{df}} \mathbf{LLL_4} =_{\mathsf{df}} \mathbf{K}$

In order to define the sets of abnormalities of the alternative adaptive logics we need a few preliminary definitions. Where $\Theta \subset \mathcal{W}^l_c$, let $\sigma(\Theta) = \bigvee\{!\bigvee\Theta' \mid \emptyset \neq \Theta' \subseteq \Theta\}$ and where $\Theta \subset \mathcal{S}$, let $\tau(\Theta) = \sigma(\Theta \cup \Theta^{\neg})$. Whenever these notations are used, it is assumed that $\Theta$ is non-empty and finite. To get some more grip on the abbreviations, consider the following examples:

- $!p = \Box p \wedge \neg p$; $!(p \vee \neg q) = \Box(p \vee \neg q) \wedge \neg(p \vee \neg q)$
- $\sigma(\{p, \neg q\}) = !p \vee !\neg q \vee !(p \vee \neg q)$
- $\tau(\{p, q\}) = !p \vee !q \vee !\neg p \vee !\neg q \vee !(p \vee q) \vee !(p \vee \neg q) \vee !(\neg p \vee q) \vee !(\neg p \vee \neg q)$

The sets of abnormalities we will use in the remainder are:

$\Omega^{\mathsf{r}}_2 =_{\mathsf{df}} \{!D \mid D \in \mathcal{W}^l_c\}$

$\Omega^{\mathsf{r}}_3 =_{\mathsf{df}} \{\tau(\Psi) \mid \emptyset \neq \Psi \subset \mathcal{S}\}$
$\Omega^{\mathsf{r}}_4 =_{\mathsf{df}} \{\sigma(\Theta) \mid \emptyset \neq \Theta \subset \mathcal{W}^l_c\}$

| strategy\\$\mathbf{LLL}, \Omega^{\mathsf{r}}_i$ | $\mathbf{Kt}, \Omega^{\mathsf{r}}_1$, | $\mathbf{Kt}, \Omega^{\mathsf{r}}_2$ | $\mathbf{K}, \Omega^{\mathsf{r}}_3$ | $\mathbf{K}, \Omega^{\mathsf{r}}_4$ |
|---|---|---|---|---|
| Reliability | $\mathbf{AR^r_1}$ | $\mathbf{AR^r_2}$ | $\mathbf{AR^r_3}$ | $\mathbf{AR^r_4}$ |
| Minimal Abnormality | $\mathbf{AR^m_1}$ | $\mathbf{AR^m_2}$ | $\mathbf{AR^m_3}$ | $\mathbf{AR^m_4}$ |

Table 9.1: An overview of the eight logics for belief revision.

Finally, where $i \in \{1, 2, 3, 4\}$ and $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$, $\mathbf{AR^x_i}$ is the flat adaptive logic defined by the triple $\langle \mathbf{LLL_i}, \Omega^{\mathsf{r}}_i, \mathbf{x}\rangle$. This gives us eight flat adaptive logics, as spelled out in Table 9.1.

In the current and next section, it is always assumed that $i \in \{1, 2, 3, 4\}$. We use $Dab_i(\Delta)$ to denote the disjunction of the members of a finite $\Delta \subset \Omega^{\mathsf{r}}_i$. Also, $Ab_i(M) = \{A \in \Omega^{\mathsf{r}}_i \mid M \Vdash A\}$. We will sometimes use $\mathbf{AR}$ as a metavariable for any of the logics in the table, $\mathbf{AR_i}$ as a metavariable for both logics of the same column, and $\mathbf{AR^r_i}$ and $\mathbf{AR^m_i}$ as metavariables for the logics in the first row, resp. the second row.

**The Revision Operations.** The revision operations that correspond to the logics from Table 9.1 are defined as follows:

**Definition 9.16** $\Upsilon \oplus_{\mathbf{AR^x_i}} \Psi = \{B \in \mathcal{W}_c \mid \Upsilon^\Box \cup \Psi \vdash_{\mathbf{AR^x_i}} B\}$.

In order to understand how the revision operations work, it may be useful to know that the following holds:

**Theorem 9.7** *If* $\Upsilon \models_{\mathbf{CL}} B$*, then each of the following holds:*

1. $\Upsilon^\Box \models_{\mathbf{Kt}} B \check{\vee} \bigvee\{!C \vee !\neg C \mid C \in E(B)\}$.
2. $\Upsilon^\Box \models_{\mathbf{K}} B \check{\vee} \bigvee\{\sigma(\Theta) \mid \Theta \subset \mathcal{W}^l_c, E(\Theta) = E(B)\}$.
3. $\Upsilon^\Box \models_{\mathbf{LLL_i}} B \check{\vee} Dab_i(\Delta)$ *for a* $\Delta \subset \Omega^{\mathsf{r}}_i$.

*Proof. Ad 1.* Immediate in view of Theorem 9.6.

*Ad 2.* Suppose $\Upsilon \models_{\mathbf{CL}} B$. Hence $\Upsilon^\Box \models_{\mathbf{K}} \Box B$. Let $C = \bigwedge_{i \in I} C_i$ be a conjunctive normal form of $B$ such that $E(C) \subseteq E(B)$. Note that $\Upsilon^\Box \models_{\mathbf{K}} B \check{\vee} (\Box B \wedge \neg B)$, whence also $\Upsilon^\Box \models_{\mathbf{K}} B \check{\vee} \bigvee_{i\in I}(\Box C_i \wedge \neg C_i)$ or shorter, $\Upsilon^\Box \models_{\mathbf{K}} B \check{\vee} \bigvee_{i \in I} !C_i$. Since for each $i \in I$, $E(C_i) \subseteq E(C) \subseteq E(B)$, the theorem follows immediately.

*Ad 3.* Immediate in view of Theorem 9.6, items 1 and 2, and the fact that $\vee$ and $\check{\vee}$ are equivalent in $\mathbf{K_t}$ and $\mathbf{K}$. ■

Note that by Theorem 9.7, whenever $B$ is a **CL**-consequence of $\Upsilon$, we may conditionally infer that $B$ is in the revision set. Whether or not a condition can be safely relied upon, depends on which logic guides our revisions. However, if $B$ is not relevant to $\Upsilon \oplus \Psi$, then its condition will always be reliable, as shown in Section 9.8.3.

**$\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}}$ versus $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}}$.** Now let us discuss in what respect the 8 revision operations differ, starting with the two most basic operations: $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}}$ and $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}}$. Consider the following example: $\Upsilon_8 = \{p \equiv q, r, s\}$ and $A_2 = \neg r \vee (p \wedge \neg q)$.

Obviously $s$ is in the revision set $\Upsilon_8 \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{i}}} A_2$ for both $i = 1$ and $i = 2$, because belief $s$ does not conflict with the new information $A_2$ in any sense of the word; it is irrelevant to the revision.[25] Whether $r$ and $p \equiv q$ are in the revision set is less obvious. $p$, $q$ and $r$ are all relevant to the revision. Moreover $r$ and $p \equiv q$ definitely cannot be both in the revision set. There seems to be no good reason to choose between one of the two, as the following is a minimal $\mathrm{Dab}_1$-consequence of $\Upsilon_8^{\Box} \cup A_2$:

$$\rho(p) \check{\vee} \rho(q) \check{\vee} \rho(r) \tag{9.6}$$

So, at first glance, it seems rational to drop both entirely. This is exactly what $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$ does: $\Upsilon_8 \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}} A_2 = Cn_{\mathbf{CL}}(\{A_2, s\})$.

However, if one looks a bit closer, one observes that there is no argument in the new information that falsifies the right to left direction of the equivalence $p \equiv q$. It is quite intuitive not to give up on this belief: it is not because the left to right direction of the equivalence $p \equiv q$ is falsified (in combination with the falsification of $r$), that one would therefore immediately possess good reasons to stop believing the right to left direction.

This intuition is formalized by $\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}$. The following is the only minimal $\mathrm{Dab}_2$-consequence of $\Upsilon_8 \cup A_2$:

$$!\neg p \check{\vee} !q \check{\vee} !r \tag{9.7}$$

Hence the literals $p$ and $\neg q$ are not considered unreliable by $\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}$, in view of $\Upsilon_8 \cup A_2$. Consequently, $\neg q \vee p \in \Upsilon_8 \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}} A_2$.

In order to explain the difference between $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$ and $\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}$ in a more general way, let us introduce the notions *positive* and *negative part*. Where $B$ is the complement of $A$ iff $B = \neg A$ or $A = \neg B$, let $A$ be a cosubformula of $C$ if a complement of $A$ is a subformula of $C$. Let a (co-)subformula $B$ of a formula $A$ be a positive part of $A$ iff there is a set $\Gamma$ of (co-)subformulas of $A$ such that $\Gamma \nvdash_{\mathbf{CL}} A$, $\Gamma \nvdash_{\mathbf{CL}} \neg B$ and $\Gamma \cup \{B\} \vdash_{\mathbf{CL}} A$. A negative part of $A$ is a complement of a positive part of $A$.[26]

While $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$ localizes conflicts merely with respect to the bare letters, no matter whether they are positive or negative parts of the conflicts, $\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}$ makes a distinction between positive or negative use of letters. Intuitively, $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$ pinpoints the conflicts between the new information and the belief base by means of the letters that cause the conflict. Any of these conflicting letters are considered harmful/unreliable, no matter whether they are negative or positive parts of beliefs. This may be considered a rather cautious attitude compared to the more detailed attitude formalized by $\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}$. One might therefore conclude that it depends on the level of prudence which one of $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}}$ and $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}}$ is the adequate operation.

[25] To see why, note that the finest splitting of $\Upsilon$ is $\mathbb{E} = \{\{p, q\}, \{r\}, \{s\}\}$.

[26] This definition of the notions positive and negative part is slightly different from the more usual definition in the literature (e.g. in [18]). More particularly, using our definition $\mathbf{CL}-$contradictions have no positive or negative parts and a formula can never be both a positive and a negative part of a formula.

However, it is not the case that for all $\Upsilon, \Psi \subseteq \mathcal{W}_r$, $\Upsilon \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}} \Psi \subseteq \Upsilon \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}} \Psi$. Consider the following example: $A_3 = ((p \wedge \neg q) \vee (q \wedge \neg p)) \wedge ((p \wedge \neg q) \vee \neg r)$. In this case $r \in \Upsilon_8 \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}} A_3$ but $r \notin \Upsilon_8 \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}} A_3$. To see why, compare the minimal Dab$_2$-consequences of $\Upsilon_8 \cup \{A_3\}$:

$$!\neg p \,\check{\vee}\, !q \,\check{\vee}\, !p \,\check{\vee}\, !\neg q \tag{9.8}$$

$$!\neg p \,\check{\vee}\, !q \,\check{\vee}\, !r \tag{9.9}$$

to the (only) minimal Dab$_1$-consequence of the same premise set:

$$\rho(p) \,\check{\vee}\, \rho(q) \tag{9.10}$$

In view of the latter Dab-consequence, $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$ localizes the conflict in $p$ and $q$ – there is no need to also consider $r$ as problematic. However, if one analyzes the situation in a more detailed way, distinguishing positive and negative formulas, one sees that considering $p$, $\neg q$, and $r$ as abnormal is not more justified than pinpointing $p$, $q$, $\neg p$, and $\neg q$ as abnormal literals (as a matter of fact, these choices are incomparable). So there is no reason to assume that $r$ is unproblematic, if one localizes conflicts according to the intuitions behind $\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}$.

**$\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}}$ versus $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{3}}}$.** Consider the belief base $\Upsilon_9 = \{p \supset q, p \supset r\}$ and new information $A_4 = p \wedge \neg q$. We clearly need to drop the belief $p \supset q$ because it is falsified by the new information. But does this also require us to give up on $p \supset r$, just because it contains $p$? If it comes to $\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}$, the answer to this question is yes and indeed $p \supset r \notin \Upsilon_9 \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}} A_4$. But obviously this is not necessary. That one particular relation between two phenomena is falsified by new information does obviously not entail that one should give up every belief about any of the related phenomena. $\mathbf{AR}^{\mathbf{r}}_{\mathbf{3}}$ solves this problem: it treats individual implications (to be more precise: disjunctions of literals) individually and therefore it manages to maintain as many implications as possible, even if some other implication involving the same letter is falsified. For this reason, $p \supset r \in \Upsilon_9 \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{3}}} A_4$

Again, one might think that $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{3}}}$ is always an extension of $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}}$. For similar reasons as the ones regarding the difference between $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}}$ and $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}}$, this is not the case. For example, let $\Upsilon_{10} = \{p \vee q, p \vee r, q \vee r, s\}$ and $A_5 = ((\neg p \wedge \neg q) \vee (\neg q \wedge \neg r) \vee \neg s) \wedge ((\neg p \wedge \neg q) \vee (\neg p \wedge \neg r))\}$. A little thought shows that $s \in \Upsilon_{10} \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}} A_5$ while $s \notin \Upsilon_{10} \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{3}}} A_5$. This illustrates once again that localizing conflicts in a more detailed way does not always allow one to uphold more of the old beliefs—see however below where we return to this point. For the same reasons, the operations $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}}$ and $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{3}}}$ are incommensurable.

While there are good arguments for the way $\mathbf{AR}^{\mathbf{r}}_{\mathbf{3}}$ treats beliefs about implications, the reader needs to be aware that this has counterintuitive effects. Suppose $\Upsilon_{11} = \{p \supset q, q \supset r\}$ and $A_6 = p \wedge \neg q$. This example is similar to the one concerning the revision of $\Upsilon_9$ by $A_4$, but it is somehow less convincing. One would intuitively not expect that the new belief $r$ is a result of revising $\Upsilon_{11}$ by $A_6$. It is perfectly natural that the first original belief $(p \supset q)$ is retracted, since it is falsified by the new information. There is no problem with the other original belief $q \supset r$, for it is a **CL**-consequence of the revision formula $A_6$. So one might expect the revision set to comprise nothing but the **CL**-consequences of $A_6$. Although intuitively, the revision does not present new arguments in favor of $r$, this formula is in the $\mathbf{AR^r_3}$ revision set because $p \supset r$ is a **CL**-consequence of the original beliefs, and there is no reason to retract $p \supset r$. Combined with the new information, retaining $p \supset r$ entails believing that $r$.

Solving this counterintuitive behavior of $\mathbf{AR^r_3}$ would require making a formal distinction between the precise ways in which a belief base can be phrased. More particularly, it would require that some **CL**-equivalent belief bases are revised differently even when they are revised with the same new information. This type of belief revision, although possibly very sensible, is beyond the scope of this chapter.[27]

**Revision by $\oplus_{\mathbf{AR^r_4}}$.** Finally, $\mathbf{AR^r_4}$ combines the ideas behind $\mathbf{AR^r_2}$ and $\mathbf{AR^r_3}$. It treats implications individually and it makes a distinction between positive and negative literals. Consequently, the mentioned differences between the operations naturally extend to the operation $\oplus_{\mathbf{AR^r_4}}$. The reader can easily check that each of the following holds:

- $s \in \Upsilon_8 \oplus_{\mathbf{AR^r_4}} A_2$,
- $q \supset p \in \Upsilon_8 \oplus_{\mathbf{AR^r_4}} A_2$,
- $r \notin \Upsilon_8 \oplus_{\mathbf{AR^r_4}} A_3$,
- $p \supset r \in \Upsilon_9 \oplus_{\mathbf{AR^r_4}} A_4$,
- $s \notin \Upsilon_{10} \oplus_{\mathbf{AR^r_4}} A_5$, and
- $r \in \Upsilon_{11} \oplus_{\mathbf{AR^r_4}} A_6$.

**Minimal Abnormality versus Reliability.** Next to the already explained operations, there are also the stronger Minimal Abnormality variants of the operations. Comparing $\mathbf{AR^m_1}$ and $\mathbf{AR^r_1}$ by means of one example should explain the difference for every particular logic. Consider $\Upsilon_{12} = \{p, q\}$ and $A_7 = \neg p \vee \neg q$. We now wonder whether $p \vee q \in \Upsilon_{12} \oplus_{\mathbf{AR^r_1}} A_7$. The answer is no. Both $p$ and $q$ are considered unreliable, because one of both causes a conflict. Nevertheless, each **Kt**-model that (set theoretically) contains the least abnormalities, verifies either $p$ or $q$. The Minimal Abnormality operations formalize this consideration and so $p \vee q \in \Upsilon_{12} \oplus_{\mathbf{AR^m_1}} A_7$ holds.

The next theorem shows that the Minimal Abnormality operations always yield revision sets that are supersets of the revision sets that are obtained by the corresponding Reliability operations.

**Theorem 9.8** *If* $\Gamma \models_{\mathbf{AR^r_i}} A$*, then* $\Gamma \models_{\mathbf{AR^m_i}} A$*. Consequently,* $\Upsilon \oplus_{\mathbf{AR^r_i}} \Psi \subseteq \Upsilon \oplus_{\mathbf{AR^m_i}} \Psi$*.*

The following theorems show that, unlike the Reliability operations, there are revision operations among the four Minimal Abnormality operations that are straightforward extensions of each other. We prove them in Section 9.7 below.

**Theorem 9.9** *If* $\Gamma \models_{\mathbf{AR^m_2}} A$*, then* $\Gamma \models_{\mathbf{AR^m_1}} A$*. Consequently,* $\Upsilon \oplus_{\mathbf{AR^m_2}} \Psi \subseteq \Upsilon \oplus_{\mathbf{AR^m_1}} \Psi$

[27] For one thing, it is not clear whether we can make such distinctions, yet still ensure that Parikh's postulate **P** is obeyed.

**Theorem 9.10** *If* $\Gamma \models_{\mathbf{AR^m_4}} A$*, then* $\Gamma \models_{\mathbf{AR^m_3}} A$*. Consequently,* $\Upsilon \oplus_{\mathbf{AR^m_4}} \Psi \subseteq \Upsilon \oplus_{\mathbf{AR^m_3}} \Psi$

This is a rather unexpected result, so let us explain it for the case of $\mathbf{AR^m_2}$ and $\mathbf{AR^m_1}$. Recall the minimal Dab-consequence in equation (9.6). Since this disjunction is minimal, there are models of this premise set that verify only one of the three disjuncts. Hence the models $M \in \mathcal{M}_{\mathbf{AR^m_1}}(\Upsilon_8^{\Box} \cup \{A_2\})$ are such that one of the following holds:

$Ab_1(M) = \{\rho(p)\}$
$Ab_1(M) = \{\rho(q)\}$
$Ab_1(M) = \{\rho(r)\}$

Since all models $M$ in the third class of models falsify the abnormality of the letters $p$ and $q$, it obviously holds that $M \Vdash p \equiv q$, whence also $M \Vdash q \supset p$. By a similar reasoning, all models $M$ in the first and second class falsify the abnormality of the letter $r$, whence they verify $r$. But in view of the new information $A_2 = \neg r \vee (p \wedge \neg q)$, this means that these models verify $p \wedge \neg q$, which implies $q \supset p$. In other words, if we minimize the sets of abnormalities verified by each selected model, we overcome the fact that positive and negative parts cannot be treated separately by the logics.

Tables 9.2 and 9.3 summarize the results of applying the 8 belief operations to the examples we presented in this section. The **CL**-consequences of the sets in the table are the actual resulting revision sets (to simplify notation, we skip set brackets in the tables). In order to prove that each of the 8 operations is unique, observe the examples in these tables.

| $\Upsilon, A$ | $\oplus_{\mathbf{AR^r_1}}$ | $\oplus_{\mathbf{AR^r_2}}$ | $\oplus_{\mathbf{AR^r_3}}$ | $\oplus_{\mathbf{AR^r_4}}$ |
|---|---|---|---|---|
| $\Upsilon_8, A_2$ | $s, A_2$ | $s, q \supset p, A_2$ | $s, A_2$ | $s, q \supset p, A_2$ |
| $\Upsilon_8, A_3$ | $s, r, A_3$ | $s, A_3$ | $s, r, A_3$ | $s, A_3$ |
| $\Upsilon_9, A_4$ | $A_4$ | $A_4$ | $p \supset r, A_4$ | $p \supset r, A_4$ |
| $\Upsilon_{10}, A_5$ | $s, A_5$ | $s, A_5$ | $A_5$ | $A_5$ |
| $\Upsilon_{11}, A_6$ | $A_6$ | $A_6$ | $r, A_6$ | $r, A_6$ |
| $\Upsilon_{12}, A_7$ | $A_7$ | $A_7$ | $A_7$ | $A_7$ |

Table 9.2: An overview of the Reliability revision sets for the examples.

| $\Upsilon, A$ | $\oplus_{\mathbf{AR^m_1}}$ | $\oplus_{\mathbf{AR^m_2}}$ | $\oplus_{\mathbf{AR^m_3}}$ | $\oplus_{\mathbf{AR^m_4}}$ |
|---|---|---|---|---|
| $\Upsilon_8, A_2$ | $s, r \vee (p \equiv q), A_2$ | $s, r \vee (p \equiv q), A_2$ | $s, r \vee (p \equiv q), A_2$ | $s, r \vee (p \equiv q), A_2$ |
| $\Upsilon_8, A_3$ | $s, r, A_3$ | $s, q \supset p, A_3$ | $s, r, A_3$ | $s, q \supset p, A_3$ |
| $\Upsilon_9, A_4$ | $A_4$ | $A_4$ | $p \supset r, A_4$ | $p \supset r, A_4$ |
| $\Upsilon_{10}, A_5$ | $s, p \vee q \vee r, A_5$ | $s, p \vee q \vee r, A_5$ | $p \vee q \vee r, A_5$ | $p \vee q \vee r, A_5$ |
| $\Upsilon_{11}, A_6$ | $A_6$ | $A_6$ | $r, A_6$ | $r, A_6$ |
| $\Upsilon_{12}, A_7$ | $p \vee q, A_7$ | $p \vee q, A_7$ | $p \vee q, A_7$ | $p \vee q, A_7$ |

Table 9.3: An overview of the Minimal Abnormality revision sets for the examples

## 9.7 Proofs of Theorems 9.9 and 9.10

Theorems 9.9 and 9.10 are an immediate consequence of the following two lemmas:

**Lemma 9.6** $\mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{2}}}(\Gamma)$

*Proof.* Suppose $M \in \mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma) - \mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{2}}}(\Gamma)$. Note that since $\mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{Kt}}(\Gamma)$, $M \in \mathcal{M}_{\mathbf{Kt}}(\Gamma)$. Hence by the Strong Reassurance of $\mathbf{AR}^{\mathbf{m}}_{\mathbf{2}}$, there is an $M' \in \mathcal{M}_{\mathbf{Kt}}(\Gamma)$ such that (†) $Ab_2(M') \subset Ab_2(M)$.

Suppose $M' \Vdash \rho(A)$ for an $A \in \mathcal{S}$. Then $M' \Vdash !A$ or $M' \Vdash !\neg A$. By (†), $M \Vdash !A$ or $M \Vdash !\neg A$. Hence $M \Vdash \rho(A)$.

So we obtain that

$$Ab_1(M') \subseteq Ab_1(M) \tag{9.11}$$

By (†), there is a $B \in \mathcal{W}^l_c$: $M \Vdash !B$ and $M' \nVdash !B$.

*Case 1.* $B \in \mathcal{S}$. Then $M \Vdash \rho(B)$. Note that $M \Vdash \neg B$, whence $M \nVdash !\neg B$. By (†), $M' \nVdash !\neg B$. Since also $M' \nVdash !B$, $M' \nVdash \rho(B)$.

*Case 2.* $B = \neg C$ for a $C \in \mathcal{S}$. Then $M \Vdash \rho(C)$. Note that $M \Vdash C$, whence $M \nVdash !C$. By (†), $M' \nVdash !C$. Since also $M' \nVdash !\neg C$, $M' \nVdash \rho(C)$. So we obtain that

$$Ab_1(M') \neq Ab_1(M) \tag{9.12}$$

By (9.11) and (9.12), $Ab_1(M') \subset Ab_1(M)$, whence $M \notin \mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma)$. But this contradicts the supposition. ■

**Lemma 9.7** $\mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{3}}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{4}}}(\Gamma)$.

*Proof.* Suppose $M \in \mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{3}}}(\Gamma) - \mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{4}}}(\Gamma)$. Note that since $\mathcal{M}_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{3}}}(\Gamma) \subseteq \mathcal{M}_{\mathbf{K}}(\Gamma)$, $M \in \mathcal{M}_{\mathbf{K}}(\Gamma)$. Hence by the Strong Reassurance of $\mathbf{AR}^{\mathbf{m}}_{\mathbf{4}}$, there is an $M' \in \mathcal{M}_{\mathbf{K}}(\Gamma)$ such that (†) $Ab_4(M') \subset Ab_4(M)$.

Suppose $M' \Vdash \tau(\Pi)$ for a $\Pi \subset \mathcal{S}$. Hence $M' \Vdash \sigma(\Theta)$, for a $\Theta \subseteq \Pi \cup \Pi^{\neg}$. By (†), $M \Vdash \sigma(\Theta)$, whence also $M \Vdash \tau(\Pi)$. So we obtain that

$$Ab_3(M') \subseteq Ab_3(M) \tag{9.13}$$

Note that by (†), there is a $\Theta \subset \mathcal{W}^l_c$ such that $M \Vdash \sigma(\Theta)$, $M' \nVdash \sigma(\Theta)$. Let $\Theta' \subseteq \Theta$ be minimal such that $M \Vdash \sigma(\Theta')$. This implies that (a) $M \Vdash \neg \bigvee \Theta'$, whence $M \Vdash \neg B$ for every $B \in \Theta'$. Hence ($\star$) there is no $A \in \mathcal{S}$ such that $A, \neg A \in \Theta'$. Finally, note that

$$M \Vdash \tau(E(\Theta')) \tag{9.14}$$

Suppose that (‡) $M' \Vdash \tau(E(\Theta'))$. Hence there is a $\Xi \subseteq \Theta' \cup \Theta'^{\neg}$ such that $M' \Vdash \sigma(\Xi)$. Let $\Xi' \subseteq \Xi$ be minimal such that $M' \Vdash \sigma(\Xi')$. Hence $M' \Vdash \neg \bigvee \Xi'$, whence $M' \Vdash \neg B$ for every $B \in \Xi'$. Hence ($\star\star$) there is no $A \in \mathcal{S} : A, \neg A \in \Xi'$.

By (†), $M \Vdash \sigma(\Xi')$. Let $\Xi'' \subseteq \Xi'$ be minimal such that $M \Vdash \sigma(\Xi'')$. Note that $M \Vdash ! \bigvee \Xi''$, whence (b) $M \Vdash \neg \bigvee \Xi''$.

Since $M' \nVdash \sigma(\Theta')$, (c) $\Xi' \nsubseteq \Theta'$. Since $E(\Xi') \subseteq E(\Xi) \subseteq E(\Theta')$ and by (c), ($\star$) and ($\star\star$), we can derive that (d) $\Theta' \nsubseteq \Xi'$.

*Case 1.* $\Xi'' \subseteq (\Theta' \cap \Xi')$. By (d), $\Xi'' \subset \Theta'$. This contradicts the fact that $\Theta'$ is a minimal subset of $\Theta$ such that $M \Vdash \sigma(\Theta')$.

*Case 2.* $\Xi'' \not\subseteq (\Theta' \cap \Xi')$. But then, since $\Xi'' \subseteq \Xi'$, there is a $B \in \Xi'' - \Theta'$.

*Case 2.1.* $B \in \mathcal{S}$. By (b), $M \Vdash \neg B$. But since $E(\Xi'') \subseteq E(\Theta')$, it follows that $\neg B \in \Theta'$, whence by (a), $M \Vdash \neg\neg B$ — a contradiction.

*Case 2.2.* $B = \neg C$ for a $C \in \mathcal{S}$. By (b), $M \Vdash \neg\neg C$. But since $E(\Xi'') \subseteq E(\Theta')$, it follows that $C \in \Theta'$, whence by (a), $M \Vdash \neg C$ — a contradiction.

As a result, (‡) fails: $M' \nVdash \tau(E(\Theta'))$. By (9.14), we obtain that

$$Ab_3(M') \neq Ab_3(M) \tag{9.15}$$

By (9.13) and (9.15), $Ab_3(M') \subset Ab_3(M)$, whence $M \notin \mathcal{M}_{\mathbf{AR_3^m}}(\Gamma)$. But this contradicts the supposition. ■

# 9.8 Some Meta-theoretic Properties

## 9.8.1 The Properties

In this section, we list the most central meta-theoretic virtues of the eight adaptive logics we defined before. Since the logics are formulated in the standard format of adaptive logics, they inherit all the properties that were proven generically for this format – we restate some of them here for the ease of reference:

**Theorem 9.11 ([21]: Th. 11.2)** $\Gamma \subseteq Cn_{\mathbf{AR_i^x}}(\Gamma)$. *(Reflexivity)*

**Theorem 9.12** $Cn_{\mathbf{LLL_i}}(Cn_{\mathbf{AR_i^x}}(\Gamma)) = Cn_{\mathbf{AR_i^x}}(\Gamma)$. *(LLL-Closure)*

Note that Theorem 9.11 entails that $\Psi \subseteq \Upsilon \oplus_{\mathbf{AR}} \Psi$, as required by the *Success* postulate (G2). Theorem 9.12 is crucial for the proof that $\oplus\,\mathbf{AR}$ obeys the *Closure* postulate (G1).

The next property on the list is Reassurance. This is crucial to prove that $\oplus_{\mathbf{AR}}$ obeys the *Success* postulate (G5).

**Theorem 9.13 ([21]: Cr. 2)** *If* $\Gamma$ *has* $\mathbf{LLL_i}$*-models, then* $\Gamma$ *has* $\mathbf{AR_i}$*-models. (Reassurance)*

A final important result for the current application is a specific case of the equivalence criteria for flat ALs (see Theorem 2.20, condition C2):

**Theorem 9.14** *If* $\Gamma$ *and* $\Gamma'$ *are* $\mathbf{LLL_i}$*-equivalent, then they are* $\mathbf{AR_i}$*-equivalent. (Equivalent premise sets)*

Theorem 9.14 renders the proof of the postulate of *Extensionality* (G6) extremely short – we refer to Section 9.8.2 for the details.

Up to this point, we only considered properties of the logics $\mathbf{AR} : \wp(\mathcal{W}_r) \to \wp(\mathcal{W}_r)$. As indicated, some of these properties are useful to establish the well-behavedness of $\oplus_{\mathbf{AR}} : \wp(\mathcal{W}_c) \times \wp(\mathcal{W}_c) \to \wp(\mathcal{W}_c)$. More particularly, the revision function $\oplus_{\mathbf{AR}}$ obeys all the 6 basic Gärdenfors postulates *and* the additional axiom of relevance. The proofs of the following two theorems are presented in Sections 9.8.2 and 9.8.3 respectively:

**Theorem 9.15** $\oplus_{\mathbf{AR}}$ *obeys* ***G1-G6***.

**Theorem 9.16** $\oplus_{\mathbf{AR}}$ *obeys* **P** *for every consistent* $\Upsilon$.

### 9.8.2 Proving the Rationality Postulates

We first state some observations, each of which follow immediately from the definition of **Kt** and **K** – recall that we use $\mathbf{LLL_i}$ as a metavariable for both.

**Fact 9.4** *Each of the following holds:*

1. *If* $\Psi$ *is* **CL**-*satisfiable, then* $\Upsilon^{\Box} \cup \Psi$ *is* $\mathbf{LLL_i}$-*satisfiable*
2. *If* $\Psi \dashv\vdash_{\mathbf{CL}} \Psi'$, *then* $\Upsilon^{\Box} \cup \Psi \dashv\vdash_{\mathbf{LLL_i}} \Upsilon'^{\Box} \cup \Psi'$
3. *Every* **Kt**-*model is a* **K**-*model*
4. $Cn_{\mathbf{CL}}(Cn_{\mathbf{LLL_i}}(\Gamma) \cap \mathcal{W}_c) = Cn_{\mathbf{LLL_i}}(\Gamma) \cap \mathcal{W}_c$

The following is an instance of [21], Theorem 7:

**Theorem 9.17** *If* $\Gamma \models_{\mathbf{LLL_i}} B \check{\vee} Dab_i(\Delta)$ *and* $\Delta \cap U^i(\Gamma) = \emptyset$, *then* $\Gamma \models_{\mathbf{AR_i^r}} B$.

With these tools, the proof of the following is rather straightforward.

**Theorem 9.18** $\oplus_{\mathbf{AR}}$ *obeys postulates G1 and G2.*

*Proof. G1.* Immediate in view of the fact that $\mathbf{AR_i}$ is closed under $\mathbf{LLL_i}$ (see Theorem 9.12), Fact 9.4.4, and Definition 9.16.

*G2.* Immediate in view of Theorem 9.11 and Definition 9.16. ■

**Definition 9.17** *An* $\mathbf{LLL_i}$-*model* $M$ *of* $\Gamma$ *is* $\mathbf{AR_i}$-normal *iff* $Ab_i(M) = \emptyset$.

**Lemma 9.8** *If* $\Upsilon \cup \Psi$ *is consistent, then* $\Upsilon^{\Box} \cup \Psi$ *has* $\mathbf{AR_i}$-*normal models.*

*Proof.* Suppose $\Upsilon \cup \Psi$ is consistent. Let $\Delta$ be a maximal consistent extension (with respect to **CL**) of $\Upsilon \cup \Psi$. Let the model $M = \langle W, R, v, w_0\rangle$ be defined as follows: (1) $W = \{w_0\}$, (2) $R = \{(w_0, w_0)\}$, and (3) for all $B \in \mathcal{S}$, $v(B, w_0) = 1$ iff $B \in \Delta$. Note that $M$ is a **Kt**-model of $\Upsilon^{\Box} \cup \Psi$. By Fact 9.4.3, $M$ is also a **K**-model of $\Upsilon^{\Box} \cup \Psi$. We safely leave it to the reader to prove that $Ab_1(M) = Ab_2(M) = Ab_3(M) = Ab_4(M) = \emptyset$. ■

Note that where $M$ is an $\mathbf{AR_i}$-normal model of $\Gamma$, $M \not\Vdash Dab_i(\Delta)$ for every $\Delta \subset \Omega_i^{\mathsf{r}}$. Hence we have:

**Fact 9.5** *If* $\Gamma$ *has* $\mathbf{AR_i}$-*normal models, then* $U^i(\Gamma) = \emptyset$.

**Theorem 9.19** $\oplus_{\mathbf{AR}}$ *obeys postulate G4.*

*Proof.* Suppose $\Upsilon \cup \Psi$ is consistent. Let $i \in \{1, 2, 3, 4\}$. By Lemma 9.8, $\Upsilon^{\Box} \cup \Psi$ has $\mathbf{AR_i}$-normal models. Hence by Fact 9.5, (†) $U^i(\Gamma) = \emptyset$.

Note that for every $B \in \Upsilon$, by Theorem 9.7.3: $\Upsilon^{\Box} \cup \Psi \vdash_{\mathbf{LLL_i}} B \check{\vee} Dab_i(\Delta)$ for a $\Delta \subset \Omega_i$. Hence by (†) and Theorem 9.17, $\Upsilon^{\Box} \cup \Psi \vdash_{\mathbf{AR_i^r}} B$. By Theorem 9.8, $\Upsilon^{\Box} \cup \Psi \vdash_{\mathbf{AR_i^m}} B$. So we obtain:

$$\text{for every } B \in \Upsilon,\ B \in \Upsilon \oplus_{\mathbf{AR}} \Psi \tag{9.16}$$

By postulate G2, also $\Psi \subseteq \Upsilon \oplus_{\mathbf{AR}} \Psi$. By postulate G1 and (9.16), we obtain that for every $C \in Cn_{\mathbf{CL}}(\Upsilon \cup \Psi) : C \in \Upsilon \oplus_{\mathbf{AR}} \Psi$.

Suppose now that $C \notin Cn_{\mathbf{CL}}(\Upsilon \cup \Psi)$. Hence there is a maximal consistent extension $\Delta_C$ of $\Upsilon \cup \Psi$ such that $C \notin \Delta_C$. Let the model $M_C$ be defined just as the model $M$ from Lemma 9.8, but replacing $\Delta$ with $\Delta_C$. It follows that (i) $M_C$ is an $\mathbf{AR_i}$-normal model of $\Upsilon^{\Box} \cup \Psi$, and (ii) $M_C \not\Vdash C$. Hence $\Upsilon^{\Box} \cup \Psi \not\models_{\mathbf{AR_i}} C$. By the soundness of $\mathbf{AR_i}$ and Definition 9.16, $C \notin \Upsilon \oplus_{\mathbf{AR_i}} \Psi$. ■

**Theorem 9.20** $\oplus_{\mathbf{AR}}$ *obeys postulates G3, G5 and G6.*

*Proof.*

*G3.* Case 1: $\Upsilon \cup \Psi$ is inconsistent. Then $Cn_{\mathbf{CL}}(\Upsilon \cup \Psi) = \mathcal{W}_c$, whence the property follows immediately. Case 2: $\Upsilon \cup \Psi$ is consistent. Then by the *Vacuity* Postulate (G4), it immediately follows that $\Upsilon \oplus_{\mathbf{AR}} \Psi = Cn_{\mathbf{CL}}(\Upsilon \cup \Psi)$.

*G5.* Suppose $\Psi$ is consistent. By Fact 9.4.1, $\Upsilon^{\Box} \cup \Psi$ has $\mathbf{LLL_i}$-models. By Theorem 9.13, $\Upsilon^{\Box} \cup \Psi$ has $\mathbf{AR_i}$-models. Hence $Cn_{\mathbf{AR_i}}(\Upsilon^{\Box} \cup \Psi)$ is $\mathbf{LLL_i}$-satisfiable. This implies that $Cn_{\mathbf{AR_i}}(\Upsilon^{\Box} \cup \Psi) \cap \mathcal{W}_c$ is $\mathbf{CL}$-satisfiable, whence by Definition 9.16, $\Upsilon \oplus_{\mathbf{AR_i}} \Psi$ is $\mathbf{CL}$-satisfiable.

*G6.* Suppose $\Psi \dashv\vdash_{\mathbf{CL}} \Psi'$. By Fact 9.4.2, $\Upsilon^{\Box} \cup \Psi$ and $\Upsilon^{\Box} \cup \Psi'$ are $\mathbf{LLL_i}$-equivalent. By Theorem 9.14, $\Upsilon^{\Box} \cup \Psi$ and $\Upsilon^{\Box} \cup \Psi'$ are $\mathbf{AR_i}$-equivalent. Hence $\Upsilon \oplus_{\mathbf{AR_i}} \Psi = \Upsilon \oplus_{\mathbf{AR_i}} \Psi'$. ■

### 9.8.3 Proving the Relevance Axiom

In this section, it is assumed that $\Upsilon$ and $\Psi$ are consistent – note that if $\Psi$ is inconsistent, then $\Upsilon^{\Box} \cup \Psi$ is $\mathbf{LLL_i}$-trivial, whence it is also $\mathbf{AR_i}$-trivial. This implies that $\Upsilon \oplus_{\mathbf{AR}} \Psi = \check{\mathcal{W}}_r$, whence the theorem follows immediately.

In the remainder, we prove for each of the operations $\oplus_{\mathbf{AR^r_2}}$, $\oplus_{\mathbf{AR^r_1}}$, $\oplus_{\mathbf{AR^r_4}}$ and $\oplus_{\mathbf{AR^r_3}}$ subsequently, that they obey $\mathbf{P}$. In view of Theorem 9.8, it immediately follows that also $\oplus_{\mathbf{AR^m_2}}$, $\oplus_{\mathbf{AR^m_1}}$, $\oplus_{\mathbf{AR^m_4}}$ and $\oplus_{\mathbf{AR^m_3}}$ obey $\mathbf{P}$. Before we start, we first need two lemmas that are used in each of the four subsections. The first one is an immediate consequence of properties of the standard format – see page 21:

**Lemma 9.9** *Where $\Gamma$ is $\mathbf{LLL_i}$-satisfiable: $A \in U^i(\Gamma)$ iff there is an $M \in \mathcal{M}_{\mathbf{AR^m_i}}(\Gamma)$ such that $M \Vdash A$.*

**Lemma 9.10** *If $M = \langle W, R, v, w_0 \rangle$, $E(\Psi) \subset \mathcal{S}$, and there is no $w \in W$ such that $(w_0, w) \in R$, then each of the following holds:*

1. $M \notin \mathcal{M}_{\mathbf{AR^m_2}}(\Upsilon^{\Box} \cup \Psi)$
2. $M \notin \mathcal{M}_{\mathbf{AR^m_4}}(\Upsilon^{\Box} \cup \Psi)$

*Proof.* Suppose the antecedent holds. If $M \notin \mathcal{M}_{\mathbf{K}}(\Upsilon^{\Box} \cup \Psi)$, the lemma follows immediately, so suppose $M \in \mathcal{M}_{\mathbf{K}}(\Upsilon^{\Box} \cup \Psi)$. Note that

$$\text{for every } A \in \mathcal{W}_c, M \Vdash \Box A \tag{9.17}$$

Let $\Delta$ be a maximal consistent extension (with respect to $\mathbf{CL}$) of $\Upsilon$. Let the model $M' = \langle W', R', v', w_0 \rangle$ be defined as follows:

(i) $W' = \{w_0, w_\Delta\}$
(ii) $R' = \{(w_0, w_\Delta)\}$
(iii) For every $B \in \mathcal{S}$: $v'(B, w_\Delta) = 1$ iff $B \in \Delta$
(iv) For every $B \in \mathcal{S} - E(\Psi)$: $v'(B, w_0) = v(B, w_\Delta)$
(v) For every $B \in E(\Psi)$: $v'(B, w_0) = v(B, w_0)$

Note that $M'$ is a **Kt**-model of $\Upsilon^\Box \cup \Psi$, whence by Fact 9.4.3, $M'$ is also a **K**-model of $\Upsilon^\Box \cup \Psi$. Note also that since $M'$ is a **Kt**-model,

$$\text{for every } \Theta \subset \mathcal{W}_c^l\text{: if } M' \Vdash \sigma(\Theta)\text{, then } M' \Vdash !B \text{ for a } B \in \Theta \tag{9.18}$$

*Ad 1.* Suppose $M' \Vdash !B$ for a $B \in \mathcal{W}_c^l$. In view of the construction, $B \in \mathcal{S} \cap E(A)$. But then by (v), $M \Vdash \neg B$, whence by (9.17), $M \Vdash !B$. Hence (†) $Ab_2(M') \subseteq Ab_2(M)$.

Let $C \in \mathcal{S} - E(\Psi)$. Note that, on the one hand, $M' \nVdash !C$ and $M' \nVdash !\neg C$. On the other hand, $M \Vdash C$ or $M \vdash \neg C$, whence by 9.17, $M \Vdash !C$ or $M \Vdash !\neg C$. As a result, (‡) $Ab_2(M') \neq Ab_2(M)$.

By (†) and ‡), $Ab_2(M') \subset Ab_2(M)$. But then since $M'$ is a **Kt**-model of $\Upsilon^\Box \cup \Psi$, $M \notin \mathcal{M}_{\mathbf{AR}_2^\mathbf{m}}(\Upsilon^\Box \cup \Psi)$ by Definition 2.2.

*Ad 2.* Suppose $M' \Vdash \sigma(\Theta)$ for a $\Theta \subset \mathcal{W}_c^l$. Hence by (9.18), $M' \Vdash !B$ for a $B \in \Theta$. By (†), it follows immediately that $M \Vdash !B$, whence also $M \Vdash \sigma(\Theta)$. As a result, (†′) $Ab_4(M') \subseteq Ab_4(M)$.

Let $C \in \mathcal{S} - E(\Psi)$ and let $\Theta = \{C\}$. By (‡), $M' \nVdash \sigma(\Theta), M' \nVdash \sigma(\Theta^\neg)$, whereas $M \Vdash \sigma(\Theta)$ or $M \Vdash \sigma(\Theta^\neg)$. As a result, (‡′) $Ab_4(M') \neq Ab_4(M)$.

By (†′) and ‡′), $Ab_4(M') \subset Ab_4(M)$. But then since $M'$ is a **K**-model of $\Upsilon^\Box \cup \Psi$, $M \notin \mathcal{M}_{\mathbf{AR}_4^\mathbf{m}}(\Upsilon^\Box \cup \{A\})$ by Definition 2.2. ■

**$\oplus_{\mathbf{AR}_2^\mathbf{r}}$ obeys P.**

**Lemma 9.11** *For every $B \in \mathcal{W}_c^l$: If $M \Vdash !B$ and $B$ is not relevant to the revision of $\Upsilon$ by $\Psi$, then $M$ is not an $\mathbf{AR}_2^\mathbf{m}$-model of $\Upsilon^\Box \cup \Psi$.*

*Proof.* Suppose the antecedent holds and $M = \langle \{w_0, w_K\}, R, v, w_0\rangle$ is a **Kt**-model of $\Upsilon^\Box \cup \{A\}$. Let $\Psi^*$ be an arbitrary least letter-set representation of $\Psi$. We prove that $M \notin \mathcal{M}_{\mathbf{AR}_2^\mathbf{m}}(\Upsilon^\Box \cup \Psi^*)$, whence by the equivalence of $\Psi$ and $\Psi^*$, also $M \notin \mathcal{M}_{\mathbf{AR}_2^\mathbf{m}}(\Upsilon^\Box \cup \Psi)$

In view of the supposition, $E(\Psi^*) \subset \mathcal{S}$. If $(w_0, w_K) \notin R$, then by Lemma 9.10, $M \notin \mathcal{M}_{\mathbf{AR}_2^\mathbf{m}}(\Upsilon^\Box \cup \Psi^*)$. So suppose moreover that $(w_0, w_K) \in R$.

Let $\mathbb{E} = \{\Lambda_i\}_{i \in I}$ be the finest splitting of $\Upsilon$. Let $\mathbb{F} = \{\Lambda_i \in \mathbb{E} \mid \Lambda_i \cap E(\Psi^*) \neq \emptyset\}$. Let $M' = \langle \{w_0, w_K\}, R, v', w_0\rangle$, where $v'$ is defined as follows:

(i) where $C \in \mathcal{S} \cap \bigcup \mathbb{F}$: $v'(C, w_0) = v(C, w_0)$ and $v'(C, w_K) = v(C, w_K)$
(ii) where $C \in \mathcal{S} - \bigcup \mathbb{F}$: $v'(C, w_0) = v'(C, w_K) = v(C, w_K)$

Each of the following holds:

(1) $M'$ is a **Kt**-model of $\Upsilon \cup \Psi^*$. To see why, let first $A \in \Psi^*$. Note that since $E(\Psi^*) \subseteq \bigcup \mathbb{F}$ and by (i), $v_{M'}(A, w_0) = v(A, w_0)$, whence $v_{M'}(A, w_0) = 1$ in view of the supposition. Second, note also that for all $C \in \mathcal{W}_c$, $v_{M'}(C, w_K) = v_M(C, w_K)$ in view of (i) and (ii). Hence also $M' \Vdash \Upsilon^\Box$ in view of the supposition.

(2) $Ab_2(M') \subseteq Ab_2(M)$. Suppose $M' \Vdash !D$ for a $D \in \mathcal{W}^l_c$. Hence $v_{M'}(D, w_K) \neq v_{M'}(D, w_0)$, whence $E(D) \in \bigcup \mathbb{F}$ in view of (ii). This implies that $v_{M'}(D, w_0) = v_M(D, w_0)$ and $v_{M'}(D, w_K) = v_M(D, w_K)$. As a result, also $M \Vdash !D$.
(3) $Ab_2(M') \neq Ab_2(M)$. Note that $M' \nVdash !B$, since $B \notin \bigcup \mathbb{F}$ and in view of (ii).

By (2) and (3), $Ab_2(M') \subset Ab_2(M)$. By (1), $M$ is not an $\mathbf{AR}^{\mathbf{m}}_{\mathbf{2}}$-model of $\Upsilon \cup \Psi^*$. ■

By Lemma 9.11 and Lemma 9.9, we obtain:

**Corollary 9.1** *If $B \in \mathcal{W}^l_c$ is not relevant to the revision of $\Upsilon$ by $\Psi$, then $!B \notin U^2(\Upsilon^\Box \cup \Psi)$.*

**Theorem 9.21** *If $B \in Cn_{\mathbf{CL}}(\Upsilon)$ and $B$ is not relevant to $\Upsilon \oplus \Psi$, then $B \in \Upsilon \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}} \Psi$.*

*Proof.* Suppose the antecedent holds. By Theorem 9.7.1,

$$\Upsilon^\Box \models_{\mathbf{Kt}} B \vee \bigvee\{!C \vee !\neg C \mid C \in E(B)\}$$

By the supposition and Definition 9.3, for every $C \in E(B)$: $C$ and $\neg C$ are not relevant to $\Upsilon \oplus \Psi$. Hence by Corollary 9.1, for every $D \in E(B)$: $!D, !\neg D \notin U^2(\Upsilon^\Box \cup \Psi)$. By Theorem 9.17, $\Upsilon^\Box \cup \Psi \models_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{2}}} B$. ■

**$\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}}$ obeys P.**

**Lemma 9.12** *For every $B \in \mathcal{S}$: If $M \Vdash \rho(B)$ and $B$ is not relevant to the revision of $\Upsilon$ by $\Psi$, then $M$ is not an $\mathbf{AR}^{\mathbf{m}}_{\mathbf{1}}$-model of $\Upsilon^\Box \cup \Psi$.*

*Proof.* Suppose the antecedent holds. Note that $M \Vdash !B$ or $M \Vdash !\neg B$, and that $B$ and $\neg B$ are irrelevant to $\Upsilon \oplus \Psi$. By Lemma 9.11, $M$ is not a $\mathbf{AR}^{\mathbf{m}}_{\mathbf{2}}$-model of $\Upsilon^\Box \cup \Psi$. But then by Lemma 9.6, $M$ is not an $\mathbf{AR}^{\mathbf{m}}_{\mathbf{1}}$-model of $\Upsilon^\Box \cup \Psi$. ■

By Lemma 9.12 and Lemma 9.9, we obtain:

**Corollary 9.2** *If $B \in \mathcal{S}$ is not relevant to the revision of $\Upsilon$ by $\Psi$, then $\rho(B) \notin U^1(\Upsilon^\Box \cup \Psi)$.*

**Theorem 9.22** *If $B \in Cn_{\mathbf{CL}}(\Upsilon)$ and $B$ is not relevant to $\Upsilon \oplus \Psi$, then $B \in \Upsilon \oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}} \Psi$.*

*Proof.* Suppose the antecedent holds. By Theorem 9.6, $\Upsilon^\Box \models_{\mathbf{Kt}} B \vee \bigvee\{\rho(C) \mid C \in E(B)\}$. By the supposition and Definition 9.3, for every $C \in E(B)$: $C$ is not relevant to $\Upsilon \oplus \Psi$. Hence by Corollary 9.2, for every $C \in E(B)$: $\rho(C) \notin U^1(\Upsilon^\Box \cup \Psi)$. By Theorem 9.17, $\Upsilon^\Box \cup \Psi \models_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{1}}} B$. ■

$\oplus_{\mathbf{AR}_4^{\mathbf{r}}}$ **obeys P.** To prepare for the subsequent proofs, we first extend the definition of relevance to sets of formulae, as follows:

**Definition 9.18** *Let $\mathbb{E}$ be the finest splitting of $\Upsilon$ and let $\Psi^*$ be a least letter-set representation of $\Upsilon$. We say that a set of formulae $\Theta$ is irrelevant to the revision of $\Upsilon$ by $\Psi$ iff for every cell $\Lambda_i \in \mathbb{E}$: $\Lambda_i \cap E(\Psi^*) = \emptyset$ or $\Lambda_i \cap E(\Theta) = \emptyset$.*

**Lemma 9.13** *If $M \Vdash \sigma(\Theta)$, where $\Theta$ is not relevant to $\Upsilon \oplus \Psi$, then $M$ is not an $\mathbf{AR}_4^{\mathbf{m}}$-model of $\Upsilon^\Box \cup \{A\}$.*

*Proof.* Suppose the antecedent holds and $M = \langle W, R, v, w_0\rangle$ is a **K**-model of $\Upsilon^\Box \cup \Psi$. Let $\Psi^*$ be an arbitrary least letter-set representation of $\Psi$. We prove that $M \notin \mathcal{M}_{\mathbf{AR}_4^{\mathbf{m}}}(\Upsilon^\Box \cup \Psi^*)$, whence by the equivalence of $\Psi$ and $\Psi^*$, also $M \notin \mathcal{M}_{\mathbf{AR}_4^{\mathbf{m}}}(\Upsilon^\Box \cup \Psi)$

In view of the supposition, $E(\Psi^*) \subset \mathcal{S}$. If there is no $w \in W$ such that $(w_0, w) \in R$, the theorem follows immediately in view of Lemma 9.10. So suppose moreover that for a $w \in W$, $(w_0, w) \in R$. Let $\mathbb{E} = \{\Lambda_i\}_{i\in I}$ be the finest splitting of $\Upsilon$ and let $\mathbb{F} = \{\Lambda_i \in \mathbb{E} \mid \Lambda_i \cap E(\Psi^*) \neq \emptyset\}$.

Let $\Delta \in \mathbb{C}_\Upsilon$.[28] Let $\Pi = \{C \in \Delta \mid E(C) \subseteq \mathcal{S} - \bigcup\mathbb{F}\}$ and let $\Pi' = \{C \in \Delta \mid E(C) \subseteq \bigcup\mathbb{F}\}$. Since $\Delta \in \mathbb{C}_\Upsilon$ and $\mathbb{E}$ is the finest splitting of $\Upsilon$, there are no $C \in \Delta$ such that $E(C) \cap \bigcup\mathbb{F} \neq \emptyset$ and $E(C) \not\subseteq \bigcup\mathbb{F}$. Thus the following holds:

$$\Delta = \Pi \cup \Pi' \tag{9.19}$$

Note that since $\Upsilon$ is assumed to be consistent, $\Delta$ is consistent and hence also $\Pi, \Pi'$ are consistent. By the supposition, $M \Vdash \Box C$ for every $C \in Cn_{\mathbf{CL}}(\Upsilon)$, whence also $M \Vdash \Box C$ for every $C \in \Delta$. More particularly,

$$M \Vdash \Box C \text{ for every } C \in \Pi' \tag{9.20}$$

Let $\Pi^+$ be a maximal consistent extension (with respect to **CL**) of $\Pi$. Let the **K**-model $M'$ be defined by $\langle W, R, v', w_0\rangle$, where $v'$ is defined as follows:

(i) where $C \in \bigcup\mathbb{F}$: $v'(C, w) = v(C, w)$ for every $w \in W$
(ii) where $C \in \mathcal{S} - \bigcup\mathbb{F}$: $v'(C, w) = 1$ iff $C \in \Pi^+$, for every $w \in W$

For all $C \in \Pi^+ \cap (\mathcal{S} - \bigcup\mathbb{F})$, $M' \Vdash \Box C$ in view of (ii). Hence for all $C \in \Pi$, $M' \Vdash \Box C$. Also, for all $C \in \Pi'$, $M' \Vdash \Box C$ in view of (i) and (9.20). By (9.19), for all $C \in \Delta$, $M' \Vdash \Box C$. Since $\Delta \dashv\vdash_{\mathbf{CL}} \Upsilon$, $M' \Vdash \Box C$ for every $C \in \Upsilon$. Furthermore, in view of (i) and the fact that $E(\Psi^*) \subseteq \bigcup\mathbb{F}$, $M' \Vdash \Psi^*$. Hence we obtain:

$$M' \text{ is a } \mathbf{K}\text{-model of } \Upsilon^\Box \cup \Psi^* \tag{9.21}$$

In view of (ii), the following holds:

$$\text{for every } \Xi \text{ with } E(\Xi) \subseteq \mathcal{S} - \bigcup\mathbb{F},\ M' \nVdash \sigma(\Xi) \tag{9.22}$$

Suppose $M' \Vdash \sigma(\Xi)$ for some $\Xi \subset \mathcal{W}_c^l$. Let $\Xi'$ be a minimal subset of $\Xi$ for which $M' \Vdash \sigma(\Xi)$. There are three cases to consider:

[28]Recall that $\mathbb{C}_\Upsilon$ is the set of canonical forms of $\Upsilon$ – see page 227 for the exact definition of this set.

*Case 1.* $E(\Xi') \subseteq \bigcup \mathbb{F}$. Then in view of (i), also $M \Vdash \sigma(\Xi)$.
*Case 2.* $E(\Xi') \subseteq \mathcal{S} - \bigcup \mathbb{F}$. This possibility is ruled out by (9.22).
*Case 3.* $E(\Xi')$ contains both members of $\bigcup \mathbb{F}$ and of $\mathcal{S} - \bigcup \mathbb{F}$. Note that for every disjunct $C$ of $\Xi'$, with $E(C) \in \mathcal{S} - \bigcup \mathbb{F}$, either $M' \Vdash \Box C$ or $M' \Vdash \Box\neg C$. In both cases, we can derive that $\Xi'$ is not a minimal subset of $\Xi$ for which $M' \Vdash \sigma(\Xi)$.

It follows that (†) $Ab_4(M') \subseteq Ab_4(M)$.

By the supposition and Definition 9.3, $E(\Theta) \subseteq \mathcal{S} - \bigcup \mathbb{F}$. By (9.22), $M' \nVdash \sigma(\Theta)$. Hence (‡) $Ab_4(M') \neq Ab_4(M)$. By (†) and (‡), $Ab_4(M') \subset Ab_4(M)$. By (9.21), $M \notin \mathcal{M}_{\mathbf{AR}_4^{\mathbf{m}}}(\Upsilon \cup \Psi^*)$. ■

By Lemma 9.9, we obtain:

**Corollary 9.3** *If $\Theta$ is not relevant to $\Upsilon \oplus \Psi$, then $\sigma(\Theta) \notin U^4(\Upsilon^{\Box} \cup \Psi)$.*

**Theorem 9.23** *If $B \in Cn_{\mathbf{CL}}(\Upsilon)$ and $B$ is not relevant to $\Upsilon \oplus \Psi$, then $B \in \Upsilon \oplus_{\mathbf{AR}_4^{\mathbf{r}}} \Psi$.*

*Proof.* Suppose the antecedent holds. By Theorem 9.7.2, $\Upsilon^{\Box} \models_{\mathbf{K}} B \vee \bigvee\{\sigma(\Theta) \mid E(\Theta) = E(B)\}$. By the supposition and Definition 9.18, for every $\Theta \subset \mathcal{W}_c^l$ such that $E(\Theta) = E(B)$: $\Theta$ is not relevant to $\Upsilon \oplus \Psi$. Hence by Corollary 9.3, for every such $\Theta$: $\sigma(\Theta) \notin U^4(\Upsilon^{\Box} \cup \Psi)$. By Theorem 9.17, $\Upsilon^{\Box} \cup \Psi \models_{\mathbf{AR}_4^{\mathbf{r}}} B$. ■

**$\oplus_{\mathbf{AR}_3^{\mathbf{r}}}$ obeys P.**

**Lemma 9.14** *Let $\Psi \subset \mathcal{S}$. If $M \Vdash \tau(\Psi)$ and $\Psi$ is not relevant to $\Upsilon \oplus \Psi$, then $M$ is not an $\mathbf{AR}_3^{\mathbf{m}}$-model of $\Upsilon^{\Box} \cup \Psi$.*

*Proof.* Suppose the antecedent holds. Then $M \Vdash \sigma(\Theta)$ for a $\Theta \subseteq \Psi \cup \Psi^{\neg}$. Note that $\Theta$ is not relevant to $\Upsilon \oplus \Psi$. By Lemma 9.13, $M \notin \mathcal{M}_{\mathbf{AR}_3^{\mathbf{m}}}(\Upsilon^{\Box} \cup \Psi)$. But then by Lemma 9.7, $M \notin \mathcal{M}_{\mathbf{AR}_4^{\mathbf{m}}}(\Upsilon^{\Box} \cup \Psi)$. ■

**Corollary 9.4** *If $\Psi \subset \mathcal{S}$ is not relevant to $\Upsilon \oplus \Psi$, then $\tau(\Psi) \notin U^3(\Upsilon^{\Box} \cup \Psi)$.*

**Theorem 9.24** *If $B \in Cn_{\mathbf{CL}}(\Upsilon)$ and $B$ is not relevant to $\Upsilon \oplus \Psi$, then $B \in \Upsilon \oplus_{\mathbf{AR}_3^{\mathbf{r}}} \Psi$.*

*Proof.* Suppose the antecedent holds. By Theorem 9.7.2, $\Upsilon^{\Box} \models_{\mathbf{K}} B \vee \bigvee\{\sigma(\Theta) \mid E(\Theta) = E(B)\}$, whence also $\Upsilon^{\Box} \models_{\mathbf{K}} B \vee \bigvee \tau(E(B))$. By the supposition and Definition 9.18, $E(B)$ is not relevant to $\Upsilon \oplus \Psi$. Hence by Corollary 9.4, $\tau(\Psi) \notin U^3(\Upsilon^{\Box} \cup \Psi)$. This implies by Theorem 9.17: $\Upsilon^{\Box} \cup \Psi \models_{\mathbf{AR}_3^{\mathbf{r}}} B$. ■

## 9.9 Concluding Remarks

Let us briefly summarize the main results of this chapter. We have argued that a realistic model of relevant belief revision should capture how people perform local analysis in view of new information, and that it should display an internal dynamics. We have shown how to model the dynamics of relevant belief revision with the aid of the dynamic proof theory of $\mathbf{AR}_{\mathbf{1}}^{\mathbf{r}}$. We have presented 8 operations for belief revision, each of them defined in terms of an adaptive logic and a straightforward translation function. An overview has been given of the relations between these operations in terms of their relative strength. Finally, we have shown that they obey all Gärdenfors' postulates for rational belief revision and Parikh's additional axiom of relevance.

Further research should focus on the application of the ideas from the current chapter to related problems, e.g. the revision of inconsistent beliefs and the revision of prioritized belief bases – these issues are considered in Appendix F, resp. Chapter 10. Another line of research could focus more on the heuristics behind dynamic belief revision, as modeled by an adaptive logic.[29] As already noted in Section 9.5, several strategies can be described which allow one to render a line in a proof (un)marked. That humans reason dynamically towards the revision set, does not mean that they do not use certain rules of thumb, which warrant success at least to some extent. A systematic study of such rules could provide further insight into the rationale behind dynamic belief revision.

[29] In [160, 18], algorithms and heuristics are presented that help determining whether a formula is an adaptive logic consequence of a premise set.

# Chapter 10

# Prioritized Belief Revision

*This chapter contains only unpublished results and ideas. I thank Giuseppe Primiero and Peter Verdée for their useful comments on and suggestions for improvements of this chapter.*

## 10.1 Introduction

In the preceding chapter, we saw how the adaptive logic approach captures the dynamics of (relevant) belief revision, and how varying several parameters in the definition of ALs for belief revision allows us to distinguish between various ways in which a revision operation can obey relevance. In this chapter, I turn to the topic of *prioritized* belief revision, i.e. the revision of a set of beliefs in view of new data, where the beliefs are ordered according to their plausibility. I will assume this order to be modular, whence we may consider a set of prioritized beliefs as a tuple $\langle \Upsilon_i \rangle_{i \in I}$, where intuitively, beliefs in $\Upsilon_i$ are more plausible (reliable, trustworthy, ...) than those in $\Upsilon_{i+1}$.

The current chapter differs from the preceding ones in that I will not only present a number of results, but also devote some space to a few hypotheses that I think deserve to be studied in further research. These concern the application of concepts from the study of (prioritized) adaptive logics to the context of belief revision, and of the relevance axiom to prioritized belief bases.

The outline of this chapter is as follows. I will first present a specific approach to prioritized belief revision in terms of what I call "superposing belief revision" (Section 10.2). I will illustrate this idea with the aid of a simple example and show that it has a number of attractive features. In Section 10.3, I define prioritized extensions of the eight adaptive logics from the previous chapter. I illustrate the way these logics capture prioritized revision by means of a simple example. Finally, I return to the topic of relevance in Section 10.4, and propose two ways to extend Parikh's axiom to the prioritized case.

**Preliminaries** I will use $\boldsymbol{\Upsilon}$ (note the bold font) as a metavariable for sequences of sets of beliefs, i.e. $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$ where $I$ is an initial subsequence of $\mathbb{N} = \{1, 2, \ldots\}$ and each $\Upsilon_i \subseteq \mathcal{W}_c$. As before, it is not assumed that the sets $\Upsilon_i$ are closed under **CL**. I will use $\bigcup \boldsymbol{\Upsilon}$ as a shortcut for $\bigcup_{i \in I} \Upsilon_i$. Also, in line with the previous chapter, revision operations are conceived as functions that map every pair $\langle \Upsilon, \Psi \rangle$ to a revision set $\Upsilon \oplus \Psi$. Prioritized revision operations are denoted in bold, by $\boldsymbol{\oplus}$. These are conceived as functions that map every pair $\langle \boldsymbol{\Upsilon}, \Psi \rangle$ to a revision set $\boldsymbol{\Upsilon} \boldsymbol{\oplus} \Psi$.

## 10.2 Superposing Revision Operations

### 10.2.1 The General Idea

Numerous proposals have been made to deal with prioritized sets of beliefs in non-monotonic logic and belief revision.[1] It is not the aim of this section to give an overview of these here. Rather, I will present a *new* way to deal with prioritized belief bases, which is inspired by the idea of superposing adaptive logics. I will show that the resulting operations are well-behaved (Section 10.2.2). After that, a particular such operation will be compared to a more familiar prioritized revision operation by means of concrete examples (Section 10.2.3).

The approach I will present is very generic: from every flat revision operation $\oplus$, we can obtain a prioritized revision operation $\boldsymbol{\oplus}^{\mathsf{s}}$. Also, "hybrid" prioritized revision operations $\boldsymbol{\oplus}^{\mathsf{s}}$ can be obtained from the sequential combination of different flat operations $\oplus_1, \oplus_2, \ldots$. All that is required is that the operation $\oplus$, resp. the operations $\oplus_1, \oplus_2, \ldots$ satisfy the *Success* postulate G2: $\Psi \subseteq \Upsilon \oplus \Psi$ (see also Chapter 9). As will be shown, several other properties of the flat revision operations can be immediately transferred to the prioritized operations obtained from them. But first, let me explain the idea behind $\boldsymbol{\oplus}^{\mathsf{s}}$.

Suppose we want to revise $\boldsymbol{\Upsilon} = \langle \Upsilon_1, \ldots, \Upsilon_n \rangle$ by $\Psi$. Then, starting from a flat revision operation $\oplus$, we may obtain an operation $\boldsymbol{\oplus}^{\mathsf{s}}$ as follows. First, we revise $\Upsilon_1$ by $\Psi$. This gives us the set $\Upsilon_1 \oplus \Psi$. Everything that is in this revision set can be considered as reliable, since it was obtained by the revision of the most plausible beliefs by the new information $\Psi$. Next, we revise the set $\Upsilon_2$ by $\Upsilon_1 \oplus \Psi$, obtaining $\Upsilon_2 \oplus (\Upsilon_1 \oplus \Psi)$. Repeating this procedure $n$ times, we obtain the following set:

$$\boldsymbol{\Upsilon} \boldsymbol{\oplus}^{\mathsf{s}} \Psi =_{\mathsf{df}} \Upsilon_n \oplus (\ldots (\Upsilon_3 \oplus (\Upsilon_2 \oplus (\Upsilon_1 \oplus \Psi))) \ldots)$$

where the right $\ldots$ denotes a sequence of right brackets. Note that, for $\boldsymbol{\oplus}^{\mathsf{s}}$ to be well-defined, $\oplus$ has to be an operation that maps a couple of sets of formulas

[1] An overview of consequence relations for prioritized belief bases can be found in [36]. Within the logic of belief revision, Gärdenfors defined so-called entrenchment-based contraction of theories, i.e. **CL**-closed sets of beliefs. As shown in [57], every partial meet contraction of a theory corresponds to an entrenchment-based contraction of the same theory, where the entrenchment order is modular. Williams adapted Gärdenfors' approach to the case of belief base contraction, using the notion of "ensconsement relation" [166]. It is also possible to apply other formalisms such as safe contraction and partial meet contraction as operations to prioritized belief revision – see [1], resp. [114]. (As explained in the previous chapter, contraction operations yield revision operations in view of the Levi identity.) In [37], revision of prioritized belief bases is studied in the framework of possibility theory and Spohn's ordinal conditional functions.

$\langle \Gamma, \Gamma' \rangle$ to another set of formulas $\Gamma''$.[2] Note also that if $\oplus$ satisfies the *Success* postulate, then the following holds:

$$\Psi \subseteq (\Upsilon_1 \oplus \Psi) \subseteq (\Upsilon_2 \oplus (\Upsilon_1 \oplus \Psi)) \subseteq \ldots \subseteq \boldsymbol{\Upsilon} \oplus^{\mathsf{s}} \Psi$$

The above construction might strike some as far-fetched or even counterintuitive. To explain why it is not, let us briefly consider the relation between belief revision and default logic, as discussed by Makinson and Gärdenfors in [95]. According to their interpretation, the operation $\Upsilon \oplus \Psi$ can also be interpreted as a kind of non-monotonic operation *on* $\Psi$. That is, we take $\Psi$ as a set of facts, and consider $\Upsilon$ as our set of defaults. Hence we try to add elements of $\Upsilon$ to $\Psi$ in a way we consider "rational", and derive **CL**-consequences from the resulting set.[3]

If we transfer this idea to the prioritized context, the result is: first add members of $\Upsilon_1$ to $\Psi$, according to a specific rationality criterion. Next, add as members of $\Upsilon_2$ to $\Upsilon_1 \oplus \Psi$, in accordance with the same criterion. Etc. So in the end, we obtain a rational extension of the set $\Psi$, again, on the assumption that $\oplus$ satisfies the *Success* postulate.

Now, just as we can construct a default logic from every revision operation, we can also construct a prioritized revision operation from every (flat) revision operation, provided that the latter is defined over the appropriate variables (i.e. as a function from a couple of sets to a third set).

Some readers may think that the above procedure is a kind of iterated belief revision (IBR). This is the process in which we stepwise revise our beliefs in view of a sequence of sets that contain ever more recent information. However, in the case of IBR, it is the new information and not the set of old beliefs that has a sequential (and possibly prioritized) character. More formally, IBR looks as follows:

$$(\ldots(((\Upsilon \oplus \Psi_1) \oplus \Psi_2) \oplus \Psi_2) \oplus \ldots) \oplus \Psi_n$$

It is hence perhaps better to say that prioritized belief revision can be seen as a kind of *inverse* iterated belief revision – I will however stick to the term *superposed revision* in the remainder of this chapter.

**Example 10.1** *Define the flat revision operation* $\oplus_{\mathbf{R}}$ *as follows:*[4]

***Definition 10.1*** $\mathbb{M}(\Upsilon, \Psi)$ *is the set of all* $\Theta \subseteq \Upsilon$*, such that (i)* $\Theta \cup \Psi$ *is consistent, and (ii) there is no* $\Theta' \subseteq \Upsilon$ *such that* $\Theta \subset \Theta'$ *and* $\Theta' \cup \Psi$ *is consistent.*

***Definition 10.2 (Reliable Revision)*** $\Upsilon \oplus_{\mathbf{R}} \Psi =_{\mathsf{df}} Cn_{\mathbf{CL}}(\bigcap \mathbb{M}(\Upsilon, \Psi) \cup \Psi)$.

[2] Technically speaking, the construction would also work if $\oplus$ maps each $\langle \Gamma, A \rangle$ to a formula $B$, but I will not consider this non-standard option here.

[3] The paper by Makinson and Gärdenfors presents much more subtle and technical results, but what matters to us here is the idea that we can consider $\Upsilon \oplus \Psi$ as an operation *on* $\Psi$ in view of $\Upsilon$.

[4] As is clear from Definitions 10.1 and 10.2, the operation $\oplus_{\mathbf{R}}$ is very similar to the operation of full meet revision from [3]. However, $\oplus_{\mathbf{R}}$ is defined in such a way that the new information is a set of formulae, and without making the detour via a contraction operation.

*So for instance,* $\mathbb{M}(\{p, \neg q, r\}, \{p \supset q\}) = \{\{p, r\}, \{\neg q, r\}\}$*. Hence* $r$ *is the only element in* $\bigcap \mathbb{M}(\{p, \neg q, r\}, \{p \supset q\})$ *and therefore,* $\{p, \neg q, r\} \oplus_{\mathbf{R}} \{p \supset q\} = Cn_{\mathbf{CL}}(\{r\} \cup \{p \supset q\})$*. It can easily be verified that* $\oplus_{\mathbf{R}}$ *obeys Gärdenfors' six postulates G1-G6 – see Chapter 9.*

*The prioritized revision operation* $\bigoplus^{\mathsf{s}}_{\mathbf{R}}$ *is obtained from the sequential superposition of revision operations by* $\oplus_{\mathbf{R}}$*. Hence, where* $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \leq n}$ *is a sequence of sets of beliefs, and* $\Psi$ *is a set that represents new information, we have:*

$$\boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}}_{\mathbf{R}} \Psi =_{\mathsf{df}} \Upsilon_n \oplus_{\mathbf{R}} (\ldots (\Upsilon_3 \oplus_{\mathbf{R}} (\Upsilon_2 \oplus_{\mathbf{R}} (\Upsilon_1 \oplus_{\mathbf{R}} \Psi))) \ldots)$$

*For every* $i \leq n$*, let* $\Lambda_i$ *denote the set* $\Upsilon_i \oplus_{\mathbf{R}} (\ldots (\Upsilon_3 \oplus_{\mathbf{R}} (\Upsilon_2 \oplus_{\mathbf{R}} (\Upsilon_1 \oplus_{\mathbf{R}} \Psi))) \ldots)$*. Note that* $\Lambda_n = \boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}}_{\mathbf{R}} \Psi$*.*

*To see how* $\bigoplus^{\mathsf{s}}_{\mathbf{R}}$ *works, consider the revision of* $\boldsymbol{\Upsilon}^a = \langle \{p \supset q, \neg r\}, \{\neg q, \neg r \vee s\}, \{\neg s\} \rangle$ *by* $\Psi^a = \{p \wedge r\}$*. This revision operation consists in three steps:*

*(i)* *Revise* $\Upsilon^a_1 = \{p \supset q, \neg r\}$ *by* $\Psi^a$*. Note that* $\mathbb{M}(\Upsilon^a_1, \Psi^a) = \{\{p \supset q\}\}$*. Hence we obtain the set* $\Lambda^a_1 = Cn_{\mathbf{CL}}(\{p \supset q, p \wedge r\})$*. Note that* $q \in \Lambda^a_1$*.*

*(ii)* *Revise* $\Upsilon^a_2 = \{\neg q, \neg r \vee s\}$ *by* $\Lambda^a_1$*. Note that* $\mathbb{M}(\Upsilon^a_2, \Lambda^a_1) = \{\{\neg r \vee \neg s\}\}$*. This gives us the set* $\Lambda^a_2 = Cn_{\mathbf{CL}}(\{\neg r \vee s\} \cup \Lambda^a_1) = Cn_{\mathbf{CL}}(p, q, r, s)$*.*

*(iii)* *Revise* $\Upsilon^a_3 = \{\neg s\}$ *by* $\Lambda^a_2$*. This gives us the set* $\Lambda^a_3 = \Lambda^a_2$*.*

*The set* $\Lambda^a_3$ *is the end result of the procedure, and equals* $\boldsymbol{\Upsilon}^a \bigoplus^{\mathsf{s}}_{\mathbf{R}} \Psi^a$*.*

As I will argue below, it may also be useful to combine different flat revision operations in order to obtain a prioritized one and to superpose an infinite number of such revision operations. Both generalizations are captured by the following definition:

**Definition 10.3** *Let* $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$ *and let* $\mathsf{s} = \langle \oplus_i \rangle_{i \in I}$ *be a sequence of revision operations, where each* $\oplus_i$ *(*$i \in I$*) satisfies the Success postulate. Then:*

*(i)* $\Xi^0_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) =_{\mathsf{df}} \Psi$*.*

*(ii) for all* $i \in I$*,* $\Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) =_{\mathsf{df}} \Upsilon_i \oplus_i \Xi^{i-1}_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$*.*

*(iii)* $\boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}} \Psi =_{\mathsf{df}} \limsup_{i \to \vec{I}} \Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) = \bigcup_{i \in I} \Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$*.*[5]

*We call* $\bigoplus^{\mathsf{s}}$ *the prioritized revision operation, obtained from the superposition of the revision operations* $\langle \oplus_i \rangle_{i \in I}$*.*[6]

Where for all $i, j \in I$, $\oplus_i = \oplus_j = \oplus_{\mathbf{X}}$, I will use $\bigoplus^{\mathsf{s}}_{\mathbf{X}}$ to denote the associated prioritized revision operation.

Before closing this section, let me briefly argue why it is sometimes sensible to combine *different* flat revision operations along the lines of Definition 10.3. In many contexts, we reason on the basis of information from various sources. Apart from the fact that we attach different degrees of plausibility to each of these sources, we also sometimes treat them differently.

[5]Note that $\Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$ converges to its limes superior, in view of the fact that each $\Xi^{i-1}_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) \subseteq \Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$ – see also Lemma 10.1 below.

[6]Recall that for finite $I$, $\vec{I}$ denotes the last element of $I$; for infinite $I$, $\vec{I} = \omega$.

For instance, when trying to learn more about the current British Prime Minister, I may rely on three sources: someone's blog on the internet, the BBC news and a professor in politics at the university of Cambridge. If the blog turns out to contradict the other two sources, I will most likely forget whatever else this blog tells me about the Prime Minister. However, if the BBC news turns out to be mistaken about a specific fact, in view of what the professor says, then I might still consider some other information provided by the same BBC news as reliable.

### 10.2.2 Rationality Postulates for Prioritized Revision

In this section, I prove some generic results concerning superpositions of revision operations. As the reader will note, most of the proofs are fairly straightforward. The upshot of this section is that superposing revision operations leads to intuitive results, whenever the flat operations in the superposition obey Gärdenfors six postulates for rational belief revision – see page 224.

The rationality postulates from Chapter 9 can be generalized to the prioritized case as follows:

**PG1** *Closure:* $\boldsymbol{\Upsilon} \oplus \Psi = Cn_{\mathbf{CL}}(\boldsymbol{\Upsilon} \oplus \Psi)$
**PG2** *Success:* $\Psi \subseteq \boldsymbol{\Upsilon} \oplus \Psi$
**PG3** *Inclusion:* $\boldsymbol{\Upsilon} \oplus \Psi \subseteq Cn_{\mathbf{CL}}(\bigcup \boldsymbol{\Upsilon} \cup \Psi)$
**PG4** *Vacuity:* If $\bigcup \boldsymbol{\Upsilon} \cup \Psi$ is consistent, then $\boldsymbol{\Upsilon} \oplus \Psi = Cn_{\mathbf{CL}}(\bigcup \boldsymbol{\Upsilon} \cup \Psi)$
**PG5** *Consistency:* If $\Psi$ is consistent, then $\boldsymbol{\Upsilon} \oplus \Psi$ is consistent
**PG6** *Extensionality:* If $\Psi \dashv\vdash_{\mathbf{CL}} \Psi'$ , then $\boldsymbol{\Upsilon} \oplus \Psi = \boldsymbol{\Upsilon} \oplus \Psi'$

To prove that $\oplus^{\mathsf{s}}$ satisfies these postulates, I will rely on a lemma which follows immediately from the supposition that all operations $\oplus_i$ in the superposition satisfy the flat *Success* postulate (G2). Recall that this postulate stipulates that the new information is a subset of the revision set: $\Psi \subseteq \Upsilon \oplus \Psi$.

**Lemma 10.1** *Let* $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$ *and let* $\mathsf{s} = \langle \oplus_i \rangle_{i \in I}$ *be a sequence of revision operations, where each* $\oplus_i$ $(i \in I)$ *satisfies the Success postulate. Then for all* $i \in I$*:* $\Psi \subseteq \Xi^{i-1}_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) \subseteq \Xi^{i}_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) \subseteq \boldsymbol{\Upsilon} \oplus^{\mathsf{s}} \Psi$.

*Proof.* Suppose the antecedent holds. By Definition 10.3.(i),

$$\Psi = \Xi^0_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$$

Since for all $i \in I$, $\oplus_i$ obeys G2, and by Definition 10.3.(ii),

$$\Xi^{i-1}_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) \subseteq \Xi^{i}_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$$

Finally, by Definition 10.3.(iii), $\boldsymbol{\Upsilon} \oplus^{\mathsf{s}} \Psi = \bigcup_{j \in I} \Xi^{j}_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$, and hence

$$\Xi^{i}_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) \subseteq \boldsymbol{\Upsilon} \oplus^{\mathsf{s}} \Psi$$

■

The following theorem is the central result of this section. It links the rationality of the flat operations $\langle \oplus_i \rangle_{i \in I}$ to the rationality of the superposed operation $\oplus^{\mathsf{s}}$, obtained from the superposition of these flat operations.

**Theorem 10.1** *Let* $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$ *and let* $\mathsf{s} = \langle \oplus_i \rangle_{i \in I}$ *be a sequence of revision operations, where each* $\oplus_i$ $(i \in I)$ *satisfies the Success postulate. Then each of the following holds:*

1. *if each operation* $\oplus_i$ $(i \in I)$ *obeys G1, then* $\bigoplus^{\mathsf{s}}$ *obeys PG1.*
2. $\bigoplus^{\mathsf{s}}$ *obeys PG2.*
3. *if each operation* $\oplus_i$ $(i \in I)$ *obeys G4, then* $\bigoplus^{\mathsf{s}}$ *obeys PG4.*
4. *if each operation* $\oplus_i$ $(i \in I)$ *obeys G3 and G4, then* $\bigoplus^{\mathsf{s}}$ *obeys PG3.*
5. *if each operation* $\oplus_i$ $(i \in I)$ *obeys G5, then* $\bigoplus^{\mathsf{s}}$ *obeys PG5.*
6. *if each operation* $\oplus_i$ $(i \in I)$ *obeys G6, then* $\bigoplus^{\mathsf{s}}$ *obeys PG6.*

*Proof.* *Ad 1.* Suppose the antecedent holds. That $\boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}} \Psi \subseteq Cn_{\mathbf{CL}}(\boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}} \Psi)$ is immediate in view of the reflexivity of **CL**. For the other direction, suppose $A \in Cn_{\mathbf{CL}}(\boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}} \Psi)$. By the compactness of **CL**, there are $B_1, \ldots, B_n \in \boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}} \Psi$ such that $\{B_1, \ldots, B_n\} \vdash_{\mathbf{CL}} A$. By Definition 10.3.(ii)-(iii), there is an $m \in I$: $\{B_1, \ldots, B_n\} \subseteq \Xi^m_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) = \Upsilon_m \oplus_m (\Upsilon_{m-1} \oplus_{m-1} (\ldots \oplus (\Upsilon_1 \oplus \Psi) \ldots))$. Since $\oplus_m$ obeys G1, $\Xi^m_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) = Cn_{\mathbf{CL}}(\Xi^m_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi))$. It follows that $A \in \Xi^m_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$. Hence, by Definition 10.3.(iii), $A \in \boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}} \Psi$.

*Ad 2.* Immediate in view of Lemma 10.1.

*Ad 3.* Suppose the antecedent holds, and that $\bigcup \boldsymbol{\Upsilon} \cup \Psi$ is consistent. I first prove that

(†) for every $i \in I$, $\Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) = Cn_{\mathbf{CL}}(\bigcup_{j \leq i} \Upsilon_j \cup \Psi)$

$(i = 1)$ By the supposition and the monotonicity of **CL**, $\Upsilon_1 \cup \Psi$ is consistent. Hence, since $\oplus_1$ obeys G4, $\Xi^1_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) = \Upsilon_1 \oplus_1 \Psi = Cn_{\mathbf{CL}}(\Upsilon_1 \cup \Psi)$.

$(i \Rightarrow i+1)$ By the supposition and the monotonicity of **CL**, $\bigcup_{j \leq i+1} \Upsilon_j \cup \Psi$ is consistent. By **CL**-properties, $\Upsilon_{i+1} \cup Cn_{\mathbf{CL}}(\bigcup_{j \leq i} \Upsilon_j \cup \Psi)$ is consistent. By the induction hypothesis, $\Upsilon_{i+1} \cup \Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$ is consistent. Since $\oplus_{i+1}$ obeys G4, it follows that $\Xi^{i+1}_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) = \Upsilon_{i+1} \oplus_{i+1} \Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) = Cn_{\mathbf{CL}}(\Upsilon_{i+1} \cup \Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)) = Cn_{\mathbf{CL}}(\bigcup_{j \leq i+1} \Upsilon_j \cup \Psi)$.

By Definition 10.3.(iii), (†) and **CL**-properties, $\boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}} \Psi = \bigcup_{i \in I} \Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi) = \bigcup_{i \in I} Cn_{\mathbf{CL}}(\bigcup_{j \leq i} \Upsilon_j \cup \Psi) = Cn_{\mathbf{CL}}(\bigcup \boldsymbol{\Upsilon} \cup \Psi)$.

*Ad 4.* Suppose the antecedent holds. If $\bigcup \boldsymbol{\Upsilon} \cup \Psi$ is inconsistent, $Cn_{\mathbf{CL}}(\bigcup \boldsymbol{\Upsilon} \cup \Psi) = \mathcal{W}_c$, whence the property follows immediately. In the other case, we have by the preceding item that $\boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}} \Psi = Cn_{\mathbf{CL}}(\bigcup \boldsymbol{\Upsilon} \cup \Psi)$.

*Ad 5.* Suppose the antecedent holds and $\Psi$ is consistent. I first prove the following:

(‡) for every $i \in I$, $\Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$ is consistent

$(i = 1)$ Immediate in view of the supposition that $\Psi$ is consistent, that $\oplus_1$ obeys G5, and Definition 10.3.(ii).

$(i \Rightarrow i+1)$ Immediate in view of the induction hypothesis, the fact that $\oplus_{i+1}$ obeys G5, and Definition 10.3.(ii).

By (‡) and the compactness of **CL**, $\bigcup_{i \in I} \Xi^i_{\mathsf{s}}(\boldsymbol{\Upsilon}, \Psi)$ is consistent. Hence, by Definition 10.3.(iii), $\boldsymbol{\Upsilon} \bigoplus^{\mathsf{s}} \Psi$ is consistent.

*Ad 6.* Suppose the antecedent holds and $\Psi \dashv\vdash \Psi'$. Then since $\oplus_1$ obeys G6, it also follows that $\Upsilon_1 \oplus_1 \Psi = \Upsilon_1 \oplus_1 \Psi'$. The rest is immediate in view of Definition 10.3.(ii)-(iii). ■

Besides these basic postulates, we can also prove a stronger variant of *Vacuity*. This postulate corresponds to the theorems about premise sets that are normal up to level $i$, which were proven for prioritized ALs in the first part of this thesis (see Chapters 3-5).

**Theorem 10.2 (Prioritized Vacuity)** *Let* $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$ *and let* $\mathsf{s} = \langle \oplus_i \rangle_{i \in I}$ *be a sequence of revision operations that satisfy the Success postulate. If* $\bigcup_{i \leq j} \Upsilon_i \cup \Psi$ *is consistent for a* $j \in I$*, then* $Cn_{\mathbf{CL}}(\bigcup_{i \leq j} \Upsilon_i \cup \Psi) \subseteq \boldsymbol{\Upsilon} \oplus \Psi$.

*Proof.* Immediate in view of the proof for item 3 of Theorem 10.1. ■

The *Prioritized Vacuity* postulate relates to a more general requirement, which I will call "prioritized rationality". Suppose that we have a number of requirements, say $X_1$ up to $X_n$, that we consider as characteristic for the class of all "rational" flat revision operations. Moreover, suppose that for all flat operations $\oplus$ that are rational according to the list of postulates $X_1, \ldots, X_n$, $A \in \Upsilon_1 \oplus \Psi$. In that case, it seems justified to require that for all rational prioritized revision operations $\boldsymbol{\oplus}$, $A \in \boldsymbol{\Upsilon} \boldsymbol{\oplus} \Psi$. That is, since we consider the beliefs in $\Upsilon_1$ as the most plausible beliefs, we should want every rational prioritized revision operation to give us at least the results that would be obtained by revising $\Upsilon_1$ by $\Psi$, in terms of a rational flat revision operation. I will return to this point in Section 10.4, where I apply it to the Axiom of Relevance.

### 10.2.3 Lexicographic versus Superposed Revision

**Maxi-Revision** Many revision operations, and more generally, many nonmonotonic consequence relations described in the literature follow a strategy that was first proposed by Nicholas Rescher and Ruth Manor [126]. Roughly speaking, their idea is to use maximal consistent subsets of a set of formulas, in order to avoid explosion in the face of mutually inconsistent beliefs or assumptions. When applied to the context of belief revision, this can lead to several different operations. One example is the operation of partial meet revision, which was recapitulated in Chapter 9. Another is the operation $\oplus_{\mathbf{R}}$, which was already defined in Section 10.2.1 (see Definitions 10.2).

A slightly stronger flat revision operation is obtained by first applying classical logic to each of the sets $\Delta \cup \Psi$ with $\Delta \in \mathbb{M}(\Upsilon, \Psi)$, and only afterwards taking their intersection. Let us call this operation *maxi-revision*:

**Definition 10.4 (Maxi-Revision)** $\Upsilon \oplus_{\mathbf{M}} \Psi = \bigcap_{\Delta \in \mathbb{M}(\Upsilon, \Psi)} Cn_{\mathbf{CL}}(\Delta \cup \Psi)$

**Theorem 10.3** *Maxi-revision obeys postulates G1-G6.*

*Proof. G1.* In view of the reflexivity of **CL**, it suffices to prove that $Cn_{\mathbf{CL}}(\Upsilon \oplus_{\mathbf{M}} \Psi) \subseteq \Upsilon \oplus_{\mathbf{M}} \Psi$. So suppose $A \in Cn_{\mathbf{CL}}(\Upsilon \oplus_{\mathbf{M}} \Psi)$. By the compactness of **CL**, there are $B_1, \ldots, B_n \in \Upsilon \oplus_{\mathbf{M}} \Psi$ such that (†) $\{B_1, \ldots, B_n\} \vdash_{\mathbf{CL}} A$. By Definition 10.4, for every $\Delta \in \mathbb{M}(\Upsilon, \Psi)$, $\{B_1, \ldots, B_n\} \subseteq Cn_{\mathbf{CL}}(\Delta)$. By (†) and the transitivity of **CL**, $A \in \Delta$ for every $\Delta \in \mathbb{M}(\Upsilon, \Psi)$. Hence, by Definition 10.4, $A \in \Upsilon \oplus_{\mathbf{M}} \Psi$.

*G2.* Immediate in view of the reflexivity of **CL** and Definition 10.4.

*G4.* Suppose $\Upsilon \cup \Psi$ is consistent. By Definition 10.1, $\mathbb{M}(\Upsilon, \Psi) = \{\Upsilon\}$, whence by Definition 10.4, $\Upsilon \oplus_{\mathbf{M}} \Psi = Cn_{\mathbf{CL}}(\Upsilon \cup \Psi)$.

*G3.* If $\Upsilon \cup \Psi$ is consistent, the property follows immediately in view of the preceding item. In the other case, $Cn_{\mathbf{CL}}(\Upsilon \cup \Psi) = \mathcal{W}_c$, whence the property also follows immediately.

*G5.* Suppose $\Psi$ is consistent. By Definition 10.1, for every $\Delta \in \mathbb{M}(\Upsilon, \Psi)$, $\Delta \cup \Psi$ is consistent. It follows that for every $\Delta \in \mathbb{M}(\Upsilon, \Psi)$, $Cn_{\mathbf{CL}}(\Delta \cup \Psi)$ is consistent. By the monotonicity of **CL**, $\bigcap_{\Delta \in \mathbb{M}(\Upsilon,\Psi)} Cn_{\mathbf{CL}}(\Delta \cup \Psi) = \Upsilon \oplus_{\mathbf{M}} \Psi$ is consistent.

*G6.* Suppose $\Psi \dashv\vdash_{\mathbf{CL}} \Psi'$. By Definition 10.1, $\mathbb{M}(\Upsilon, \Psi) = \mathbb{M}(\Upsilon, \Psi')$. The rest is immediate in view of Definition 10.4. ■

**Example 10.2** *Consider again the revision of $\{p, \neg q, r\}$ by $\{p \supset q\}$ (see also Example 10.1). As was the case with $\oplus_{\mathbf{R}}$, the belief $r$ is upheld. However, this time we are also able to do something with the beliefs $p$ and $\neg q$. That is, the two maximal subsets of $\{p, \neg q, r\}$ that are selected are $\{p, r\}$ and $\{\neg q, r\}$. Each of these entail $p \vee \neg q$. Hence, although we have to give up both $p$ and $\neg q$, we can still uphold the (implicit) belief $p \vee \neg q$.*[7]

In what follows, I will consider two prioritized variants of $\oplus_{\mathbf{M}}$: one that is obtained by selecting a subset of $\mathbb{M}(\bigcup \boldsymbol{\Upsilon}, \Psi)$ in view of the priority of the members of $\bigcup \boldsymbol{\Upsilon}$, and another that is obtained by superposing revisions by $\oplus_{\mathbf{M}}$. As I will show, the latter operation is often stronger, and thus allows us to retain more of the original beliefs when revising a prioritized belief base $\boldsymbol{\Upsilon}$.

**Lexicographic Maxi-Revision** In order to obtain a prioritized variant of $\oplus_{\mathbf{M}}$, we can define a selection function, which allows us to select those $\Delta \in \mathbb{M}(\bigcup \boldsymbol{\Upsilon}, \Psi)$ that contain the most plausible beliefs from $\bigcup \boldsymbol{\Upsilon}$. Following Nebel [114], this is done as follows. We first define a lexicographic order $\ll$ on the sets $\Delta \in \mathbb{M}(\bigcup \boldsymbol{\Upsilon}, \Psi)$:[8]

**Definition 10.5** *Let $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$. Where $\Delta, \Delta' \in \mathbb{M}(\bigcup \boldsymbol{\Upsilon}, \Psi)$: $\Delta \ll \Delta'$ iff there is an $i \in I$ such that (a) for all $j < i$, $\Delta \cap \Upsilon_j = \Delta' \cap \Upsilon_j$ and (b) $\Delta' \cap \Upsilon_i \subset \Delta \cap \Upsilon_i$.*

By $\ll$, we obtain the set $\mathbb{P}(\boldsymbol{\Upsilon}, \Psi) =_{\mathsf{df}} \{\Delta \in \mathbb{M}(\bigcup \boldsymbol{\Upsilon}, \Psi) \mid \text{ for no } \Delta' \in \mathbb{M}(\bigcup \boldsymbol{\Upsilon}, \Psi) : \Delta' \ll \Delta\}$. Finally, we define the prioritized revision operation $\oplus_{\mathbf{P}}$ in a way analogous to $\oplus_{\mathbf{M}}$, but replacing $\mathbb{M}(\Upsilon, \Psi)$ with $\mathbb{P}(\boldsymbol{\Upsilon}, \Psi)$:

**Definition 10.6** $\Upsilon \oplus_{\mathbf{P}} \Psi =_{\mathsf{df}} \bigcap_{\Delta \in \mathbb{P}(\boldsymbol{\Upsilon},\Psi)} Cn_{\mathbf{CL}}(\Delta \cup \Psi)$

It can easily be shown that $\oplus_{\mathbf{P}}$ satisfies the prioritized rationality postulates PG1-PG6. The proof is obtained by small variations on that for Theorem 10.3 – I safely leave this to the reader. Also, $\oplus_{\mathbf{P}}$ satisfies the postulate of *Prioritized Vacuity*:

[7] As this example shows, the difference between $\oplus_{\mathbf{R}}$ and $\oplus_{\mathbf{M}}$ resembles the difference between adaptive logics that use the Reliability Strategy, and their Minimal Abnormality-variants.

[8] This definition of $\ll$ is exactly the same as Nebel's; however, Nebel uses this preference relation to define a contraction operation in terms of remainder sets, whereas I use it to select a subset of $\mathbb{M}(\bigcup \boldsymbol{\Upsilon}, \Psi)$.

**Theorem 10.4** *Let $\mathbf{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$. If $\bigcup_{i \leq j} \Upsilon_i \cup \Psi$ is consistent for a $j \in I$, then $Cn_{\mathbf{CL}}(\bigcup_{i \leq j} \Upsilon_i \cup \Psi) \subseteq \mathbf{\Upsilon} \oplus_{\mathbf{P}} \Psi$.*

*Proof.* Suppose the antecedent holds. Let $\Theta \in \mathbb{M}(\bigcup \mathbf{\Upsilon}, \Psi)$ be such that $\bigcup_{i \leq j} \Upsilon_i \subseteq \Theta$ — it can easily be verified that $\Theta$ exists in view of the supposition. Let $\Theta' \in \mathbb{M}(\bigcup \mathbf{\Upsilon}, \Psi)$ be such that $\bigcup_{i \leq j} \Upsilon_i \not\subseteq \Theta'$. It follows immediately that $\Theta \ll \Theta'$. As a result, for every $\Lambda \in \mathbb{P}(\mathbf{\Upsilon}, \Psi)$, $\bigcup_{i \leq j} \Upsilon_i \subseteq \Lambda$. Hence, for every $\Lambda \in \mathbb{P}(\mathbf{\Upsilon}, \Psi)$, $Cn_{\mathbf{CL}}(\bigcup_{i \leq j} \Upsilon_i \cup \Psi) \subseteq Cn_{\mathbf{CL}}(\Lambda)$. The rest is immediate in view of Definition 10.6. ■

**Example 10.3** *Let $\mathbf{\Upsilon}^b = \langle \{p, q\}, \{r \supset \neg p, \neg s\}, \{\neg q \vee s\} \rangle$. Consider the revision of this base by $\Psi^b = \{r\}$. First of all, $\mathbb{M}(\bigcup \mathbf{\Upsilon}^b, \Psi^b)$ consists of five sets:*

$\Delta_1 = \{p, q, \neg s\}$
$\Delta_2 = \{p, q, \neg q \vee s\}$
$\Delta_3 = \{p, \neg s, \neg q \vee s\}$
$\Delta_4 = \{q, r \supset \neg p, \neg s\}$
$\Delta_5 = \{q, r \supset \neg p, \neg q \vee s\}$
$\Delta_6 = \{r \supset \neg p, \neg s, \neg q \vee s\}$

*From these, only $\Delta_1$ is $\ll$-minimal: $\Delta_1 \ll \Delta_2 \ll \Delta 3 \ll \Delta_6$ and $\Delta_1 \ll \Delta_2 \ll \Delta_4 \ll \Delta_5 \ll \Delta_6$. It follows that $\mathbf{\Upsilon}^b \oplus_{\mathbf{P}} \Psi^b = Cn_{\mathbf{CL}}(\Delta_1 \cup \{r\}) = Cn_{\mathbf{CL}}(\{p, q, \neg s, r\})$.*

**Example 10.4** *Let $\mathbf{\Upsilon}^c = \langle \{p, q\}, \{r\} \rangle$ and let $\Psi^c = \{\neg p \vee \neg q, \neg p \vee \neg r\}$. In this case, there are two sets in $\mathbb{M}(\bigcup \mathbf{\Upsilon}^c, \Psi^c)$:*

$\Theta_1 = \{p\}$
$\Theta_2 = \{q, r\}$

*Neither of these sets is "better" than the other in terms of the lexicographic order $\ll$. That is, $\Theta_1 \not\ll \Theta_2$ and $\Theta_2 \not\ll \Theta_1$, since the sets $\Theta_1 \cap \Upsilon_1^c$ and $\Theta_2 \cap \Upsilon_1^c$ are incomparable. As a result, $p \vee q$ and $p \vee r$ are in the revision set $\mathbf{\Upsilon}^c \oplus_{\mathbf{P}} \Psi^c$, but neither $p$ nor $q$ or $r$ can be upheld.*

**Superposing Maxi-Revision** The operation $\oplus^{\mathsf{s}}_{\mathbf{M}}$ is obtained by superposing revisions by $\oplus_{\mathbf{M}}$. Since $\oplus_{\mathbf{M}}$ obeys the flat rationality postulates G1-G6, and in view of Theorem 10.1, $\oplus^{\mathsf{s}}_{\mathbf{M}}$ obeys the rationality postulates from the preceding section. Moreover, as the following examples show, $\oplus^{\mathsf{s}}_{\mathbf{M}}$ is a very strong operation.

**Example 10.5** *Consider again the revision of $\mathbf{\Upsilon}^b = \langle \{p, q\}, \{r \supset \neg p, \neg s\}, \{\neg q \vee s\} \rangle$ by $\Psi^b = \{r\}$. Let $\Upsilon_1^b = \{p, q\}, \Upsilon_2^b = \{r \supset \neg p, \neg s\}, \Upsilon_3^b = \{\neg q \vee s\}$. First of all, note that $\Psi^b \cup \Upsilon_1^b$ is consistent, which implies that all the beliefs in $\Upsilon_1^b$ are upheld: $\Delta_1^b = \Upsilon_1^b \oplus_{\mathbf{M}} \Psi^b = Cn_{\mathbf{CL}}(\{p, q\} \cup \{r\})$.*

*The second step in the revision operation consists in revising $\Upsilon_2^b$ by $\Delta_1^b$. This means that the belief $r \supset \neg p \in \Upsilon_2^b$ has to be removed; the belief $\neg s \in \Upsilon_2^b$ can be upheld without problems. So we obtain $\Delta_2^b = Cn_{\mathbf{CL}}(\{p, q, r, \neg s\})$. As a result, also the belief $\neg q \vee s$ will have to be removed in the third and last step of the revision. So we obtain the following revision set:*

$$\mathbf{\Upsilon}^b \oplus_{\mathbf{M}} \Psi^b = Cn_{\mathbf{CL}}(\{p, q, r, \neg s\})$$

*Note that, although the procedure by which it was obtained is rather different, this revision set is the same as the one obtained by the operation* $\oplus_{\mathbf{P}}$.

**Example 10.6** *Consider again the revision of* $\mathbf{\Upsilon}^c = \langle\{p, q\}, \{r\}\rangle$ *by* $\Psi^c = \{\neg p \vee \neg q, \neg p \vee \neg r\}$. *As before, we use* $\Upsilon^c_1$ *and* $\Upsilon^c_2$ *to denote the two sets in the sequence* $\mathbf{\Upsilon}^c$. *We start with revising* $\Upsilon^c_1$ *by* $\Psi^c$. *There are two sets in* $\mathbb{M}(\Upsilon^c_1, \Psi^c)$*:* $\{p\}$ *and* $\{q\}$. *This means that both the belief* $p$ *and the belief* $q$ *have to be dropped. However, since each of the following hold:*

$p \vee q \in Cn_{\mathbf{CL}}(\{p\} \cup \Psi^c)$
$p \vee q \in Cn_{\mathbf{CL}}(\{q\} \cup \Psi^c)$

*we do have that* $p \vee q \in \Upsilon^c_1 \oplus_{\mathbf{M}} \Psi^c$. *In fact,* $\Upsilon^c_1 \oplus_{\mathbf{M}} \Psi^c = \Delta^c_1 = Cn_{\mathbf{CL}}(\{p \vee q, \neg p \vee \neg q, \neg p \vee \neg r\})$.

*Now consider the revision of* $\Upsilon^c_2$ *by* $\Delta^c_1$. *Note that* $r$ *is compatible with* $\Delta^c_1$. *As a result, there is only one set in* $\mathbb{M}(\Upsilon^c_2, \Delta^c_1)$, *i.e.* $\Upsilon^c_2 = \{r\}$. *But this means that* $\Delta^c_2 = \Upsilon^c_1 \oplus_{\mathbf{M}} \Psi^c = Cn_{\mathbf{CL}}(\{r\} \cup \Delta^c_1) = Cn_{\mathbf{CL}}(\{\neg p, q, r\})$.

*So, in contrast to the operation* $\oplus_{\mathbf{P}}$, *the operation* $\oplus^{\mathsf{s}}_{\mathbf{M}}$ *allows us to uphold the belief* $r$, *and thereby also the belief* $q$ *(since* $\{\neg p \vee \neg r, r\} \vdash_{\mathbf{CL}} \neg p$ *and* $\{p \vee q, \neg p\} \vdash_{\mathbf{CL}} q$*).*

These and other, more complex examples (which I omit here for reasons of space) motivate the first conjecture of this chapter, which is inspired by the results from Section 6.7 in Chapter 6. There it was proven that every superposition of flat adaptive logics that use the Minimal Abnormality strategy is at least as strong as the corresponding lexicographic adaptive logic, whenever $\Gamma$ gives rise to only finitely many equivalence classes of minimally abnormal models.

The operation $\oplus_{\mathbf{P}}$ is defined in terms of a lexicographic selection of sets $\Delta \in \mathbb{M}(\bigcup \mathbf{\Upsilon}, \Psi)$, whereas the operation $\oplus^{\mathsf{s}}_{\mathbf{M}}$ is defined from the superposition of corresponding flat revision operations. Hence, translating the results from Chapter 6 to the current context, we obtain the following conjecture:

**Conjecture 10.1** *If* $\mathbb{M}(\bigcup \mathbf{\Upsilon}, \Psi)$ *is finite, then* $\mathbf{\Upsilon} \oplus_{\mathbf{P}} \Psi \subseteq \mathbf{\Upsilon} \oplus^{\mathsf{s}}_{\mathbf{M}} \Psi$.

## 10.3 Prioritized Adaptive Revision

### 10.3.1 A Class of Logics

Recall the **K**-based logics from Chapters 3-5: $\mathbf{SK^r}, \mathbf{SK^m}, \mathbf{HK^r}, \mathbf{HK^m}, \mathbf{K}^{\mathbf{m}}_{\sqsubset}, \ldots$. These allowed us to conditionally derive $A$ from $\Diamond^i A$, for every $A \in \mathcal{W}^l_c$. Although I introduced these systems in order to illustrate the various formats of prioritized ALs, and to establish some facts about those formats, they can be considered as logics for prioritized belief revision as well. Let me briefly explain why this is so.

Where $\mathbf{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$, let $\mathbf{\Upsilon}^{\Diamond} = \{\Diamond^i A \mid A \in \Upsilon_i, i \in I\}$. Let **PK** be a metavariable for the prioritized adaptive logics from Chapters 3-5. Then we may define the prioritized revision operation $\oplus_{\mathbf{PK}}$ as follows:

$\boldsymbol{\Upsilon} \oplus_{\mathbf{PK}} \Psi =_{\mathsf{df}} Cn_{\mathbf{PK}}(\boldsymbol{\Upsilon}^{\Diamond} \cup \Psi) \cap \mathcal{W}_c$

In words, the revision set of $\boldsymbol{\Upsilon}$ by $\Psi$ in view of the logic **PK** is the set of all non-modal formulas that can be finally **PK**-derived from the premise set $\boldsymbol{\Upsilon}^{\Diamond} \cup \Psi$.

One clear advantage of the resulting revision operations is that they are very conflict-tolerant, in the sense that they do not require the sets $\Upsilon_i$ in a tuple $\boldsymbol{\Upsilon}$ to be internally consistent. All that is required is that each belief $A$ is itself consistent – if not, the logics yield a trivial consequence set.[9]

In the current section, I will however concentrate on another group of adaptive logics for prioritized belief revision. These logics can be considered as adaptive characterizations of the revision operations obtained by the superposition of the revision operations defined in the preceding Chapter – I return to this point at the end of this section.

One crucial remark should be made from the start. In Chapter 9, it was assumed that $\Upsilon$ is a consistent set of beliefs – otherwise, the revision set obtained by the adaptive logics simply equals $Cn_{\mathbf{CL}}(\Psi)$. A similar restriction is necessary to apply the systems from this section in a sensible way. That is, where $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$, we need to assume that each set $\Upsilon_i$ is in itself consistent.

The logics from this section are the result of a superposition of flat ALs, each of which are variants of the logics from the previous chapter. The variants are obtained by introducing a set of infinitely many different modalities $\Box_i$ $(i \in \mathbb{N})$, which allow us to represent a prioritized belief base in the object language of the logics. Intuitively speaking, $\Box_i A$ expresses that the belief $A$ has priority degree $i$. The $i$th logic of the superposition then allows us to conditionally derive $A$ from $\Box_i A$. In the next few paragraphs, I will make this idea formally precise.

**The Language** $\mathcal{L}_\omega$ is obtained from $\mathcal{L}_c$ by adding an infinite set of necessity-operators $\Box_i$ $(i \in \mathbb{N})$. The associated set of well-formed formulas, $\mathcal{W}_\omega$, is defined in a way completely analogous to the definition of $\mathcal{W}_r$:

(i) $\mathcal{W}_c \subset \mathcal{W}_\omega$
(ii) Where $A \in \mathcal{W}_c$, $\Box_i A \in \mathcal{W}_\omega$ for every $i \in \mathbb{N}$
(iii) Where $A, B \in \mathcal{W}_\omega$, $\neg A, A \wedge B, A \vee B, A \supset B, A \equiv B \in \mathcal{W}_\omega$

As before, we need a translation to obtain a premise set from a prioritized set of beliefs $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$. The following definition should not come as a surprise, in view of the preceding chapters:

$$\boldsymbol{\Upsilon}^{\Box} =_{\mathsf{df}} \{\Box_i A \mid A \in \Upsilon_i, i \in I\}$$

Analogous to the case of flat revision, $\boldsymbol{\Upsilon}^{\Box} \cup \Psi$ is the premise set we feed into the adaptive logic whenever we are concerned with the revision of $\boldsymbol{\Upsilon}$ by $\Psi$. The adaptive logics $\mathbf{SAR}_{\mathbf{i}}^{\mathbf{x}}$, which are characterized below, define a revision operation as follows:

**Definition 10.7** $\boldsymbol{\Upsilon} \oplus_{\mathbf{SAR}_{\mathbf{i}}^{\mathbf{x}}} \Psi =_{\mathsf{df}} \{B \in \mathcal{W}_c \mid \boldsymbol{\Upsilon}^{\Box} \cup \Psi \vdash_{\mathbf{SAR}_{\mathbf{i}}^{\mathbf{x}}} B\}$.

[9] In other words, the **K**-based logics from Chapters 3-5 are as conflict-tolerant as the logic $\mathbf{MP}_{\sqsubset}$ from Chapter 7.

**Multi-Modal Lower Limit Logics** The logics $\mathbf{MK_s}$ and $\mathbf{MKt_s}$ are straightforward multi-modal extensions of $\mathbf{K_s}$, resp. $\mathbf{Kt_s}$ from the previous chapter. I will start with the $\mathbf{MK_s}$-semantics and next explain how the semantics of $\mathbf{MKt_s}$ can be obtained from it.

A $\mathbf{MK_s}$-model is a quadruple $\langle W, \mathcal{R}, v, w_0\rangle$, where $W$ is a set of possible worlds, $\mathcal{R} = \{R_i \mid i \in \mathbb{N}\}$ a set of accessibility relations, $v : \mathcal{S} \times W \to \{0, 1\}$ an assignment function and $w_0$ the actual world. The valuation $v_M : \mathcal{W}_\omega \to \{0, 1\}$ defined by the model $M$ is characterized by the following clauses:

C1 where $A \in \mathcal{S}$, $v_M(A, w) = v(A, w)$
C2 $v_M(\neg A, w) = 1$ iff $v_M(A, w) = 0$
C3 $v_M(A \vee B, w) = 1$ iff $v_M(A, w) = 1$ or $v_M(B, w) = 1$
C4 $v_M(A \wedge B, w) = 1$ iff $v_M(A, w) = 1$ and $v_M(B, w) = 1$
C5 $v_M(A \supset B, w) = 1$ iff $v_M(A, w) = 0$ or $v_M(B, w) = 1$
C6 $v_M(\Box_i A, w) = 1$ iff, $v_M(A, w') = 1$ for all $w'$ such that $R_i ww'$

Let $M \Vdash A$ iff $v_M(A, w_0) = 1$. In order to obtain the $\mathbf{MKt_s}$-semantics, it suffices to add the following condition on the definition of a model:

**S-MKt$_\mathbf{s}$** if $R_i ww'$ and $R_i ww''$, then $w' = w''$

This condition ensures that from every possible world (including the actual world), only one other world (at most) is accessible by a given relation $R_i \in \mathcal{R}$. As a result, $\Box_i A \vee \Box_i \neg A$ is true in every $\mathbf{MKt_s}$-model, for every $A \in \mathcal{W}_c$.

The syntax of $\mathbf{MK_s}$ and $\mathbf{MKt_s}$ is obtained by equally straightforward variations on $\mathbf{K}$ and $\mathbf{Kt}$: we extend the axiomatization of $\mathbf{CL}$ with the following axiom (where $A, B \in \mathcal{W}_c$ and $i \in \mathbb{N}$):

A1 $\Box_i(A \supset B) \supset (\Box_i A \supset \Box_i B)$

and close it under modus ponens (MP) and the following rule (where $A, B \in \mathcal{W}_c$):

RN if $\vdash A$, then $\vdash \Box_i A$

Where $\vdash_{\mathbf{MK_s}}$ indicates membership in the set of $\mathbf{MK_s}$-axioms, we define $\Gamma \vdash_{\mathbf{MK_s}} A$ iff there are $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_{\mathbf{MK_s}} (B_1 \wedge \ldots \wedge B_n) \supset A$. Note that according to these definitions, $\mathbf{MK_s}$ is a compact Tarski-logic. As before, Soundness and Completeness are a matter of routine and left to the reader.

Finally, $\mathbf{MKt_s}$ is obtained by adding the following axiom to those of $\mathbf{MK_s}$:

A2 $\Box_i A \vee \Box_i \neg A$

$\vdash_{\mathbf{MKt_s}} A$ and $\Gamma \vdash_{\mathbf{MKt_s}} A$ are defined in the same way as $\vdash_{\mathbf{MK_s}} A$ and $\Gamma \vdash_{\mathbf{MK_s}} A$.

To obtain lower limit logics from $\mathbf{MK_s}$ and $\mathbf{MKt_s}$, we enrich them with axioms for the checked connectives. As before, this can be done generically, since both logics are supra-classical – see Section 2.4.2 in Chapter 2 where this was explained. This gives us the logics $\mathbf{MK}$ and $\mathbf{MKt}$.

**The Adaptive Logics $\mathbf{AR^x_{i,j}}$** With the above preliminary definitions, we can define variants of the adaptive logics from the preceding chapter. First of all, we introduce the metavariable $\mathbf{LLL^p_i}$ to range over the lower limit logics **MKt** and **MK**:

Let $\mathbf{LLL^p_1} =_{\mathsf{df}} \mathbf{LLL^p_2} =_{\mathsf{df}} \mathbf{MKt}$
Let $\mathbf{LLL^p_3} =_{\mathsf{df}} \mathbf{LLL^p_4} =_{\mathsf{df}} \mathbf{MK}$

Next, where $i \in \{1,2,3,4\}$ and $j \in \mathbb{N}$, let the set $\Omega^{\mathsf{r}}_{i,j}$ be obtained by replacing every occurrence of $\Box$ in $\Omega^{\mathsf{r}}_i$ with $\Box_j$. So for instance, $\Omega^{\mathsf{r}}_{2,1} =_{\mathsf{df}} \{\Box_1 A \wedge \neg A \mid A \in \mathcal{W}^l_c\}$ and $\Omega^{\mathsf{r}}_{1,4} =_{\mathsf{df}} \{(\Box_4 A \wedge \neg A) \vee (\Box_4 \neg A \wedge A) \mid A \in \mathcal{W}^l_c\}$.

Finally, every logic $\mathbf{AR^x_{i,j}}$ is a flat adaptive logic, characterized by the triple $\langle \mathbf{LLL^p_i}, \Omega^{\mathsf{r}}_{i,j}, \mathbf{x}\rangle$. Where $i \in \{1,2,3,4\}$ and $j \in \mathbb{N}$, the logic $\mathbf{AR^x_{i,j}}$ allows for the defeasible inference from $\Box_j A$ to $A$, in the same way as the logic $\mathbf{AR^x_i}$ allowed us to infer $A$ from $\Box A$ – see Theorems 9.6 and 9.7 in Chapter 9.

**The Prioritized Adaptive Logics $\mathbf{SAR^x_i}$** The final step of the generic procedure is to combine the flat ALs. As expected, each logic $\mathbf{SAR^x_i}$ is obtained by the superposition of the logics $\langle \mathbf{AR^x_{i,j}}\rangle_{j\in\mathbb{N}}$. So, for instance:

$$Cn_{\mathbf{SAR^r_1}} =_{\mathsf{df}} \limsup_{i\to\omega} Cn_{\mathbf{AR^r_{1,i}}}(\ldots(Cn_{\mathbf{AR^r_{1,2}}}(Cn_{\mathbf{AR^r_{1,1}}}(\Gamma)))\ldots)$$

In Section 10.3.2, I will illustrate the generic procedure for superposing the revision operations by means of the logic $\mathbf{SAR^r_1}$, and give an example of an object-level proof.

**Variations in Several Respects** The preceding definitions yield eight prioritized revision operations, which are variants of the revision operations from the preceding chapter. However, in view of the metatheoretic results from Chapter 3, it is possible to obtain a much larger number of logics for prioritized belief revision. Let me briefly explain why.

First of all, the strategies of the logics in a superposition of ALs need not all be the same. If, for some reason, we want to be more cautious when we revise the beliefs with a specific priority $k$, then we may use the Reliability Strategy for this part of the revision operation.

Second, since all logics $\mathbf{AR^x_{1,i}}$ and $\mathbf{AR^x_{2,j}}$ have the same lower limit logic, we may also superpose them and still obtain a regular logic in the format studied in chapter 3. Likewise, superpositions of logics $\mathbf{AR^x_{3,i}}$ and $\mathbf{AR^x_{4,j}}$ pose no metatheoretic problems.

Third, by varying the definition of the lower limit logics **MK** and **MKt**, we can even obtain combinations of revision operations such as $\mathbf{AR^r_1}$ and $\mathbf{AR^r_3}$. Recall that the format of superpositions of adaptive logics requires that all flat ALs in the superposition have the same lower limit logic. Hence if we want to combine $\mathbf{AR^r_1}$ and $\mathbf{AR^r_3}$, we need to make sure that our lower limit logic is a hybrid logic itself, i.e. that some of its modal operators behave according to **K**, whereas others behave according to **Kt**. To obtain this result at the semantic level, the condition $\mathbf{S\text{-}MKt_s}$ has to be restricted to a subset of relations $R_i \subset \mathcal{R}$. On the syntactic level, it the validity of axiom A2 is restricted to the corresponding operators $\Box_i$. As a result, we can apply the adaptive mechanisms from $\mathbf{AR}^{\mathbf{x}}_{\mathbf{1}}$ and $\mathbf{AR}^{\mathbf{x}}_{\mathbf{2}}$ to the corresponding sets $\Upsilon_i$, whereas we use the mechanisms from $\mathbf{AR}^{\mathbf{x}}_{\mathbf{3}}$ or $\mathbf{AR}^{\mathbf{x}}_{\mathbf{4}}$ for the remaining sets $\Upsilon_j$.

To avoid technicalities, I will not present the above-mentioned "hybrid" superpositions in detail here. However, the upshot is that we can obtain superpositions that "correspond" (see below) to any combination of the revision operations from the preceding chapter.

**The Rationality Postulates** It can easily shown, by proofs similar to those from Section 9.8.2 in the preceding chapter, that the operators $\oplus_{\mathbf{SAR}^{\mathbf{x}}_{\mathbf{i}}}$ obey the six rationality postulates PG1-PG6. Let me briefly point out how each of the postulates can be proven:

- The *Closure* postulate follows almost immediately from the fact that all superpositions of ALs are closed under their lower limit logic (see Theorem 3.1.2), and that the logics $\mathbf{LLL}^{\mathbf{P}}_{\mathbf{i}}$ are modal extensions of **CL**.
- The *Success* postulate follows from the reflexivity of each logic $\mathbf{SAR}^{\mathbf{x}}_{\mathbf{i}}$.
- As in the flat case, the *Vacuity* postulate requires a more lengthy proof, which basically relies on (i) the compactness of **CL**, (ii) the fact that for all $i \in \{1,2,3,4\}$, whenever $A \in \Upsilon_j$ for a $j \in I$, then $\mathbf{\Upsilon}^{\Box} \vdash_{\mathbf{LLL}^{\mathbf{P}}_{\mathbf{i}}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subset \Omega^{\mathsf{r}}_{i,j}$, (iii) the fact that whenever $\bigcup \mathbf{\Upsilon} \cup \Psi$ is consistent, then $\mathbf{\Upsilon}^{\Box} \cup \Psi$ is a normal premise set, and (iv) Theorem 3.15, which states that for normal premise sets, every logic **SAL** is equivalent to its upper limit logic. For the proof of (iii), one needs to construct an $\mathbf{LLL}^{\mathbf{P}}_{\mathbf{i}}$-model with exactly one world $w_0$, in which all formulas $A \in Cn_{\mathbf{CL}}(\bigcup \mathbf{\Upsilon} \cup \Psi)$ are true.
- For the *Inclusion* postulate, we can reason by cases: either $\bigcup \mathbf{\Upsilon} \cup \Psi$ is consistent, whence by the *Vacuity* postulate, $\mathbf{\Upsilon} \oplus_{\mathbf{SAR}^{\mathbf{x}}_{\mathbf{i}}} \Psi = Cn_{\mathbf{CL}}(\bigcup \mathbf{\Upsilon} \cup \Psi)$; or $\bigcup \mathbf{\Upsilon} \cup \Psi$ is inconsistent, whence $Cn_{\mathbf{CL}}(\bigcup \mathbf{\Upsilon} \cup \Psi) = \mathcal{W}_c$.
- The *Consistency* postulate follows from the Syntactic Reassurance of all logics $\mathbf{SAR}^{\mathbf{x}}_{\mathbf{i}}$ (see Theorem 3.3).
- The *Extensionality* postulate follows from (i) the fact that whenever $\Psi$ and $\Psi'$ are **CL**-equivalent, then $\mathbf{\Upsilon}^{\Box} \cup \Psi$ and $\mathbf{\Upsilon}^{\Box} \cup \Psi'$ are $\mathbf{LLL}^{\mathbf{P}}_{\mathbf{i}}$-equivalent, and (ii) Theorem 3.4.

In view of the behavior of superpositions with respect to premise sets that are normal up to a certain level $i$ (see Theorem 3.13), we can also safely infer that all revision operations $\oplus_{\mathbf{SAR}^{\mathbf{x}}_{\mathbf{i}}}$ satisfy the *Prioritized Vacuity* postulate that was introduced in this chapter. So in general, the revision operations defined in the current section are well-behaved.

**Another Conjecture** It seems plausible that the following holds for all $\mathbf{\Upsilon}$ and $\Psi$, for all $i \in \{1,2,3,4\}$ and $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$:

**Conjecture 10.2** $\mathbf{\Upsilon} \oplus_{\mathbf{SAR}^{\mathbf{x}}_{\mathbf{i}}} \Psi = \mathbf{\Upsilon} \oplus_{\mathbf{AR}^{\mathbf{x}}_{\mathbf{i}}} \Psi$

In words, It seems plausible that the revision operation obtained by the superposition of the logics $\langle \mathbf{AR^x_{i,j}} \rangle$ is equivalent to the superposition of revision operations obtained from $\mathbf{AR^x_i}$. This is so for several reasons. First of all, although each logic $\mathbf{AR^x_{i,j}}$ ranges over all the modal operators $\Box_i$ ($i \in \mathbb{N}$), its adaptive behavior seems to be restricted to the operator $\Box_j$. If this is right, then in the superposition, each logic $\mathbf{AR^x_{i,j+1}}$ gets as input a set that reduces to the following: (i) all the $\mathbf{LLL^p_i}$-consequences of $\boldsymbol{\Upsilon}^{\Box}$, and (ii) a set of non-modal formulas, which corresponds to the revision set obtained by the logic $\mathbf{AR^x_{i,j}}$ from the preceding step. In view of (i) and (ii), the revision obtained by the $(j+1)$th step is equivalent to $\Upsilon_{j+1} \oplus (\Upsilon_j \oplus (\ldots \oplus \Upsilon_1 \oplus \Psi))\ldots)$.

## 10.3.2 An Example: The Logic $\mathbf{SAR^r_1}$

**An Example** To illustrate the logic $\mathbf{SAR^r_1}$, let us consider the revision of $\boldsymbol{\Upsilon}^d = \langle \{p \wedge q\}, \{\neg p \wedge r\}, \{r \supset s\} \rangle$ by $\Psi^d = \{\neg q\}$. As spelled out in the preceding section, we first need to translate the beliefs, using the modal operators $\Box_i$. This gives us:

$$\boldsymbol{\Upsilon}^{d\Box} = \{\Box_1(p \wedge q), \Box_2(\neg p \wedge r), \Box_3(r \supset s)\}$$

In the remainder of this section, let $\Gamma = \boldsymbol{\Upsilon}^{d\Box} \cup \Psi^d$.

Now recall that each logic $\mathbf{AR^r_{1,i}}$ in the superposition of logics $\mathbf{SAR^r_1}$ is defined by the triple $\langle \mathbf{LLL^p_1}, \Omega^{\mathsf{r}}_{1,i}, \mathbf{r} \rangle$. To simplify notation, I will use $\rho^i(A)$ as an abbreviation for $(\Box_i A \wedge \neg A) \vee (\Box_i \neg A \wedge A)$. Then for all $i \in \mathbb{N}$, $\Omega^{\mathsf{r}}_{1,i} = \{\rho^i(A) \mid A \in \mathcal{S}\}$.

Each logic $\mathbf{AR^r_{1,i}}$ allows for the defeasible derivation of $A$ from $\Box_i A$, on the condition $\{\rho^i(B) \mid B \in E(A)\}$ – the proof of this fact proceeds wholly analogous to that of Theorem 9.6. Let us use the name $\mathrm{RD}^i$ for the associated (derivable) inference rule. Since in an $\mathbf{SAR^r_1}$-proof, we may apply all these logics whenever it suits us, we can e.g. start by deriving $\neg p \wedge r$ from $\Box_2(\neg p \wedge r)$:[10]

| | | | |
|---|---|---|---|
| 1 | $\neg q$ | PREM | $\emptyset$ |
| 2 | $\Box_1(p \wedge q)$ | PREM | $\emptyset$ |
| 3 | $\Box_2(\neg p \wedge r)$ | PREM | $\emptyset$ |
| 4 | $\Box_3(r \supset s)$ | PREM | $\emptyset$ |
| 5 | $\neg p \wedge r$ | 3;RD$^2$ | $\{\rho^2(p), \rho^2(r)\}$ |

Relying on the last step of this stage, and the premise $\Box_3(r \supset s)$, we may moreover derive $s$:

| | | | |
|---|---|---|---|
| 6 | $r \supset s$ | 4;RD$^3$ | $\{\rho^3(r), \rho^3(r)\}$ |
| 7 | $s$ | 5,6;RU | $\{\rho^2(p), \rho^2(r), \rho^3(r), \rho^3(s)\}$ |

However, not all of the preceding derivations are unproblematic. In view of the most plausible belief, i.e. $p \wedge \neg q$, we should uphold $p$ unless this belief contradicts the revision set. But this means that $\neg p \wedge r$, and anything that was derived from it, has to be retracted:

[10] Note that for all $i, j \in \mathbb{N}$ with $i \neq j$, $\Omega^{\mathsf{r}}_{1,i} \cap \Omega^{\mathsf{r}}_{1,j} = \emptyset$, whence we can use sets instead of sequences of sets, in the conditions of lines – see Section 3.5 in Chapter 3.

| | | | | |
|---|---|---|---|---|
| 5 | $\neg p \wedge r$ | 3;RD$^2$ | $\{\rho^2(p), \rho^2(r)\}$ | $\checkmark^2$ |
| 6 | $r \supset s$ | 4;RD$^3$ | $\{\rho^3(r), \rho^3(s)\}$ | |
| 7 | $s$ | 5,6;RU | $\{\rho^2(p), \rho^2(r), \rho^3(r), \rho^3(s)\}$ | $\checkmark^2$ |
| 8 | $\Box_1 p$ | 2;RU | $\emptyset$ | |
| 9 | $p$ | 8;RD$^1$ | $\{\rho^1(p)\}$ | |
| 10 | $\Box_2 \neg p$ | 3;RU | $\emptyset$ | |
| 11 | $\rho^2(p)$ | 9,10;RU | $\{\rho^1(p)\}$ | |

Note that ${}^{\mathbf{S}}\Sigma^1_{11}(\Gamma) = \emptyset$, whence no lines are 1-marked at stage 11. But this means that the formula on line 11 is a minimal Dab$_2$-formula at stage 11. As a result, ${}^{\mathbf{S}}U^2_{11}(\Gamma) = \{\rho^2(p)\}$, which implies that lines 5 and 7 are 2-marked at stage 11.

Nevertheless, by analyzing the belief $\neg p \wedge r$, we are able to uphold $r$ and hence derive $s$, as the following continuation of the proof illustrates:

| | | | |
|---|---|---|---|
| 12 | $\Box_2 r$ | 3;RU | $\emptyset$ |
| 13 | $r$ | 12;RD$^2$ | $\{\rho^2(r)\}$ |
| 14 | $s$ | 6,13;RU | $\{\rho^2(r), \rho^3(s)\}$ |

It can easily be verified that line 14 is unmarked at stage 14, and that $p$, $\neg q$, $r$ and $s$ are finally derived in this proof.

**Adaptive Revision and Superposed Revision.** Let us now compare this result to the set $\boldsymbol{\Upsilon}^d \oplus_{\mathbf{AR}^{\mathrm{r}}_{\mathbf{1}}} \Psi^d$, which is obtained from the superposition of the operation $\oplus_{\mathbf{AR}^{\mathrm{r}}_{\mathbf{1}}}$. This revision consists in three steps:

(i) Revision of $\Upsilon^d_1$ in view of $\Psi^d$, by the operation $\oplus_{\mathbf{AR}^{\mathrm{r}}_{\mathbf{1}}}$. Obviously, we have to retract the belief $q$. However, since $p \in Cn_{\mathbf{CL}}(\Upsilon^d_1)$ is not relevant to the revision, we can uphold it. More generally, it can easily be shown that

$$\Upsilon^d_1 \oplus_{\mathbf{AR}^{\mathrm{r}}_{\mathbf{1}}} \Psi^d = Cn_{\mathbf{CL}}(\{p, \neg q\})$$

(ii) Revise $\Upsilon^d_2$ in view of $\Delta^d_1 = Cn_{\mathbf{CL}}(\{p, \neg q\})$, by $\oplus_{\mathbf{AR}^{\mathrm{r}}_{\mathbf{1}}}$. Note that in this case, we have to retract the (implicit) belief in $\neg p$, but we can still uphold the belief in $r$. As a result, we obtain the set $\Delta^d_2 = Cn_{\mathbf{CL}}(\{p, \neg q, r\})$.

(iii) Revise $\Upsilon^d_3 = \{r \supset s\}$ in view of $\Delta^d_2$, by $\oplus_{\mathbf{AR}^{\mathrm{r}}_{\mathbf{1}}}$. Since these two sets are mutually compatible, we obtain the following end result:

$$\boldsymbol{\Upsilon}^d \oplus_{\mathbf{AR}^{\mathrm{r}}_{\mathbf{1}}} \Psi^d = Cn_{\mathbf{CL}}(\{p, \neg q, r, r \supset s\}) = Cn_{\mathbf{CL}}(\{p, \neg q, r, s\})$$

Note that the fact that every operation $\Upsilon \oplus_{\mathbf{AR}^{\mathrm{r}}_{\mathbf{1}}} \Psi$ obeys the relevance axiom in a specific way, allows us to obtain this result – otherwise, $r$ would not be in the revision set.

## 10.4 Priorities and Relevance

In this section, I return to the axiom of relevance **P** from the preceding chapter. More specifically, I will address the question whether we can define an appropriate variant of **P** for prioritized belief bases. I will first consider the most straightforward way to do this, arguing that it leads to rather counterintuitive results. This will bring me to a number of requirements on sensible relevance axioms. After that, I will propose two slightly more complicated proposals, and explain why they both seem worthy of further inquiry.

### 10.4.1 A Failing Proposal

As explained at the beginning of the previous chapter, the central motor behind the axiom of relevance is the idea that whenever a belief $A$ is in the original set of beliefs, and $A$ is not "relevant" to the new information (in view of the other beliefs in $\Upsilon$), then it should be upheld. So it remains to be specified what is meant by the phrase "$A$ is not relevant to the revision of $\boldsymbol{\Upsilon}$ by $\Psi$." As before, I will do this in terms of letter sets of $A$ and $\Psi$, and partitions of the set of letters $\mathcal{S}$ that are obtained from a given set $\Delta \subseteq \bigcup \boldsymbol{\Upsilon}$.

A straightforward way to define relevance to the revision of $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$ by $\Psi$, is by letting $\bigcup \boldsymbol{\Upsilon}$ play the role of $\Upsilon$ in the original definition of relevance. Formally, this proposal reads as follows:

**Definition 10.8** *Let $\mathbb{E}$ be the finest splitting of $\bigcup \boldsymbol{\Upsilon}$. We say that $B$ is* simply p-relevant *to the revision of $\boldsymbol{\Upsilon}$ by $\Psi$ iff there is a $\Delta_i \in \mathbb{E}$ such that $E(B) \cap \Delta_i \neq \emptyset$ and $E^*(\Psi) \cap \Delta_i \neq \emptyset$.*[11]

Hence, simple p-relevance (the "p" stands for "prioritized") with respect to $\boldsymbol{\Upsilon} \oplus \Psi$ is equivalent to relevance with respect to $\bigcup \boldsymbol{\Upsilon} \oplus \Psi$. In other words, the concept of simple p-relevance does not take into account the priority of the beliefs in $\bigcup \boldsymbol{\Upsilon}$, and simply looks at the prioritized belief base as if it were one set of beliefs.

Under this interpretation, it should be assumed that $\bigcup \boldsymbol{\Upsilon}$ is consistent; otherwise we end up with the same problems as in the case of the original axiom **P** and inconsistent sets of beliefs – see also Chapter F in the appendix. Although this is already a strong assumption, let us grant it here for the sake of argument.

In line with the preceding chapter, the resulting relevance axiom reads: if $B \in Cn_{\mathbf{CL}}(\bigcup \boldsymbol{\Upsilon})$ is not simply p-relevant to the revision (contraction) of $\boldsymbol{\Upsilon}$ by $\Psi$, then $B \in \boldsymbol{\Upsilon} \oplus \Psi$ ($B \in \boldsymbol{\Upsilon} \ominus \Psi$). Let us call this axiom **Ps**.

Let us take a look at a very simple example, to illustrate **Ps**. Consider the prioritized belief base $\boldsymbol{\Upsilon}^e = \langle \{p \vee q, r\}, \{p, \neg q\} \rangle$, and its revision by $\{\neg p \vee \neg r\}$. Note that $\bigcup \boldsymbol{\Upsilon}^e \vdash_{\mathbf{CL}} p, \neg q, r$. It follows that $\mathbb{E}^e = \{\{p\}, \{q\}, \{r\}\}$ is the finest splitting of $\bigcup \boldsymbol{\Upsilon}$.[12] Hence the belief $\neg q$ is not relevant to the new information, i.e. $\neg p \vee \neg r$. This means that one should uphold this belief, in order to obey **Ps**.

Now consider again the postulate of *Prioritized Vacuity* from Section 10.2.2. This postulate states that if $\Upsilon_1 \cup \ldots \cup \Upsilon_n$ is consistent with $\Psi$, then we are not allowed to retract any belief in $\Upsilon_1 \cup \ldots \cup \Upsilon_n$ (irrespective of the members of any $\Upsilon_m$ with $m > n$). Still in other words, beliefs with a lower priority degree should not be able to rule out beliefs with a higher priority degree, if the latter

[11] Recall that $E^*(\Psi)$ is the least letter-set of $\Psi$ – see Section 9.2 in Chapter 9.

[12] As in the previous chapter, I omit letters that do not occur in the initial formulation of a set $\Upsilon$, whenever I give the finest splitting of this set.

are totally unproblematic with regards to the new information. This intuition lies at the core of prioritized belief revision.

Now let us take a closer look at $\boldsymbol{\Upsilon}^e$, and the priorities of the different beliefs in it. Note that $\{p \vee q, r\}$, i.e. the set of beliefs with the highest priority, does not conflict with the new information $\{\neg p \vee \neg r\}$. Hence, if want to obey the Prioritized Vacuity postulate, we should get $\neg p, r, q$ in our revision set. But in that case we can no longer put the belief $\neg q$ (which has a lower priority degree) in the revision set, if we want to avoid an inconsistent revision set. So it seems that in this and similar cases, we face a dilemma: either we cannot obey **Ps**, or we have to violate the (very intuitive) postulate of Prioritized Vacuity.

### 10.4.2 Some Requirements for Axioms of Relevance

Before I introduce the alternative proposal, let me briefly spell out some requirements which any sensible axiom of p-relevance should fulfill. The first was already called upon in the preceding section, i.e. that the resulting relevance axiom should not conflict with such basic requirements as the *Prioritized Vacuity* postulate, or the requirement that revision sets are consistent.

Another requirement follows immediately from the idea of prioritized rationality, as spelled out at the end of Section 10.2.2. It states that whenever $A \in \Upsilon_1$ is not relevant to the revision of $\Upsilon_1$ by $\Psi$, then neither should it be p-relevant to the revision of $\boldsymbol{\Upsilon} = \langle \Upsilon_1, \Upsilon_2, \ldots \rangle$ by $\Psi$. However, as explained in the preceding chapter, there are many different ways to satisfy relevance in the flat case.

Consider the following example: we revise $\boldsymbol{\Upsilon}^d = \langle \Upsilon_1^d, \Upsilon_2^d \rangle = \langle \{p \vee q, q \vee r\}, \{\neg r\} \rangle$ by $\Psi^d = \{\neg p, \neg q\}$. Let us first focus on the flat revision of $\Upsilon_1^d$ by $\Psi$. Note that this operation corresponds to the revision of $\Upsilon_9$ by $A_4$, as discussed in Section 9.6 of Chapter 9. As explained there, both $p \vee q$ and $q \vee r$ are relevant to the revision by $\Psi^d$. However, it depends on the specific revision operation whether $q \vee r$ is upheld, and hence whether $r \in \Upsilon_1^d \oplus \Psi^d$. That is, compare the flat revision operation defined from $\mathbf{AR_1^r}$ to that defined from $\mathbf{AR_3^r}$, for this specific case:

$$r \notin \Upsilon_1^d \oplus_{\mathbf{AR_1^r}} \Psi^d$$
$$r \in \Upsilon_1^d \oplus_{\mathbf{AR_3^r}} \Psi^d$$

Hence, depending on the way we obey flat relevance with regards to $\Upsilon_1^d \oplus \Psi^d$, we may or may not be able to uphold the belief $\neg r \in \Upsilon_2^d$. Note that $\neg r$ is not relevant to $\Upsilon_2^d \oplus (\Upsilon_1^d \oplus_{\mathbf{AR_1^r}} \Psi^d)$, whereas it is relevant to $\Upsilon_2^d \oplus (\Upsilon_1^d \oplus_{\mathbf{AR_3^r}} \Psi^d)$.

This brings me to a final requirement for any axiom of p-relevance. An obvious feature of postulates is that they are operation-independent, i.e. they are means to compare different revision operations. Hence, a useful axiom of p-relevance should also be independent of the specific operation at hand. However, another question is if *p-relevance* itself should also be defined in such a way that it is operation-independent. This also relates to the question whether we want to define an independent notion of relevance (and afterwards spell out the axiom of relevance on the basis of this notion), or to simply formulate an axiom of relevance immediately – this distinction will become more clear below.

In the next two sections, I will consider two proposals for a relevance axiom: one that relies on an independent notion of relevance, and another that defines relevance of a revision operation in a more direct way. In both sections, it is not assumed that $\bigcup \boldsymbol{\Upsilon}$ is consistent, but only that each set $\Upsilon_i$ is consistent.

### 10.4.3 An Operation-Independent Notion of P-relevance

Recall the example from Section 10.4.1. The reason why the belief $\neg q \in \bigcup \boldsymbol{\Upsilon}^e$ could not be upheld in the revision of $\langle\{p \vee q, r\}, \{p, \neg q\}\rangle$ by $\{\neg p \vee \neg r\}$ is that it conflicts with the more plausible information in the sets $\Upsilon_1^e$ and $\Psi^e$, which were mutually compatible. More generally, in the case of prioritized revision, a belief $A \in \bigcup \boldsymbol{\Upsilon}$ may be retracted not just because it relates to the new information, but also because it relates to more plausible original beliefs. In that case, we may consider a belief $A \in \Upsilon_i$ as relevant to the revision of $\boldsymbol{\Upsilon}$ by $\Psi$ iff it is relevant to the revision of $\Upsilon_i$ by either $\Psi$ or $\Upsilon_1$ or $\ldots$ or $\Upsilon_{i-1}$.

In order to stay in line with the more generic definition of relevance from the previous chapter – which defines relevance for *all* formulas $A \in \mathcal{W}_c$, not just for all $A \in \Upsilon$ –, it is convenient to introduce the following notion:

**Definition 10.9 (p1$_k$-relevance)** *Let $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$ and $k \in I$. Let $\mathbb{E}$ be the finest splitting of $\Upsilon_k$. We say that $B$ is* p1$_k$-relevant *to the revision of $\boldsymbol{\Upsilon}$ by $\Psi$ iff there is a $\Delta \in \mathbb{E}$ such that (a) $E(B) \cap \Delta \neq \emptyset$ and (b) $E(\Psi^*) \cap \Delta \neq \emptyset$ or $E(\Upsilon_1^*) \cap \Delta \neq \emptyset$ or $\ldots$ or $E(\Upsilon_{k-1}^*) \cap \Delta \neq \emptyset$.*

Hence according to Definition 10.9, $B$ is p1$_k$-relevant to $\boldsymbol{\Upsilon} \oplus \Psi$ if and only if it is either relevant to $\Upsilon_k \oplus \Psi$, or to $\Upsilon_k \oplus \Upsilon_1$, or $\ldots$ or to $\Upsilon_k \oplus \Upsilon_{k-1}$. Note that this means that as the index $k$ gets higher, it becomes more likely that a given formula is p1$_k$-relevant to the revision of $\boldsymbol{\Upsilon}$ by $\Psi$, since an increasing number of sets is taken into account. Thus we have:

**Fact 10.1** *Where $k, k+1 \in I$: if $A$ is p1$_k$-relevant to $\boldsymbol{\Upsilon} \oplus \Psi$, then $A$ is also p1$_{k+1}$-relevant to $\boldsymbol{\Upsilon} \oplus \Psi$.*

The following is immediate in view of Definition 10.9:

**Fact 10.2** *$A$ is p1$_1$-relevant to $\boldsymbol{\Upsilon} \oplus \Psi$ iff $A$ is relevant to $\Upsilon_1 \oplus \Psi$.*

The relevance axiom that is associated with the notion of p1$_k$-relevance can be stated as follows:

**P1** For every $k \in I$ and $A \in Cn_{\mathbf{CL}}(\Upsilon_k)$: if $A$ is not p1$_k$-relevant to $\boldsymbol{\Upsilon} \oplus \Psi$, then $A \in \boldsymbol{\Upsilon} \oplus \Psi$.

In view of Fact 10.2, it follows immediately that if $\oplus$ satisfies p1-relevance, and $A \in \Upsilon_1$ is not relevant to $\Upsilon_1 \oplus \Psi$, then $A \in \boldsymbol{\Upsilon} \oplus \Psi$.

To see how the definition of p1$_k$-relevance works, let us consider a simple example:

**Example 10.7** *Consider the revision of* $\boldsymbol{\Upsilon}^f = \langle\{p \vee q, r\}, \{p, \neg q\}, \{\neg r \wedge s\}\rangle$ *by* $\Psi^f = \{\neg p \vee \neg r\}$ *– note that* $\Psi^f = \Psi^e$ *and that* $\boldsymbol{\Upsilon}^f$ *is an extension of* $\Upsilon^e$*. Note also that* $\Psi^f$ *is a least letter-set representation of itself, and where* $i \in \{1, 2, 3\}$*, each set* $\Upsilon^f_i$ *is a least letter-set representation of itself.*

*Let us start with the belief* $\neg q \in \Upsilon^f_2$*. This belief is* $p1_2$*-relevant to this revision. That is,*

- *the finest splitting of* $\Upsilon^f_2$ *is* $\mathbb{E}^f_2 = \{\{p\}, \{q\}, \{r\}, \{s\}\}$
- *there is no* $\Delta \in \mathbb{E}^f_2 = \{\{p\}, \{q\}\}$*, such that* $E(\neg q) \cap \Delta \neq \emptyset$ *and* $E(\Psi^f) \cap \Delta) \neq \emptyset$*, but*
- *there is a* $\Delta \in \mathbb{E}^f_2$ *such that* $E(\neg q) \cap \Delta \neq \emptyset$ *and* $E(\Upsilon^f_1) \cap \Delta) \neq \emptyset$*, viz.* $\Delta = \{q\}$

*Hence, it is not required by the axiom* **P1** *that* $\neg q \in \boldsymbol{\Upsilon}^e \oplus \Psi^e$*.*

*Consider now the belief* $s \in Cn_{\mathbf{CL}}(\Upsilon^f_3)$*. Note that the finest splitting of* $\Upsilon^f_3$ *is* $\mathbb{E}^f_3 = \{\{p\}, \{q\}, \{r\}, \{s\}\}$*. From this, it can easily be verified that* $s$ *is not* $p1_3$*-relevant to* $\boldsymbol{\Upsilon}^f \oplus \Psi^f$*. Hence the relevance axiom* **P1** *stipulates that* $s$ *should be upheld.*

### 10.4.4 P-Relevance in a Direct Way

The second alternative proposal for a prioritized relevance axiom makes reference to the specific operation $\oplus$ under consideration. It requires certain beliefs to be upheld in the revision of $\boldsymbol{\Upsilon}$ by $\Psi$, in view of other beliefs that are upheld in the same revision. Note that the rationality postulate G1 also has this property: it states that whenever $A$ is classically entailed by the revision set, then $A$ should itself also be in the revision set.

This means that, unlike the previous proposal, the axiom of p2-relevance is not specified in terms of an operation-independent notion of relevance to "any revision" of $\boldsymbol{\Upsilon}$ by $\Psi$. Rather, the axiom is spelled out in a direct way, as will become clear below.

The idea behind the second alternative is that, whenever $A \in Cn_{\mathbf{CL}}(\Upsilon_k)$, then this belief should be upheld, unless it is relevant to either the new information, or other beliefs that have a higher priority, and that are themselves upheld. Formally, where $\oplus$ is an operation of prioritized revision and $\boldsymbol{\Upsilon} = \langle \Upsilon_i \rangle_{i \in I}$, this axiom requires the following:

**P2** Where $A \in Cn_{\mathbf{CL}}(\Upsilon_k)$ for a $k \in I$, let $\Delta_k = (\boldsymbol{\Upsilon} \oplus \Psi) \cap \bigcup_{j<k} Cn_{\mathbf{CL}}(\Upsilon_j)$. If $A$ is not relevant to the revision of $\Upsilon_k$ by $\Delta_k \cup \Psi$, then $A \in \boldsymbol{\Upsilon} \oplus \Psi$.

**Example 10.8** *I will first consider the operation* $\oplus_{\mathbf{P}}$*, and show that it does not satisfy the axiom of p2-relevance. Consider again the revision of* $\boldsymbol{\Upsilon}^f = \langle\{p \vee q, r\}, \{p, \neg q\}, \{\neg r \wedge s\}\rangle$ *by* $\Psi^f = \{\neg p \vee \neg r\}$*. Note that* $\mathbb{P}(\bigcup \boldsymbol{\Upsilon}^f, \Psi)$ *contains only one set, i.e.* $\{p \vee q, r\}$*. So* $\boldsymbol{\Upsilon}^f \oplus_{\mathbf{P}} \Psi^f = Cn_{\mathbf{CL}}(\{p \vee q, r\} \cup \{\neg p \vee \neg r\}) = Cn_{\mathbf{CL}}(\{\neg p, r, q\})$*.*

*Let us first consider the belief* $\neg q \in \Upsilon^f_2$*. In order to see what the relevance axiom tells us about this belief, we have to consider the set* $\boldsymbol{\Upsilon}^f \oplus_{\mathbf{P}} \Psi^f \cap Cn_{\mathbf{CL}}(\Upsilon^f_1) = \{p \vee q, r\}$*. Next, we should ask whether* $\neg q$ *is relevant to the revision of* $\Upsilon^f_2$ *by*

$\Psi^f \cup \{p \vee q, r\}$. *This holds trivially so since* $\neg q$ *shares a letter with* $p \vee q$. *It follows that* $\neg q$ *should not be upheld.*

*However, note that also* $s \notin \mathbf{\Upsilon}^f \oplus_{\mathbf{P}} \Psi^f$. *Note that* $\mathbf{\Upsilon}^f \oplus \Psi^f \cap (Cn_{\mathbf{CL}}(\Upsilon_1^f) \cup Cn_{\mathbf{CL}}(\Upsilon_2^f)) = Cn_{\mathbf{CL}}(\{\neg p, r, q\}) \cap Cn_{\mathbf{CL}}(\{p \vee q, r\})$ *(none of the beliefs from* $\Upsilon_2^f$ *are retained in the revision operation). So it remains to be checked whether* $s$ *is relevant to the revision of* $\{\neg r \wedge s\}$ *by* $\{p \vee q, r\} \cup \{\neg p \vee \neg r\}$. *Since the finest splitting of* $\{\neg r \wedge s\}$ *is* $\mathbb{E}_2^f = \{\{p\}, \{q\}, \{r\}, \{s\}\}$, *it immediately follows that* $s$ *is not relevant to this revision. Hence in this case, the relevance axiom stipulates that* $s$ *should be upheld, which is not the case in* $\mathbf{\Upsilon}^f \oplus_{\mathbf{P}} \Psi^f$.

*Hence* $\oplus_{\mathbf{P}}$ *does not obey the axiom* **P2**. *This is an immediate consequence of the fact that* $\oplus_{\mathbf{P}}$ *is defined in terms of maximal consistent subsets of the set of beliefs,* as they are initially formulated. *For similar reasons, also* $\oplus^{\mathsf{s}}_{\mathbf{M}}$ *does not obey the axioms of P1-relevance or P2-relevance.*

In general, it seems that **P2** is a stronger requirement than **P1**. That is, in the definition of p$1_k$-relevance, *all* the beliefs in the sets $\Upsilon_1, \ldots, \Upsilon_{k-1}$ are taken into account. In axiom **P2**, only those more plausible beliefs are taken into account that are in the revision set.

Several questions can be asked about the two axioms **P1** and **P2** presented here. First of all, although they were motivated in terms of intuitive principles, and seem to lead to expected results for simple examples, one might still wonder whether it can be shown that both axioms are compatible with the postulates PG1-PG6, and the *Prioritized Vacuity* postulate. Second, it could be asked to what extent **P1** and **P2** are co-extensive, if one of both always implies the other, or if there are cases in which they are mutually exclusive. Finally, one might wonder in what way these axioms relate to the revision operations obtained by the superposition of flat operations and to the adaptive logics from Section 10.3. For instance, can we show that whenever each flat operation $\oplus_i$ in the sequence $\mathsf{s} = \langle \oplus_i \rangle_{i \in I}$ obeys the flat relevance axiom, then the operation $\oplus^{\mathsf{s}}$ obeys **P1**? Or can it be shown that all logics $\mathbf{SAR}^{\mathsf{x}}_{\mathbf{i}}$ from Section 10.3 satisfy axiom **P2**?

## 10.5 In Conclusion

In the preceding, we saw how it is possible to obtain well-behaved prioritized revision operations by superposing flat revision operations. In an analogous way, prioritized extensions were defined of the adaptive logics for belief revision from the preceding chapter. It was shown that a straightforward extension of the Axiom of Relevance to the prioritized case leads to counterintuitive results. Finally, I proposed two alternative ways to deal with the notion of relevance in a prioritized setting, which seem to lead to intuitive results and interesting questions.

More than any other chapter in this thesis, the current one ends with a number of open problems. First and foremost, we encountered a number of conjectures, which are still in need of a proof. Also, it seems that more research is needed to uncover the relations between prioritized belief revision and relevance in detail. Finally, as with many results from this thesis, it remains to be seen whether the insights from the current chapter can be transferred to the context in which our information is (merely) partially ordered.

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# Appendix A

# Overview of Languages

| | | |
|---|---|---|
| **CL** | $\mathcal{L}_c$ | $\mathcal{W}_c =_{\mathsf{df}} \langle \mathcal{S} \cup \{\bot\} \rangle \mid \neg\langle \mathcal{W}_c \rangle \mid \langle \mathcal{W}_c \rangle \vee \langle \mathcal{W}_c \rangle \mid \langle \mathcal{W}_c \rangle \wedge \langle \mathcal{W}_c \rangle \mid \langle \mathcal{W}_c \rangle \supset \langle \mathcal{W}_c \rangle \mid \langle \mathcal{W}_c \rangle \equiv \langle \mathcal{W}_c \rangle$ |
| $\mathbf{LLL_s}$ | $\mathcal{L}_s$ | $\mathcal{W}_s$ |
| **LLL**, **AL**, **ULL** | $\check{\mathcal{L}}_s$ | $\check{\mathcal{W}}_s =_{\mathsf{df}} \langle \mathcal{W}_s \rangle \mid \check{\neg}\langle \check{\mathcal{W}}_s \rangle \mid \langle \check{\mathcal{W}}_s \rangle \check{\vee} \langle \check{\mathcal{W}}_s \rangle \mid \langle \check{\mathcal{W}}_s \rangle \check{\wedge} \langle \check{\mathcal{W}}_s \rangle \mid \langle \check{\mathcal{W}}_s \rangle \check{\supset} \langle \check{\mathcal{W}}_s \rangle \mid \langle \check{\mathcal{W}}_s \rangle \check{\equiv} \langle \check{\mathcal{W}}_s \rangle$ |
| **CLuN** | $\mathcal{L}_\sim$ | $\mathcal{W}_\sim =_{\mathsf{df}} \langle \mathcal{S} \cup \{\bot\} \rangle \mid \sim\langle \mathcal{W}_\sim \rangle \mid \langle \mathcal{W}_\sim \rangle \vee \langle \mathcal{W}_\sim \rangle \mid \langle \mathcal{W}_\sim \rangle \wedge \langle \mathcal{W}_\sim \rangle \mid \langle \mathcal{W}_\sim \rangle \supset \langle \mathcal{W}_\sim \rangle \mid \langle \mathcal{W}_\sim \rangle \equiv \langle \mathcal{W}_\sim \rangle$ |
| **CLuN+**, $\mathbf{CLuN^x}$ | $\check{\mathcal{L}}_\sim$ | $\check{\mathcal{W}}_\sim =_{\mathsf{df}} \langle \mathcal{W}_\sim \rangle \mid \check{\neg}\langle \check{\mathcal{W}}_s \rangle \mid \langle \check{\mathcal{W}}_s \rangle \check{\vee} \langle \check{\mathcal{W}}_s \rangle \mid \langle \check{\mathcal{W}}_s \rangle \check{\wedge} \langle \check{\mathcal{W}}_s \rangle \mid \langle \check{\mathcal{W}}_s \rangle \check{\supset} \langle \check{\mathcal{W}}_s \rangle \mid \langle \check{\mathcal{W}}_s \rangle \check{\equiv} \langle \check{\mathcal{W}}_s \rangle$ |
| $\mathbf{K_s}$ | $\mathcal{L}_m$ | $\mathcal{W}_m =_{\mathsf{df}} \langle \mathcal{W}_c \rangle \mid \Box\langle \mathcal{W}_m \rangle \mid \neg\langle \mathcal{W}_m \rangle \mid \langle \mathcal{W}_m \rangle \vee \langle \mathcal{W}_m \rangle \mid \langle \mathcal{W}_m \rangle \wedge \langle \mathcal{W}_m \rangle \mid \langle \mathcal{W}_m \rangle \supset \langle \mathcal{W}_m \rangle \mid \langle \mathcal{W}_m \rangle \equiv \langle \mathcal{W}_m \rangle$ |
| $\mathbf{MP_s}$ | $\mathcal{L}_o$ | $\mathcal{W}_o =_{\mathsf{df}} \langle \mathcal{W}_c \rangle \mid O\langle \mathcal{W}_c \rangle \mid O_i\langle \mathcal{W}_c \rangle \mid \neg\langle \mathcal{W}_o \rangle \mid \langle \mathcal{W}_o \rangle \vee \langle \mathcal{W}_o \rangle \mid \langle \mathcal{W}_o \rangle \wedge \langle \mathcal{W}_o \rangle \mid \langle \mathcal{W}_o \rangle \supset \langle \mathcal{W}_o \rangle \mid \langle \mathcal{W}_o \rangle \equiv \langle \mathcal{W}_o \rangle$ |
| $\mathbf{T_s}$ | $\mathcal{L}_t$ | $\mathcal{W}_t =_{\mathsf{df}} \langle \mathcal{W}_f \rangle \mid \Box\langle \mathcal{W}_f \rangle \mid \neg\langle \mathcal{W}_t \rangle \mid \langle \mathcal{W}_t \rangle \vee \langle \mathcal{W}_t \rangle \mid \langle \mathcal{W}_t \rangle \wedge \langle \mathcal{W}_t \rangle \mid \langle \mathcal{W}_t \rangle \supset \langle \mathcal{W}_t \rangle \mid \langle \mathcal{W}_t \rangle \equiv \langle \mathcal{W}_t \rangle$ |
| $\mathbf{K_s}, \mathbf{Kt_s}$ | $\mathcal{L}_r$ | $\mathcal{W}_r =_{\mathsf{df}} \langle \mathcal{W}_c \rangle \mid \Box\langle \mathcal{W}_c \rangle \mid \neg\langle \mathcal{W}_r \rangle \mid \langle \mathcal{W}_r \rangle \vee \langle \mathcal{W}_r \rangle \mid \langle \mathcal{W}_r \rangle \wedge \langle \mathcal{W}_r \rangle \mid \langle \mathcal{W}_r \rangle \supset \langle \mathcal{W}_r \rangle \mid \langle \mathcal{W}_r \rangle \equiv \langle \mathcal{W}_r \rangle$ |
| $\mathbf{MK_s}$, $\mathbf{MKt_s}$ | $\mathcal{L}_\omega$ | $\mathcal{W}_\omega =_{\mathsf{df}} \langle \mathcal{W}_c \rangle \mid \Box_i\langle \mathcal{W}_c \rangle \mid \neg\langle \mathcal{W}_\omega \rangle \mid \langle \mathcal{W}_\omega \rangle \vee \langle \mathcal{W}_\omega \rangle \mid \langle \mathcal{W}_\omega \rangle \wedge \langle \mathcal{W}_\omega \rangle \mid \langle \mathcal{W}_\omega \rangle \supset \langle \mathcal{W}_\omega \rangle \mid \langle \mathcal{W}_\omega \rangle \equiv \langle \mathcal{W}_\omega \rangle$ |

Table A.1: (Metavariables for) Logics and their respective languages, in their order of occurrence in this thesis.

To understand this table, recall that $\mathcal{S} = \{p, q, r, \ldots\}$ is the set of propositional letters. Also, as explained in Chapter 8 (see page 193), $\mathcal{W}_f$ is the set of closed formulas in the standard language of classical first order predicate logic $\mathcal{L}_f$, where the latter is obtained from $\mathcal{S}$, a set of constants $\mathcal{C} = \{a, b, c, \ldots\}$, a set of variables

$\mathcal{V} = \{x, y, z, \ldots\}$, a set of predicates $\mathcal{P} = \{P, Q, R, \ldots\}$, the connectives $\neg, \vee, \wedge, \supset$ $, \equiv$ and the quantifiers $\forall$ and $\exists$.

The languages of **K**, **MP**, **T**, **Kt**, **MK** and **MKt** are defined generically – see the relation between $\mathcal{L}_s$ ($\mathcal{W}_s$) and $\check{\mathcal{L}}_s$ ($\check{\mathcal{W}}_s$). The same applies to the respective adaptive logics that have **K**, **MP**, **T**, **Kt**, **MK** or **MKt** as their lower limit logic.

# Appendix B

# Axiomatization of CL

**Propositional CL** For the purposes of the present thesis, it is useful to divide the axioms of propositional **CL** in two fragments: (i) the positive fragment of **CL**

| | |
|---|---|
| **A⊃1** | $A \supset (B \supset A)$ |
| **A⊃2** | $((A \supset B) \supset A) \supset A)$ |
| **A⊃3** | $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$ |
| **A⊥** | $\perp \supset A$ |
| **A∧1** | $(A \wedge B) \supset A$ |
| **A∧2** | $(A \wedge B) \supset B$ |
| **A∧3** | $A \supset (B \supset (A \wedge B))$ |
| **A∨1** | $A \supset (A \vee B)$ |
| **A∨2** | $B \supset (A \vee B)$ |
| **A∨3** | $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$ |
| **A≡1** | $(A \equiv B) \supset (A \supset B)$ |
| **A≡2** | $(A \equiv B) \supset (B \supset A)$ |
| **A≡3** | $(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$ |

and (ii) the axioms for the classical negation:

| | |
|---|---|
| **A¬1** | $\neg\neg A \supset A$ |
| **A¬2** | $(A \supset B) \supset ((A \supset \neg B) \supset \neg B)$ |

The set of all axioms of propositional **CL** is obtained by closing the above list under modus ponens (MP).

**First order predicative CL without identity** Note that I only consider **CL** in terms of closed formulas, i.e. **CL** : $\wp(\mathcal{W}_f) \rightarrow \wp(\mathcal{W}_f)$. The first order fragment of **CL** without identity, is obtained by adding the following three axioms to propositional **CL**:

| | |
|---|---|
| **A∀** | $\forall\alpha A(\alpha) \supset A(\beta)$ |
| **A∃** | $A(\beta) \supset \exists\alpha A(\alpha)$ |

and closing the resulting set under MP and the rules R∀ and R∃:

**R∀** To derive $\vdash A \supset \forall\alpha B(\alpha)$ from $\vdash A \supset B(\beta)$, provided $\beta$ does not occur in either $A$ or $B(\alpha)$.

**R∃** To derive $\vdash \exists\alpha A(\alpha) \supset B$ from $\vdash A(\beta) \supset B$, provided $\beta$ does not occur in either $A(\alpha)$ or $B$.

# Appendix C

# Minimal Abnormality-variants: Some Negative Results

*Sections B.1 and B.3 of this appendix are based on the appendix from the paper "Three Formats of Prioritized Adaptive Logics. A Comparative Study" (Logic Journal of the IGPL 2012, doi:10.1093/jigpal/JZS004), which was co-authored by Christian Straßer. I thank two anonymous referees for their valuable comments on that paper.*

In this appendix, I use the **K**-based prioritized adaptive logics from Chapters 3-5, to prove some negative results about their respective formats. These negative results also illustrate several negative claims made in Chapters 3 and 4. One of the reasons why I decided to put them in the appendix, is since that way I can rely on all the positive results from Chapters 3-5, which greatly simplifies certain proofs.

As the reader will note, only the Minimal Abnormality-variants of the logics are considered. The reason is that for the Reliability-variants, often fairly simple examples suffice to show various negative results – these were given in the main text –, whereas for Minimal Abnormality, rather complex constructions are required. To facilitate the reading, I briefly recapitulate the definitions of the various specific logics. Let $i \in \mathbb{N}$. Then we have:

- $\Diamond^i A$ denotes $A$, preceded by $i$ $\Diamond$s
- $!^i A =_{\mathsf{df}} \Diamond^i A \wedge \neg A$
- $\Omega_i^{\mathbf{K}} =_{\mathsf{df}} \{!^i A \mid A \in \mathcal{W}_c^l\}$
- $\Omega_{(i)}^{\mathbf{K}} =_{\mathsf{df}} \Omega_1^{\mathbf{K}} \cup \ldots \cup \Omega_i^{\mathbf{K}}$
- $\mathbf{K_i^m}$ is the flat AL, defined by $\langle \mathbf{K}, \Omega_i^{\mathbf{K}}, \mathbf{x} \rangle$
- $\mathbf{K_{(i)}^m}$ is the flat AL, defined by $\langle \mathbf{K}, \Omega_{(i)}^{\mathbf{K}}, \mathbf{x} \rangle$
- $\mathbf{SK2^m}$ is obtained from the superposition of the logics $\langle \mathbf{K_i^m} \rangle_{i \in \{1,2\}}$.
- $\mathbf{SK2_{(2)}^m}$ is obtained from the superposition of the logics $\langle \mathbf{K_{(i)}^m} \rangle_{i \in \{1,2\}}$.
- $\mathbf{HK2^m}$ is the hierarchic AL, obtained from the combination of $\mathbf{K_{(1)}^m}$ and $\mathbf{K_{(2)}^m}$

- $\mathbf{K2^m_\sqsubset}$ is the lexicographic AL, defined by the triple $\langle \mathbf{K}, \langle \Omega_i^{\mathbf{K}} \rangle_{i \in \{1,2\}}, \mathbf{x} \rangle$.

To avoid clutter, let me introduce some more notational conventions for this section. I use $\Phi^{\sqsubset_2}(\Gamma)$ to refer to the $\sqsubset$-minimal choice sets of $\Sigma^{(2)}(\Gamma)$.Also, where $M \in \mathcal{M}_{\mathbf{K}}$, let $Ab^{(i)}(M) = \{A \in \Omega_{(i)}^{\mathbf{K}} \mid M \Vdash A\}$. Slightly abusing notation, I write $M \Vdash \Delta$ to denote that $M \Vdash A$ for every $A \in \Delta$. As before, $\Delta^{\check{\neg}} = \{\check{\neg} A \mid A \in \Delta\}$.

# C.1 HK2$^{\mathbf{m}}$ Is Not Complete

As a result of Corollary 6.12 from Chapter 6, whenever $\Gamma$ satisfies $\bigstar_{\mathbf{HAL}}$, then it holds that $A \in Cn_{\mathbf{HK2^m}}(\Gamma)$ iff $\Gamma \models_{\mathbf{HK2^m}} A$. We will now give an example that shows that $\mathbf{HK2^m}$ is not in general complete with respect to its semantics.

Let $\Gamma_1 = \Gamma_1^1 \cup \Gamma_1^2 \cup \Gamma_1^3 \cup \Gamma_1^4$, where

$$\begin{aligned}
\Gamma_1^1 &= \{!^1 p_i \vee !^2 q_j \mid i, j \in \mathbb{N}, i \geq j\} \\
\Gamma_1^2 &= \{!^2 q_i \vee !^2 q_j \mid i, j \in \mathbb{N}, i \neq j\} \\
\Gamma_1^3 &= \{!^2 q_i \vee !^2 r \mid i \in \mathbb{N}\} \\
\Gamma_1^4 &= \{s \vee !^2 r\}
\end{aligned}$$

**Lemma C.1** $\Phi^{\sqsubset_2}(\Gamma_1) = \{!^2 q_i \mid i \in \mathbb{N}\}$.

*Proof.* First of all, note that $\Phi^{(2)}(\Gamma_1) = \Upsilon_1 \cup \Upsilon_2$, where

$$\begin{aligned}
\Upsilon_1 &= \{\{!^2 q_i \mid i \in \mathbb{N}\}\} \\
\Upsilon_2 &= \{\varphi_k \mid k \in \mathbb{N}\} = \{\{!^2 q_i, !^1 p_j, !^2 r \mid i \in \mathbb{N} - \{k\}, j \geq k\} \mid k \in \mathbb{N}\}
\end{aligned}$$

By Theorem 5.2, $\Phi^{\sqsubset_2}(\Gamma_1) \subseteq \Phi^{(2)}(\Gamma_1)$. Note that for every $\varphi \in \Upsilon_2$, $\{!^2 q_i \mid i \in \mathbb{N}\} \sqsubset \varphi$. Hence $\Phi^{\sqsubset_2}(\Gamma_1) = \{\{!^2 q_i \mid i \in \mathbb{N}\}\}$. ■

By Theorem 6.7.7, we can derive:

**Corollary C.1** ${}^{\mathsf{c}}\Phi^{\sqsubset_{(2)}}(\Gamma_1)$ *has no infinite minimal choice sets.*

**Lemma C.2** $\Gamma_1 \models_{\mathbf{HK2^m}} s$.

*Proof.* We prove that $\Gamma_1 \models_{\mathbf{K2^m_\sqsubset}} s$ – the rest is immediate in view of Corollary 6.1. By Lemma C.1 and Theorem 5.3, for every $M \in \mathcal{M}_{\mathbf{K2^m_\sqsubset}}(\Gamma)$, $Ab(M) = \{!^2 q_i \mid i \in \mathbb{N}\}$. But then for every such $M$, $M \nVdash !^2 r$, whence in view of $\Gamma_1^4$, $M \Vdash s$. ■

To show that $s$ is not in the $\mathbf{HK2^m}$-consequence set of $\Gamma_1$, we will need a slightly longer proof. Note that there is no $\Theta \subset \Omega_{(1)}^{\mathbf{K}}$ such that $\Gamma_1 \vdash_{\mathbf{K}} Dab(\Theta)$. Hence $\Gamma_1$ is normal with respect to $\Omega_{(1)}^{\mathbf{K}}$. By Theorem 2.17, we have:

**Lemma C.3** $Cn_{\mathbf{K_1^m}}(\Gamma_1) = Cn_{\mathbf{K}}(\Gamma_1 \cup \{\check{\neg} A \mid A \in \Omega_{(1)}^{\mathbf{K}}\})$.

**Lemma C.4** $s \notin Cn_{\mathbf{HK2^m}}(\Gamma_1)$.

*Proof.* Suppose $s \in Cn_{\mathbf{HK2^m}}(\Gamma_1)$. By Definition 4.1, $Cn_{\mathbf{K^m_1}}(\Gamma_1) \cup Cn_{\mathbf{K^m_{(2)}}}(\Gamma_1) \vdash_{\mathbf{K}} s$. By Lemma C.3, $Cn_{\mathbf{K}}(\Gamma_1 \cup \Omega^{\mathbf{K}}_{(1)}{}^{\check{\neg}}) \cup Cn_{\mathbf{K^m_{(2)}}}(\Gamma_1) \vdash_{\mathbf{K}} s$. Since $\mathbf{K}$ is monotonic, transitive and reflexive, we can derive that $\Gamma_1 \cup \Omega^{\mathbf{K}}_{(1)}{}^{\check{\neg}} \cup Cn_{\mathbf{K^m_{(2)}}}(\Gamma_1) \vdash_{\mathbf{K}} s$. Since $\mathbf{K^m_{(2)}}$ is reflexive, $\Omega^{\mathbf{K}}_{(1)}{}^{\check{\neg}} \cup Cn_{\mathbf{K^m_{(2)}}}(\Gamma_1) \vdash_{\mathbf{K}} s$. By the compactness of $\mathbf{K}$, $\Theta^{\check{\neg}} \cup Cn_{\mathbf{K^m_{(2)}}}(\Gamma_1) \vdash_{\mathbf{K}} s$ for a finite $\Theta \subset \Omega^{\mathbf{K}}_{(1)}$. But then, by the Deduction Theorem, $Cn_{\mathbf{K^m_{(2)}}}(\Gamma_1) \vdash_{\mathbf{K}} s \check{\vee} Dab(\Theta)$. By Theorem 2.18, $\Gamma_1 \vdash_{\mathbf{K^m_{(2)}}} s \check{\vee} Dab(\Theta)$.

Since $\Theta$ is finite, there is a $k \in \mathbb{N}$ such that, for every $l \geq k$: $!^1 p_l \notin \Theta$. Let $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_1)$ be such that each of the following holds:[1]

($C_1$) $Ab^{(2)}(M) = \varphi_k$
($C_2$) $M \not\Vdash s$

By Theorem 2.1 and Lemma C.1, $M \in \mathcal{M}_{\mathbf{K^m_{(2)}}}(\Gamma_1)$. By (C1), $M \not\Vdash Dab(\Theta)$, whence by (C2), also $M \not\Vdash s \check{\vee} Dab(\Theta)$. By the soundness of $\mathbf{K^m_{(2)}}$, $\Gamma_1 \nvdash_{\mathbf{K^m_{(2)}}} s \check{\vee} Dab(\Theta)$ — a contradiction. ■

By Corollary C.1, Lemma C.2 and Lemma C.4, we immediately have:

**Proposition C.1** *There are* $\Gamma, A$ *for which* ${}^{\mathsf{c}}\Phi^{\sqsubset(2)}(\Gamma)$ *has no infinite minimal choice sets and* $\Gamma \models_{\mathbf{HK2^m}} A$*, but* $A \notin Cn_{\mathbf{HK2^m}}(\Gamma)$.

Likewise, by Lemma C.2, Corollary 6.1 and Proposition C.1, it follows that:

**Proposition C.2** *There are* $\Gamma$ *such that* $Cn_{\mathbf{K2^m_{\sqsubset}}}(\Gamma) \not\subseteq Cn_{\mathbf{HK2^m}}(\Gamma)$.

## C.2 $\mathbf{HK2^m}$ Is Not Cumulatively Transitive

The premise set $\Gamma_1$ from the preceding section can also serve as a counterexample to the cumulative transitivity and idempotence of $\mathbf{HK2^m}$. That is, remark that $\Gamma_1' = \{\neg !^1 p_i \mid i \in \mathbb{N}\} \subseteq Cn_{\mathbf{K^m_1}}(\Gamma) \subseteq Cn_{\mathbf{HK2^m}}(\Gamma)$. Now consider the set $\Gamma_1 \cup \Gamma_1'$. Note that the only minimal $\mathrm{Dab}_{(2)}$-consequences of $\Gamma_1 \cup \Gamma_1'$ are all formulas $!^2 q_i$ ($i \in \mathbb{N}$). It follows that $!^2 r \notin \varphi$ for every $\varphi \in \Phi^{(2)}(\Gamma_1 \cup \Gamma_1')$. In view of $\Gamma_1^4$, we can derive that $\Gamma_1 \cup \Gamma_1' \vdash_{\mathbf{K^m_{(2)}}} s$, which means that also $s \in Cn_{\mathbf{HK2^m}}(\Gamma_1 \cup \Gamma_1')$. So we have:

**Proposition C.3** *There are* $\Gamma, \Gamma'$ *such that* $\Gamma' \subseteq Cn_{\mathbf{HK2^m}}(\Gamma)$*, but* $Cn_{\mathbf{HK2^m}}(\Gamma \cup \Gamma') \not\subseteq Cn_{\mathbf{HK2^m}}(\Gamma)$.

By the same reasoning, we can also infer that $\mathbf{HK2^m}$ is not in general idempotent. Note that since $\Gamma_1' \subseteq Cn_{\mathbf{HK2^m}}(\Gamma)$, and by the monotonicity of $\mathbf{K}$ and reflexivity of $\mathbf{HK2^m}$, the set of minimal $\mathrm{Dab}_{(2)}$-consequences of $Cn_{\mathbf{HK2^m}}(\Gamma_1)$ is a subset of the set of minimal Dab-consequences of $\Gamma_1 \cup \Gamma_1'$. Hence $!^2 r$ is also a reliable abnormality in view of $Cn_{\mathbf{HK2^m}}(\Gamma_1)$, whence $s \in Cn_{\mathbf{HK2^m}}(Cn_{\mathbf{HK2^m}}(\Gamma_1)) - Cn_{\mathbf{HK2^m}}(\Gamma_1)$.

[1] See Lemma C.1 for the definition of $\varphi_k$.

## C.3 $\mathbf{HK2^m}$ and $\mathbf{SK2^m_{(2)}}$ are Incomparable

**Lemma C.5** *$s \in Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma_1)$.*

*Proof.* From Lemma C.2 and Corollary 6.1, we can infer that $\Gamma_1 \models_{\mathbf{SK2^m_{(2)}}} s$. By Corollaries 6.11.1 and C.1, it follows that $s \in Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma_1)$. ■

By Lemma C.4 and Lemma C.5, we obtain the following:

**Proposition C.4** *There are $\Gamma$ such that $Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma) \nsubseteq Cn_{\mathbf{HK2^m}}(\Gamma)$.*

We will now prove that also the converse holds. Let $\Gamma_2 = \Gamma_2^1 \cup \Gamma_2^2 \cup \Gamma_2^3 \cup \Gamma_2^4 \cup \Gamma_2^5$, where

$$\begin{aligned}
\Gamma_2^1 &= \{!^1 p_i \vee !^1 p_j \mid i, j \in \mathbb{N}, i \neq j\} \\
\Gamma_2^2 &= \{!^1 p_i \vee !^2 t_j \mid i, j \in \mathbb{N}, i \leq j\} \\
\Gamma_2^3 &= \{!^1 p_i \vee !^2 s \mid i \in \mathbb{N}\} \\
\Gamma_2^4 &= \{r \vee !^1 p_i \vee !^2 q_i \mid i \in \mathbb{N}\} \\
\Gamma_2^5 &= \{r \vee !^2 s\}
\end{aligned}$$

**Lemma C.6** *$r \in Cn_{\mathbf{HK2^m}}(\Gamma_2)$.*

*Proof.* Note that $\Phi^{(2)}(\Gamma_2) = \Psi_1 \cup \Psi_2$, where

$$\begin{aligned}
\Psi_1 &= \{\varphi_0\} = \{\{!^1 p_i \mid i \in \mathbb{N}\}\} \\
\Psi_2 &= \{\varphi_j \mid j \in \mathbb{N}\} = \{\{!^1 p_i \mid i \in \mathbb{N} - \{j\}\} \cup \{!^2 t_k \mid k \geq j\} \cup \{!^2 s\} \mid j \in \mathbb{N}\}
\end{aligned}$$

In view of $\Gamma_2^4$, for every $\varphi_j \in \Psi_2$, there is a $\Theta_j = \{!^1 p_j, !^2 q_j\}$, such that $\Gamma_2 \vdash_{\mathbf{K}} r \check{\vee} Dab(\Theta_j)$ and $\Theta_j \cap \varphi_j = \emptyset$. Also, in view of $\Gamma_2^5$, $\Gamma_2 \vdash_{\mathbf{K}} r \check{\vee} Dab(\{!^2 s\})$, and $\{!^2 s\} \cap \varphi_0 = \emptyset$. By Theorem 2.7.2, $r \in Cn_{\mathbf{K^m_{(2)}}}(\Gamma_2)$. Hence by Definition 4.1 and the reflexivity of **LLL**, $r \in Cn_{\mathbf{HK2^m}}(\Gamma_2)$. ■

By Theorem 4.6 and Corollary 6.1, we immediately obtain the following:

**Lemma C.7** *$\Gamma_2 \models_{\mathbf{SK2^m_{(2)}}} r$.*

We will now prove that $r$ is not a member of the $\mathbf{SK2^m_{(2)}}$-consequence set of $\Gamma_1$. The proof relies on the following lemma:

**Lemma C.8** *There is no $\Theta \subset \Omega^{\mathbf{K}}_{(2)} - (\{!^1 p_i \mid i \in \mathbb{N}\} \cup \{!^2 s\})$, such that $Cn_{\mathbf{K^m_1}}(\Gamma_2)$ $\vdash_{\mathbf{K}} r \check{\vee} Dab(\Theta)$.*

*Proof.* Suppose that there is a $\Theta \subset \Omega^{\mathbf{K}}_{(2)} - (\{!^1 p_i \mid i \in \mathbb{N}\} \cup \{!^2 s\})$, such that $Cn_{\mathbf{K^m_1}}(\Gamma_2) \vdash_{\mathbf{K}} r \check{\vee} Dab(\Theta)$. By Theorem 2.18, (†) $r \check{\vee} Dab(\Theta) \in Cn_{\mathbf{K^m_1}}(\Gamma_2)$.

Since $\Theta$ is finite, there is a $k \in \mathbb{N}$ such that each of the following holds:

(i) $!^2 t_l \notin \Theta$ for every $l \geq k$
(ii) $!^2 q_l \notin \Theta$ for every $l \geq k$

From this and the supposition, we can derive:

$$\Theta \subseteq \Omega^{\mathbf{K}}_{(2)} - (\{!^1 p_i \mid i \in \mathbb{N}\} \cup \{!^2 s\} \cup \{!^2 t_l, !^2 q_l \mid l \geq k\}) \tag{C.1}$$

Let $M \in \mathcal{M}_{\mathbf{K}}(\Gamma_2)$ be such that each of the following holds:

($C_1$) $M \Vdash !^1 p_i$ for every $i \neq k$
($C_2$) $M \Vdash !^2 t_l$ for every $l \geq k$
($C_3$) $M \Vdash !^2 q_l$ for every $l \geq k$
($C_4$) $M \Vdash !^2 s$
($C_5$) $M \nVdash r$
($C_6$) $M \nVdash A$ for every $A \in \Omega^{\mathbf{K}}_1 - \{!^1 p_i \mid i \in \mathbb{N} - \{k\}\}$
($C_7$) $M \nVdash A$ for every $A \in \Omega^{\mathbf{K}}_2 - \{!^2 t_l, !^2 q_l, !^2 s \mid l \geq k\}$

Note that by ($C_1$), $M \Vdash \Gamma^1_2$; by ($C_1$) and ($C_2$), $M \Vdash \Gamma^2_2$; by ($C_1$) and ($C_3$), $M \Vdash \Gamma^4_2$; finally, by ($C_4$), $M \Vdash \Gamma^3_2 \cup \Gamma^5_2$. Suppose there is an $M' \in \mathcal{M}_{\mathbf{K}}(\Gamma_2)$ such that $Ab^{(1)}(M') \subset Ab^{(1)}(M)$. In that case, $M' \nVdash !^1 p_i$ for an $i \neq k$. But then, in view of $\Gamma^1_1$, $M' \Vdash !^1 p_k$, whence $!^1 p_k \in Ab^{(1)}(M') - Ab^{(1)}(M)$ — a contradiction. It follows that $M \in \mathcal{M}_{\mathbf{K^m_1}}(\Gamma_2)$.

Note that by ($C_6$) and ($C_7$), $M \nVdash Dab(\Lambda)$ for every $\Lambda \subseteq \Omega^{\mathbf{K}}_{(2)} - (\{!^1 p_i \mid i \in \mathbb{N}\} \cup \{!^2 t_l, !^2 q_l, !^2 s \mid l \geq k\})$, whence by (C.1), $M \nVdash Dab(\Theta)$. Together with ($C_5$), this implies that $M \nVdash r \check{\vee} Dab(\Theta)$. By the completeness of $\mathbf{K^m_1}$, $\Gamma_2 \nvdash_{\mathbf{K^m_1}} r \check{\vee} Dab(\Theta)$ — a contradiction. ■

**Lemma C.9** *$r \notin Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma_2)$.*

*Proof.* First of all, note that $\Phi^{(1)}(\Gamma_2) = \{\{!^1 p_i \mid i \in \mathbb{N} - \{j\} \mid j \in \mathbb{N}\}$. In view of $\Gamma^4_2$, for every $\varphi \in \Phi^{(1)}(\Gamma_2)$, $\Gamma_2 \vdash_{\mathbf{K}} !^2 s \check{\vee} Dab(\Theta)$ for a $\Theta \subset \Omega^{\mathbf{K}}_1$ such that $\varphi \cap \Theta = \emptyset$. Hence by Theorem 2.7.2, $!^2 s \in Cn_{\mathbf{K^m_1}}(\Gamma_2)$. This implies that $\Phi^{(2)}(Cn_{\mathbf{K1^m}}(\Gamma_2)) = \Xi_1 \cup \Xi_2$, where

$$\Xi_1 = \{\varphi_\star\} = \{\{!^1 p_i \mid i \in \mathbb{N}\} \cup \{!^2 s\}\}$$
$$\Xi_2 = \{\{!^1 p_i \mid i \in \mathbb{N} - \{j\}\} \cup \{!^2 t_k \mid k \geq j\} \cup \{!^2 s\} \mid j \in \mathbb{N}\}$$

By Lemma C.8, there is no $\Theta \subseteq \Omega^{\mathbf{K}}_{(2)} - \varphi_\star$, such that $Cn_{\mathbf{K^m_1}}(\Gamma_2) \vdash_{\mathbf{K}} r \check{\vee} Dab(\Theta)$. By Theorem 2.7.2, $r \notin Cn_{\mathbf{K^m_{(2)}}}(Cn_{\mathbf{K^m_1}}(\Gamma_2)) = Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma_2)$. ■

By Lemma C.6 and Lemma C.9, we immediately have:

**Proposition C.5** *There are $\Gamma$ such that $Cn_{\mathbf{HK2^m}}(\Gamma) \not\subseteq Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma)$.*

Since the $\mathbf{SK2^m_{(2)}}$-semantics and the $\mathbf{HK2^m}$-semantics are equivalent, and in view of Theorem 4.6, it follows immediately that $\mathbf{SK2^m_{(2)}}$ is not complete with respect to its semantics:

**Proposition C.6** *There are $\Gamma, A$ such that $\Gamma \models_{\mathbf{SK2^m_{(2)}}} A$, but $A \notin Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma)$.*

Finally, by Corollary 6.1, Proposition C.6 and the soundness and completeness of $\mathbf{K2^m_{\sqsubset}}$:

**Proposition C.7** *There are $\Gamma$ such that $Cn_{\mathbf{K2^m_{\sqsubset}}}(\Gamma) \not\subseteq Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma)$.*

## C.4 $\mathbf{HK2^m}$ Is not Cautiously Monotonic

By the same example from the previous section, we can show that $\mathbf{HK2^m}$ is not (in general) cautiously monotonic. Recall that $r \in Cn_{\mathbf{HK2^m}}(\Gamma_2)$, due to the fact that $r \in Cn_{\mathbf{K^m_{(2)}}}(\Gamma)$. However, it was also shown in the preceding section that $!^2 s \in Cn_{\mathbf{K^m_1}}(\Gamma)$. The reader can easily verify that $\Phi^{(2)}(\Gamma_2 \cup \{!^2 s\}) = \Phi^{(2)}(Cn_{\mathbf{K^m_1}}(\Gamma_2))$, whence the set $\varphi_\star$ is a member of $\Phi^{(2)}(\Gamma_2 \cup \{!^2 s\})$.

By Lemma C.8 and the fact that $\mathbf{K^m_1}$ is always at least as strong as $\mathbf{K}$, we can infer that there is no $\Theta \subset \Omega^{\mathbf{K}}_{(2)} - \varphi_\star$, such that $\Gamma_2 \vdash_{\mathbf{K}} r \check{\vee} Dab(\Theta)$. It follows that $r \notin Cn_{\mathbf{K^m_{(2)}}}(\Gamma_2 \cup \{!^2 s\})$. More generally, by a reasoning similar to the one from Section C.1, we can derive that $r \notin Cn_{\mathbf{HK2^m}}(\Gamma_2 \cup \{!^2 s\})$. So we have:

**Proposition C.8** *There are* $\Gamma, \Gamma'$ *such that* $\Gamma' \subseteq Cn_{\mathbf{HK2^m}}(\Gamma)$ *and* $Cn_{\mathbf{HK2^m}}(\Gamma) \not\subseteq Cn_{\mathbf{HK2^m}}(\Gamma \cup \Gamma')$.

## C.5 $\mathbf{SK2^m_{(2)}}$ Is Not Cumulatively Transitive

For the proof that $\mathbf{SK2^m_{(2)}}$ is not cumulatively transitive, we can use the premise set $\Gamma_{\mathbf{c}}$, which was introduced on page 52 in order to show that $\mathbf{SAL^m}$-logics are not in general complete, and was reconsidered on page 83 to show that the same logics are not in general cumulatively transitive or idempotent. The reasoning is completely analogous to the one for $\mathbf{SK2^m}$.

## C.6 $\mathbf{HK^m}$ and $\mathbf{K^m}$ Are Incomparable

Recall that $\mathbf{HK^m}$ is a hierarchic AL, obtained from the flat ALs $\mathbf{K^m_{(1)}} = \langle \mathbf{K}, \Omega^{\mathbf{K}}_{(1)}, \mathbf{m}\rangle$ and $\mathbf{K^m_{(2)}} = \langle \mathbf{K}, \Omega^{\mathbf{K}}_{(2)}, \mathbf{m}\rangle$, and that $\mathbf{K^m}$ is a flat AL defined by the triple $\langle \mathbf{K}, \bigcup_{i\in\mathbb{N}} \Omega_i, \mathbf{m}\rangle$. In this section, I construct a premise set $\Gamma$ for which $Cn_{\mathbf{K^m}}(\Gamma) \not\subseteq Cn_{\mathbf{HK^m}}(\Gamma)$, and hence prove Proposition 4.1 from Chapter 4 – see page 107. It can also be shown that $Cn_{\mathbf{K^m}}(\Gamma) \not\subseteq Cn_{\mathbf{SK2^m_{(2)}}}(\Gamma)$ — the reasoning is completely analogous and left to the reader.

Let $\Gamma = \Delta_1 \cup \Delta_2 \cup \Delta_3$, where

$$\begin{aligned}
\Delta_1 &= \{!^n p_i \check{\vee} !^n p_j \mid i, j, n \in \mathbb{N}, i \neq j\}\\
\Delta_2 &= \{!^n p_i \check{\vee} !^m p_j \mid i, j, n, m \in \mathbb{N}, n < m, i \leq j\}\\
\Delta_3 &= \{q \check{\vee} !^n p_1 \mid n \in \mathbb{N}\}
\end{aligned}$$

I will first show why $q \in Cn_{\mathbf{K^m}}(\Gamma)$ in an informal way. Where $M \in \mathcal{M}^{\mathbf{K}}_\Gamma$, we can represent $Ab(M)$ as a series of dots in a two-dimensional space. An example of one such $M$ is shown in Figure C.1. Each point $(n, i)$ represents the abnormality $!^n p_i$, with $n, i \in \mathbb{N}$. The white dots represent the abnormalities that are an element of $Ab(M)$. Connecting the black dots in the space, we get a "path" $\mathrm{P}_M$ that represents $\Omega - Ab(M)$. This representation allows us to grasp relations between models in a fairly straightforward way, as I will now explain.

Let us take a closer look at $\Gamma$, and start with $\Delta_1$. This set ensures that for every $n \in \mathbb{N}$, every $\mathbf{K}$-model of $\Gamma$ can falsify at most one abnormality $!^n p_i$. To

| | $!^1$ | $!^2$ | $!^3$ | $!^4$ | $!^5$ | $!^6$ | $!^7$ | $!^8$ | ... |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | |
| $p_2$ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | |
| $p_3$ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | |
| $p_4$ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | |
| $p_5$ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ○ | |
| $p_6$ | ● | ○ | ○ | ○ | ○ | ○ | ○ | ○ | |
| ... | | | | | | | | | |

Figure C.1: A representation of a model $M \in \mathcal{M}_{\mathbf{K^m}}(\Gamma)$ for which: $Ab(M) = \{!^i p_j \mid i, j \in \mathbb{N}\} - \{!^1p_6, !^3p_5, !^4p_4, !^5p_3, !^6p_1\}$.

see why, suppose $M$ falsifies $!^n p_i$ for $n, i \in \mathbb{N}$. By $\Delta_1$, every $!^n p_j$ must be true in this model, for any $j \in \mathbb{N}, j \neq i$.

As concerns the graphic representation, $\Delta_1$ implies that no path that represents a set $Ab(M)$ with $M \in \mathcal{M}_{\mathbf{K}}(\Gamma)$ can contain two black dots of the same column. Note that this restriction is satisfied in Figure 1: since $M \nVdash !^1p_6$, $M \Vdash !^1p_i$ for any $i \in \mathbb{N}, i \neq 6$.

Now let us look at $\Delta_2$, in the face of the example in Figure 1. We can see that $M \nVdash !^4p_4$. This implies by $\Delta_2$ that $M \Vdash !^m p_j$ for any $m > 4, j \geq 4$. So, if there is a black dot in the representation of a set $Ab(M)$ for an $M \in \mathcal{M}_{\mathbf{K}}(\Gamma)$, we can infer that all dots that appear after it in the same row have to be white, *and* that all dots underneath those dots have to be white too. In other words, the path that connects the black dots can only go upwards. As a result, every model of $\Gamma$ can falsify a finite number of abnormalities only.

A third remark concerns the minimal abnormal models, in view of $\mathbf{K^m}$. Suppose a selection of abnormalities in view of $\Gamma$ is represented by the path $\mathrm{P}_M$, and there is a path $\mathrm{P}_{M'}$ such that this path *extends* $\mathrm{P}_M$ with one more black dot. This implies that the model $M'$ falsifies one extra abnormality. As a result, $Ab(M') \subset Ab(M)$, which means that $M$ is not minimal abnormal.

From these facts, we can infer that every path $\mathrm{P}_M$ that represents $\Omega - Ab(M)$ for a minimal abnormal model $M$, must necessarily end at some point $!^n p_1$, hence at a point in the first row – if it would not end there, we could extend $\mathrm{P}_M$ with at least one more point, and hence $Ab(M)$ would not correspond to a minimal selection of abnormalities. Hence, every $\mathbf{K^m}$-model of $\Gamma$ falsifies one abnormality $!^n p_1$, which means that it verifies $q$. By the completeness of $\mathbf{K^m}$, $\Gamma \vdash_{\mathbf{K^m}} q$.

In the remainder, I prove that $\Gamma \nvdash_{\mathbf{HK^m}} q$. To simplify the proof, it is convenient to rely on the equivalence results from Chapter 6. In the remainder, let every logic $\mathbf{K}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}$ ($i \in \mathbb{N}$) be a lexicographic adaptive logic characterized by the triple $\langle \mathbf{K}, \langle \Omega_j^{\mathbf{K}} \rangle_{j \leq i}, \mathbf{m} \rangle$, and let $\sqsubset_{(i)}$ be the lexicographic order associated with this logic.

**Lemma C.10** *For every $i \in \mathbb{N}$: there is an $M \in \mathcal{M}_{\mathbf{K}^{\mathbf{m}}_{\sqsubset_{(\mathbf{i})}}}(\Gamma)$ such that $M \nVdash q$.*

*Proof.* Let $i \in \mathbb{N}$. Let $M \in \mathcal{M}_{\mathbf{K}}(\Gamma)$ be such that $Ab(M) = \{!^i p_j \mid i, j \in \mathbb{N}\} - \{!^j p_{i+2-j} \mid 1 \leq j \leq i\}$ and $M \nVdash q$. Assume that there is an $M' \in \mathcal{M}_{\mathbf{K}}(\Gamma)$ such that $Ab(M') \cap \Omega_{(i)} \sqsubset_{(i)} Ab(M) \cap \Omega_{(i)}$.

Let $j \leq i$ be such that (1) for all $k \leq j$, $Ab(M') \cap \Omega_k = Ab(M) \cap \Omega_k$ and (2) $Ab(M') \cap \Omega_j \subset Ab(M) \cap \Omega_j$. By (2), $M' \nVdash !^j p_{i+2-j}$. But then, since $M' \Vdash \Delta_1$, $M' \Vdash !^j p_k$ for every $k \in \mathbb{N}, k \neq i+2-j$. Hence $Ab(M') \cap \Omega_j = Ab(M) \cap \Omega_j$ — a contradiction. ■

In view of the equivalence of the semantics of lexicographic ALs and the corresponding hierarchic ALs, we have:

**Corollary C.2** *For every $i \in \mathbb{N}$: there is an $M \in \bigcap_{j \leq i} \mathcal{M}_{\mathbf{K}^{\mathbf{m}}_{(\mathbf{j})}}(\Gamma)$ such that $M \nVdash q$.*

**Lemma C.11** *$q \notin Cn_{\mathbf{HK^m}}(\Gamma)$.*

*Proof.* Assume that $q \in Cn_{\mathbf{HK^m}}(\Gamma)$. Hence $q \in Cn_{\mathbf{K}}(\bigcup_{i \in \mathbb{N}} Cn_{\mathbf{K}^{\mathbf{m}}_{(\mathbf{i})}}(\Gamma))$. By the compactness of $\mathbf{K}$, there is an $i \in \mathbb{N}$ such that $q \in Cn_{\mathbf{K}}(\bigcup_{j \leq i} Cn_{\mathbf{K}^{\mathbf{m}}_{(\mathbf{j})}}(\Gamma))$. But then, by the soundness of hierarchic adaptive logics, $q$ is true in every model $M \in \bigcap_{j \leq i} \mathcal{M}_{\mathbf{K}^{\mathbf{m}}_{(\mathbf{j})}}(\Gamma)$. This however contradicts Corollary C.2. ■

## C.7 SK2$^{\mathbf{m}}$ versus K2$^{\mathbf{m}}_{\sqsubset}$, HK2$^{\mathbf{m}}$ and SK2$^{\mathbf{m}}_{(\mathbf{2})}$

I have not yet been able to come up with examples of a $\Gamma$ such that $Cn_{\mathbf{SK2}^{\mathbf{m}}_{(\mathbf{2})}}(\Gamma) \nsubseteq Cn_{\mathbf{SK2^m}}(\Gamma)$, although it is plausible – in view of the other incomparability results from this appendix – that there are such premise sets.

For an example where $A \in Cn_{\mathbf{HK2^m}}(\Gamma) - Cn_{\mathbf{SK2^m}}(\Gamma)$, I refer to the example $\Gamma_2$ from Section C.3. There it was proven that $r \in Cn_{\mathbf{HK2^m}}(\Gamma_2)$. That $r \notin Cn_{\mathbf{SK2^m}}(\Gamma_2)$, is immediate in view of the following two lemmas – I omit their very simple proofs for reasons of space:[2]

**Lemma C.12** $\Phi^2(Cn_{\mathbf{SK2^m}}(\Gamma_2)) = \{\{!^2 s\}\}$.

**Lemma C.13** *There is no $\Delta \subseteq \Omega_2 - \{!^2 s\}$ such that $r \check{\vee} Dab(\Delta) \in Cn_{\mathbf{K}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma_2)$.*

There is also a relatively simple example that shows that in some cases, $\mathbf{K2}^{\mathbf{m}}_{\sqsubset}$ is stronger than $\mathbf{SK2^m}$. This example was already given in Section 3.3.3 of Chapter 3 (see page 52):

$$\Gamma_{\mathsf{c}} = \{!^1 p_i \vee !^1 p_j \mid i, j \in \mathbb{N}, i \neq j\} \cup \{!^1 p_i \vee !^2 q_i \vee r \mid i \in \mathbb{N}\}.$$

[2]The second lemma is proven in a similar way as Lemma C.8 in Section C.3. That is, for every finite $\Delta \subseteq \Omega_2 - \{!^2 s\}$, we can define a model $M \in \mathcal{M}_{\mathbf{K}^{\mathbf{m}}_{\mathbf{1}}}(\Gamma_2)$ such that (i) $M \nVdash s$, and (ii) $M \nVdash Dab(\Delta)$.

As shown there, $r \notin Cn_{\mathbf{SK2^m}}(\Gamma_{\mathsf{c}})$. To see why $r \in Cn_{\mathbf{K2^m_\sqsubset}}(\Gamma_{\mathsf{c}})$, note that $\Phi^{\sqsubset_2}(\Gamma_{\mathsf{c}}) = \{\varphi_j \mid j \in \mathbb{N}\} = \{\{!^1 p_i \mid i \in \mathbb{N} - \{j\}\} \mid j \in \mathbb{N}\}$. For every $\varphi_j \in \Phi^{\sqsubset_2}(\Gamma_{\mathsf{c}})$, we can derive $r$ on a condition $\Delta_j = \{!^1 p_j, !^2 q_j\}$ such that $\varphi_j \cap \Delta_j = \emptyset$.

# Appendix D

# The Complexity of Superpositions

As explained at the end of Chapter 6, most questions concerning the computational complexity of the various formats of prioritized ALs are still to be adressed in future research. However, on the basis of some preliminary work, it seems plausible that many superpositions of $n$ logics $\langle \mathbf{AL^r_i} \rangle_{i \leq n}$ have a worst case complexity of at least $\Sigma^0_{2n+1}$. Let me try to spell out why this is the case.

**Two Priority Levels** Recall that $\mathbf{SK2^r}$ is obtained by the superposition of $\mathbf{K^r_2}$ on $\mathbf{K^r_1}$, or more formally:

$$Cn_{\mathbf{SK2^r}}(\Gamma) =_{\mathsf{df}} Cn_{\mathbf{K^r_2}}(Cn_{\mathbf{K^r_1}}(\Gamma))$$

In what follows, I give the outline for a proof that $\mathbf{SK2^r}$ has a worst case complexity of at least $\Sigma^0_5$. That is, I will define a recursive premise set $\Gamma \subseteq \mathcal{W}_m$ such that, for all $i \in \mathbb{N}$, $u_i \in Cn_{\mathbf{SK2^r}}(\Gamma)$ iff $i$ is a member of a $\Sigma^0_5$-complete set. The example I use is obtained by an extension of the example given by Horsten & Welch, in order to show that flat adaptive logics with the Reliability strategy are $\Sigma^0_3$-complex (see [79, p. 56]).

Let the set $I_R \subset \mathbb{N}$ be $\Sigma^0_5$-complete, i.e. where $i \in \mathbb{N}$:

$$i \in I_R \text{ iff } \exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 : (x_1, \ldots, x_5, i) \in R$$

with $R$ recursive.

Let $\Gamma = \Delta_1 \cup \ldots \cup \Delta_6$, where

$$\begin{aligned}
\Delta_1 &= \{p^i_{x_1,\ldots,x_5} \mid (x_1, \ldots, x_5, i) \in R\}\\
\Delta_2 &= \{p^i_{x_1,\ldots,x_5} \supset\, !^1 q^i_{x_1,\ldots,x_4} \mid x_1, \ldots, x_5, i \in \mathbb{N}\}\\
\Delta_3 &= \{!^1 q^i_{x_1,\ldots,x_4} \vee\, !^1 r^i_{x_1,x_2,x_3} \mid x_1, \ldots, x_4, i \in \mathbb{N}\}\\
\Delta_4 &= \{!^1 r^i_{x_1,x_2,x_3} \vee\, !^2 s^i_{x_1,x_2} \mid x_1, x_2, x_3, i \in \mathbb{N}\}\\
\Delta_5 &= \{!^2 s^i_{x_1,x_2} \vee\, !^2 t^i_{x_1} \mid x_1, x_2, i \in \mathbb{N}\}\\
\Delta_6 &= \{u_i \vee\, !^2 t^i_{x_1} \mid x_1, i \in \mathbb{N}\}
\end{aligned}$$

We need to prove that $u_i \in Cn_{\mathbf{SK2^r}}(\Gamma)$ iff $i \in I_R$. This follows immediately from the following lemma:

**Lemma D.1** *Where $i, x_1, \ldots, x_5 \in \mathbb{N}$, each of the following holds:*

1. $u_i \in Cn_{\mathbf{K^r_2}}(Cn_{\mathbf{K^r_1}}(\Gamma))$ *iff there is an* $x_1 \in \mathbb{N}$ *such that* $!^2 t^i_{x_1} \notin U^2(Cn_{\mathbf{K^r_1}}(\Gamma))$.
2. $!^2 t^i_{x_1} \notin U^2(Cn_{\mathbf{K^r_1}}(\Gamma))$ *iff for all* $x_2 \in \mathbb{N}$, $!^2 s^i_{x_1,x_2} \in Cn_{\mathbf{K^r_1}}(\Gamma)$.
3. $!^2 s^i_{x_1,x_2} \in Cn_{\mathbf{K^r_1}}(\Gamma)$ *iff there is an* $x_3 \in \mathbb{N}$*, such that* $!^1 r^i_{x_1,x_2,x_3} \notin U^1(\Gamma)$.
4. $!^1 r^i_{x_1,x_2,x_3} \notin U^1(\Gamma)$ *iff for all* $x_4 \in \mathbb{N}$, $\Gamma \vdash_{\mathbf{K}} !^1 q^i_{x_1,\ldots,x_4}$.
5. $\Gamma \vdash_{\mathbf{K}} !^1 q^i_{x_1,\ldots,x_4}$ *iff there is an* $x_5 \in \mathbb{N}$ *such that* $p^i_{x_1,\ldots,x_5} \in \Gamma$.

Some explanation might help to understand why each of these claims are plausible. The right-left direction of item 1 is immediate in view of $\Delta_6$ and Theorem 2.6. The other direction can be proven with the aid of an additional lemma, which states that (†) each $u_i$ ($i \in \mathbb{N}$) can only be derived from $Cn_{\mathbf{K^r_1}}(\Gamma)$ on conditions $\Delta \subseteq \Omega^{\mathbf{K}}_2$ such that $!^2 t^i_{x_1} \in \Delta$. If (†) holds, then in order for $u_i$ to be finally $\mathbf{K^r_2}$-derivable from $Cn_{\mathbf{K^r_1}}(\Gamma)$ on such a condition $\Delta$, it should be the case that $!^2 t^i_{x_1} \notin U^2(Cn_{\mathbf{K^r_1}}(\Gamma))$.

The left-right direction of item 2 follows from the fact that, for every $i, x_1, x_2 \in \mathbb{N}$, $!^2 s^i_{x_1,x_2} \check{\vee} !^2 t^i_{x_1}$ is a Dab$_2$-consequence of $Cn_{\mathbf{K^r_1}}(\Gamma)$ (since $\mathbf{K^r_1}$ is reflexive and in view of $\Delta_5$). Hence, if $!^2 t^i_{x_1}$ is reliable in view of $Cn_{\mathbf{K^r_1}}(\Gamma)$, then none of these Dab$_2$-consequences are minimal, and hence, all formulas $!^2 s^i_{x_1,x_2}$ should be **K**-derivable from $Cn_{\mathbf{K^r_1}}(\Gamma)$. For the other direction, we need to show that there are no Dab-consequences of $Cn_{\mathbf{K^r_1}}(\Gamma)$ which contain $!^2 t^i_{x_1}$, except formulas of the form $!^2 s^i_{x_1,x_2} \check{\vee} !^2 t^i_{x_1} \check{\vee} Dab(\Delta)$.

The reasoning for item 3 is analogous to that for item 1: we need to prove that the only conditions on which we can $\mathbf{K^r_1}$-derive $!^2 s^i_{x_1,x_2}$ from $\Gamma$, are sets $\Delta \subset \Omega^{\mathbf{K}}_1$ for which $!^1 r^i_{x_1,x_2,x_3} \in \Delta$, for an $x_3 \in \mathbb{N}$. So $!^2 s^i_{x_1,x_2} \in Cn_{\mathbf{K^r_1}}(\Gamma)$ iff one of these abnormalities are reliable in view of $\Gamma$.

The reasoning for item 4 is analogous to that for item 2: since all formulas $!^1 q^i_{x_1,\ldots,x_4} \check{\vee} !^1 r^i_{x_1,x_2,x_3}$ are Dab$_1$-consequences of $\Gamma$, an abnormality $!^1 r^i_{x_1,x_2,x_3}$ is reliable if and only if every abnormality $!^1 q^i_{x_1,\ldots,x_4}$ is derivable in itself.

Item 5 can be shown as follows. If $p^i_{x_1,\ldots,x_5} \in \Gamma$ for an $x_5 \in \mathbb{N}$, then $!^1 q^i_{x_1,\ldots,x_4} \in Cn_{\mathbf{K}}(\Gamma)$ in view of $\Delta_2$. On the other hand, if there is no $x_5 \in \mathbb{N}$ such that $p^i_{x_1,\ldots,x_5} \in \Gamma$, then we can easily see that there is an $M \in \mathcal{M}_{\mathbf{K}}(\Gamma)$ such that $M \not\Vdash !^1 q^i_{x_1,\ldots,x_4}$ — this model is such that it verifies all formulas $!^1 r^i_{x_1,x_2,x_3}$ with $i, x_1, x_2, x_3 \in \mathbb{N}$.

**More Than Two Levels** The above example can easily be adapted to show that, whenever a **K**-based prioritized logic uses $n$ priority levels, then the worst case complexity of this logic is at least $\Sigma^0_{2n+1}$. For instance, for $n = 3$, we need to adjust the construction as follows:

Let the set $I_{R'} \subset \mathbb{N}$ be $\Sigma^0_7$-complete, i.e. where $i \in \mathbb{N}$:

$$i \in I_{R'} \text{ iff } \exists x_1 \forall x_2 \exists x_3 \forall x_4 \exists x_5 \forall x_6 \exists x_7 : (x_1, \ldots, x_7, i) \in R'$$

with $R'$ recursive.

Let $\Gamma' = \Delta'_1 \cup \ldots \cup \Delta'_8$, where

$$\begin{aligned}
\Delta'_1 &= \{p^i_{x_1,\ldots,x_7} \mid (x_1,\ldots,x_7,i) \in R'\}\\
\Delta'_2 &= \{p^i_{x_1,\ldots,x_7} \supset !^1 q^i_{x_1,\ldots,x_6} \mid x_1,\ldots,x_7,i \in \mathbb{N}\}\\
\Delta'_3 &= \{!^1 q^i_{x_1,\ldots,x_6} \vee !^1 r^i_{x_1,\ldots,x_5} \mid x_1,\ldots,x_6,i \in \mathbb{N}\}\\
\Delta'_4 &= \{!^1 r^i_{x_1,\ldots,x_5} \vee !^2 s^i_{x_1,\ldots,x_4} \mid x_1,\ldots,x_5,i \in \mathbb{N}\}\\
\Delta'_5 &= \{!^2 s^i_{x_1,\ldots,x_4} \vee !^2 m^i_{x_1,x_2,x_3} \mid x_1,\ldots,x_4,i \in \mathbb{N}\}\\
\Delta'_6 &= \{!^2 m^i_{x_1,x_2,x_3} \vee !^3 n^i_{x_1,x_2} \mid x_1,x_2,x_3,i \in \mathbb{N}\}\\
\Delta'_7 &= \{!^3 n^i_{x_1,x_2} \vee !^3 t^i_{x_1} \mid x_1,x_2,i \in \mathbb{N}\}\\
\Delta'_8 &= \{u_i \vee !^3 t^i_{x_1} \mid x_1,i \in \mathbb{N}\}
\end{aligned}$$

Let the logic $\mathbf{SK3^r}$ be obtained by the sequential superposition of the logics $\mathbf{K^r_1}$, $\mathbf{K^r_2}$ and $\mathbf{K^r_3}$. By a reasoning analogous to that for $\Gamma$, we can show that, for all $i \in \mathbb{N}$, $u_i \in Cn_{\mathbf{SK3^r}}(\Gamma')$ iff $i \in I_{R'}$.

# Appendix E

# Adaptive Belief Revision: Two Supplementary Postulates

Beside the 6 basic rationality postulates for belief revision (see Chapter 9), two supplementary postulates are usually considered in the AGM-framework, i.e. those of super- and subexpansion (see e.g. [74, Section 3]). Given the notational conventions from Chapter 9, they can be formulated as follows:

**G7** *Superexpansion:* $\Upsilon \oplus \{A \wedge B\} \subseteq Cn_{\mathbf{CL}}((\Upsilon \oplus \{A\}) \cup \{B\})$
**G8** *Subexpansion:* If $\neg B \notin Cn_{\mathbf{CL}}(\Upsilon \oplus \{A\})$, then $Cn_{\mathbf{CL}}((\Upsilon \oplus \{A\}) \cup \{B\}) \subseteq \Upsilon \oplus \{A \wedge B\}$

Note that if $\oplus$ satisfies the *Closure* postulate (G1), the antecedent of G8 is equivalent to $\neg B \notin \Upsilon \oplus \{A\}$. As shown in [3], adding G7, or both G7 and G8 to the basic postulates, one obtains characterizations of specific classes of partial meet revisions, which are (proper) subsets of the class of all partial meet revision. Although G7 and G8 are perhaps not as intuitive as the 6 basic postulates – see e.g. [117] for a critical discussion –, it nevertheless seems worthwhile to see whether the logics $\mathbf{AR}^{\mathbf{x}}_{\mathbf{i}}$ obey them.

In this appendix, I show each of the following:

(i) Where $i \in \{1, 2, 3, 4\}$, all revision operations defined from $\oplus_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{i}}}$ satisfy *Superexpansion*
(ii) Where $i \in \{1, 2, 3, 4\}$, there are revision operations defined from $\oplus_{\mathbf{AR}^{\mathbf{r}}_{\mathbf{i}}}$ which do not satisfy *Superexpansion*
(iii) Where $i \in \{1, 2, 3, 4\}$, there are revision operations defined from $\oplus_{\mathbf{AR}^{\mathbf{x}}_{\mathbf{i}}}$ which do not satisfy satisfy *Subexpansion*

**Proof of (i)** Let $i \in \{1, 2, 3, 4\}$. Suppose that $C \in \Upsilon \oplus_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{i}}} \{A \wedge B\}$. Hence by Definition 9.16,

$$C \in Cn_{\mathbf{AR}^{\mathbf{m}}_{\mathbf{i}}}(\Upsilon^{\Box} \cup \{A \wedge B\}) \cap \mathcal{W}_c$$

Note that, since $\mathbf{LLL_i}$ is a conservative extension of $\mathbf{CL}$, $\Upsilon^{\Box} \cup \{A \wedge B\}$ is $\mathbf{LLL_i}$-equivalent to $\Upsilon^{\Box} \cup \{A\} \cup \{B\}$. Hence by Theorem 9.14:

$$C \in Cn_{\mathbf{AR_i^m}}(\Upsilon^{\Box} \cup \{A\} \cup \{B\}) \cap \mathcal{W}_c$$

Since $\mathbf{AR_i^m}$ satisfies the Deduction Theorem (see Theorem 2.11), and since $\supset$ and $\check{\supset}$ are equivalent in $\mathbf{LLL_i}$, we have:

$$(B \supset C) \in Cn_{\mathbf{AR_i^m}}(\Upsilon^{\Box} \cup \{A\})$$

Note that $B, C \in \mathcal{W}_c$ and thus also $B \supset C \in \mathcal{W}_c$. Hence, by Definition 9.16:

$$(B \supset C) \in \Upsilon \oplus_{\mathbf{AR_i^m}} \{A\}$$

By modus ponens,

$$C \in Cn_{\mathbf{CL}}((\Upsilon \oplus_{\mathbf{AR_i^m}} \{A\}) \cup \{B\})$$

■

**Example of (ii)** Let $\Upsilon = \{p, q\}$, $A = \neg p \vee \neg q$ and $B = \neg p$. I will only discuss the behavior of $\mathbf{AR_1^r}$ for this example – the reasoning for $\mathbf{AR_2^r}$, $\mathbf{AR_3^r}$ and $\mathbf{AR_4^r}$ and their respective revision operations proceeds wholly analogous.

Consider first $\Upsilon \oplus_{\mathbf{AR_1^r}} \{A \wedge B\}$. Let $\Gamma = \Upsilon^{\Box} \cup \{A \wedge B\} = \{\Box p, \Box q, \neg p \vee \neg q, \neg p\}$. Note that this set has only one minimal $\mathrm{Dab}_1$-consequence:

$$\rho(p)$$

As a result, $U^1(\Gamma) = \{\rho(p)\}$. Hence we can finally derive $q$ in an $\mathbf{AR_1^r}$-proof from $\Gamma$, on the condition $\{\rho(q)\}$. So we have:

$$q \in \Upsilon \oplus_{\mathbf{AR_1^r}} \{A \wedge B\}$$

Consider now $\Upsilon \oplus_{\mathbf{AR_1^r}} \{A\}$. Let $\Gamma' = \Upsilon^{\Box} \cup \{A\} = \{\Box p, \Box q, \neg p \vee \neg q\}$. The only minimal $\mathrm{Dab}_1$-consequence of $\Gamma'$ is

$$\rho(p) \check{\vee} \rho(q)$$

Hence $U^1(\Gamma') = \{\rho(p), \rho(q)\}$. This means that there are models $M \in \mathcal{M}_{\mathbf{AR_1^r}}(\Gamma')$ for which $M \Vdash \neg p$ and $M \Vdash \neg q$. As a result, $q \notin \Upsilon \oplus_{\mathbf{AR_1^r}} \{A\}$, but also $p \vee q \notin \Upsilon \oplus_{\mathbf{AR_1^r}} \{A\}$. More generally, $\Upsilon \oplus_{\mathbf{AR_1^r}} \{A\} = Cn_{\mathbf{CL}}(\{A\}) = Cn_{\mathbf{CL}}(\{\neg p \vee \neg q\})$. This implies that $q \notin Cn_{\mathbf{CL}}((\Upsilon \oplus_{\mathbf{AR_1^r}} \{A\}) \cup \{B\}) = Cn_{\mathbf{CL}}(\{\neg p\})$.

The difference with $\mathbf{AR_1^m}$ is that the latter allows us to uphold the implicit belief $p \vee q$. As a result, when we expand the revision set $\Upsilon \oplus_{\mathbf{AR_1^m}} \{A\}$ with $B = \neg p$, we can regain $q$ by applying disjunctive syllogism to $p \vee q$ and $\neg p$.

**Example of (iii)** Let $\Upsilon = \{p, q, r\}$, $A = \neg p \vee \neg q$ and $B = \neg p \vee \neg r$. As in the preceding paragraph, I will only explain what happens in the case of $\mathbf{AR^r_1}$ and $\mathbf{AR^m_1}$ – the reasoning for the other logics, resp. revision operations is again analogous.

Consider first $\Upsilon \oplus_{\mathbf{AR^r_1}} \{A\}$. This set is obtained by closing $\Gamma = \{\Box p, \Box q, \Box r, \neg p \vee \neg q\}$ under the logic $\mathbf{AR^r_1}$. Note that $\Gamma$ has one minimal Dab$_1$-consequence, i.e.

$$\rho(p) \check{\vee} \rho(q)$$

As a result, $p$ and $q$ are lost in the revision operation defined by $\mathbf{AR^r_1}$. However, the belief $r$ can be upheld (note that this belief is not relevant to the revision of $\Upsilon$ by $A$). So we have:

$$r \in \Upsilon \oplus_{\mathbf{AR^r_1}} \{A\}$$

Since $\mathbf{AR^m_1}$ is at least as strong as $\mathbf{AR^r_1}$ (see Theorem 9.8), also $r \in \Upsilon \oplus_{\mathbf{AR^m_1}} \{A\}$. Hence, where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$:

$$\neg p \in Cn_{\mathbf{CL}}((\Upsilon \oplus_{\mathbf{AR^x_1}} \{A\}) \cup \{\neg p \vee \neg q\}) = Cn_{\mathbf{CL}}((\Upsilon \oplus_{\mathbf{AR^x_1}} \{A\}) \cup \{B\})$$

Consider now $\Upsilon \oplus_{\mathbf{AR^m_1}} \{A \wedge B\}$. Let $\Gamma' = \Upsilon^{\Box} \cup \{A \wedge B\}$. Note that the revision set equals $Cn_{\mathbf{AR^m_1}}(\Upsilon^{\Box} \cup \{A \wedge B\}) \cap \mathcal{W}_c$. In view of the additional information $B$, we are now dealing with two minimal Dab-consequences:

$$\rho(p) \check{\vee} \rho(q)$$

and

$$\rho(p) \check{\vee} \rho(r)$$

As a result, $\Gamma'$ has two kinds of $\mathbf{AR^m_1}$-models: those $M \in \mathcal{M}_{\mathbf{Kt}}(\Gamma')$ for which $Ab^1(M) = \{\rho(p)\}$, and those $M \in \mathcal{M}_{\mathbf{Kt}}(\Gamma')$ for which $Ab^1(M) = \{\rho(q), \rho(r)\}$. Models of the second kind verify $\Box p$ and falsify $\rho(p)$, whence they verify $p$. By the soundness of $\mathbf{AR^m_1}$, $\neg p \notin Cn_{\mathbf{AR^r_1}}(\Gamma')$. Hence by Definition 9.16:

$$\neg p \notin \Upsilon \oplus_{\mathbf{AR^m_1}} \{A \wedge B\}$$

Since $\mathbf{AR^m_1}$ is at least as strong as $\mathbf{AR^r_1}$, also $\neg p \notin \Upsilon \oplus_{\mathbf{AR^r_1}} \{A \wedge B\}$.

# Appendix F

# Subclassical Relevance

*This chapter is based on the paper "Subclassical Relevance: Broadening the Scope of Parikh's Concept" (under review, submitted August 2011). I am greatly indebted to Peter Verdée for his comments on an earlier draft of that paper, and to David Makinson for his helpful suggestions concerning some specific formulations and proofs.*

## F.1 Aim and Content of This Chapter

In Chapter 9, several adaptive logics for belief revision were presented, each of which satisfy Parikh's relevance postulate. In the current chapter, I will address the question whether, and if so, how the relevance axiom may be applied to inconsistent beliefs. The main results of this chapter are:

(i) as it is formulated in Chapter 9, Parikh's relevance axiom trivializes inconsistent belief bases;

(ii) however, if we weaken our standard of deduction to a paraconsistent logic, we obtain a new, fairly strong yet non-trivializing axiom of relevance; and

(iii) more generally, we may replace the standard of deduction by various subclassical logics, and extend many important results from the literature to the resulting axioms of relevance.

Besides these basic results, the chapter briefly discusses the relation between (paraconsistent) relevance and the notion of local change from [54, 75], and provides a generic way to prove Makinson's least letter-set theorem for a large class of logics.

**Outline of this Chapter.** Since relevance is a function of the classical logic consequences of the set of initial beliefs, the relevance axiom trivializes (mutually) inconsistent beliefs. This problem will be explained in Section F.2.1. The solution, as outlined in Section F.2.2, is to replace classical logic as the standard of deduction by a paraconsistent logic.

I will argue that this solves the problem of inconsistent relevance, using the well-known paraconsistent logic **CLuNs** [28, 27, 25, 161] to illustrate this point (Sections F.3.1 and F.3.2). In Section F.3.3, the resulting axiom of relevance is compared to Hansson & Wasserman's notion of Local Change [75] and to Fuhrmann's requirement that not every inconsistency is removed whenever one performs a revision or contraction [54, 127].

The remainder of the chapter is more of a technical nature. In Section F.4, nine other subclassical logics are defined, including intuitionistic logic. In the subsequent section, it is shown that many results from [87], [150] and [93] can easily be generalized to each of these systems and **CLuNs**. First and foremost, where **L** is one of these logics, every set of formulas has a finest **L**-splitting. Also, with the aid of a specific set $Min_{\mathbf{L}}(\Gamma)$, one can generalize the least letter-set theorem from [93] to all logics **L** under consideration (Section F.5.3). As a result, it is possible to define an axiom of **L**-relevance for each of these logics **L** (Section F.5.1). In Section F.5.4, the notion of a canonical form is generalized to that of an **L**-canonical form, and it is proven that a specific partial meet contraction of any **L**-canonical form of a set of formulas obeys the axiom of **L**-relevance. Finally, it is shown that $Min_{\mathbf{L}}(\Gamma)$ is an **L**-canonical form of $\Gamma$ (Section F.5.2).

**Preliminaries.** All results from this chapter are situated at the propositional level. Since I will need a negation that is often weaker than that of classical logic, I will use the language schema $\mathcal{L}_\sim$ of the logic **CLuN**, which was defined on page 25. Let $\mathcal{W}^l_\sim = \mathcal{S} \cup \{\sim A \mid A \in \mathcal{S}\}$.

For all logics **L** defined in this chapter, it is stipulated that $\vdash_{\mathbf{L}} \bot \supset A$ for any $A \in \mathcal{W}_\sim$. Also, $\Gamma \vdash_{\mathbf{L}} A$ iff there are $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_{\mathbf{L}} (B_1 \wedge \ldots \wedge B_n) \supset A$.[1]

Let $\mathcal{W}^{\not\bot}_\sim$ refer to the set of all formulas that do not contain $\bot$. A logic **L** will be called *paraconsistent* iff there are $A, B \in \mathcal{W}_\sim$ such that $A \wedge \sim A \nvdash_{\mathbf{L}} B$. **L** is *fully paraconsistent* iff there are no $A \in \mathcal{W}^{\not\bot}_\sim$ such that $A \wedge \sim A \vdash_{\mathbf{L}} B$ for every $B \in \mathcal{W}_\sim$. A monotonic logic **L** is *maximally paraconsistent* iff adding a **CL**-axiom $A$ for which $\nvdash_{\mathbf{L}} A$ to **L** yields full **CL**.

As in the previous chapter, a distinct set of metavariables $\Upsilon, \Upsilon_1, \ldots, \Upsilon', \ldots$ is used to refer to sets of beliefs. In this notation, $\Upsilon$ may be closed under a logic **L** or not. Where $\Upsilon$ is closed under a logic **L**, we say that it is an **L**-*theory*; otherwise, $\Upsilon$ is called a *base*.

# F.2 Relevance and Inconsistency

## F.2.1 The Problem with Inconsistent Beliefs

As pointed out by several authors, inconsistent belief bases are a fact of life – see e.g. [75, 54, 127, 46]. Especially when building large databases, it is very hard to avoid that inconsistencies creep in. Likewise, it is commonly acknowledged that

[1] For all logics defined in this chapter, proof theories are either available or can easily be obtained – to define them here would distract us from the main purpose of this chapter.

even our most reliable scientific theories can turn out to be inconsistent. It is therefore reasonable to try to adapt achievements in the field of belief revision to a paraconsistent setting, i.e. a setting in which the presence of inconsistencies is taken seriously, and does not lead to absurd outcomes. Examples of this paraconsistent turn can be found in [54, 127, 46], where results from the AGM approach are generalized to approaches based on paraconsistent, relevantist and inconsistency-adaptive logics respectively.

In a similar vein, one may ask whether the idea of relevant belief change can be reasonably applied to inconsistent belief bases or belief sets. However, if we take the relevance axiom literally, the outcome seems fairly negative. Let me explain why this applies to both Parikh's original axiom **P**, and my more general formulation $\mathbf{P}_g$.

Consider an $\Upsilon$ such that $\Upsilon = Cn_{\mathbf{CL}}(\Upsilon)$ and $\Upsilon \vdash_{\mathbf{CL}} \bot$. Hence also $A \in \Upsilon$ for every $A \in \mathcal{W}^l_\sim$, whence $\mathcal{W}^l_\sim \dashv\vdash_{\mathbf{CL}} \Upsilon$. It follows that the finest splitting of $\Upsilon$ is $\mathbf{E} = \{\{A\} \mid A \in \mathcal{E}\}$.

In this case, relevance to $\Upsilon \oplus \Psi$ reduces to mere letter-sharing with the least letter-set $E^*(\Psi)$ of $\Psi$: $A \in \Upsilon$ is relevant to $\Upsilon \oplus \Psi$ iff $E(A) \cap E^*(\Psi) \neq \emptyset$. As a result, a revision operation $\Upsilon \oplus \Psi$ that obeys **P** would result in (a superset of) the set $\{A \in \mathcal{W}^l_\sim \mid E(A) \cap E^*(\Psi) = \emptyset\}$. Hence, such a revision operation would result in something close to plain triviality.[2]

So how about the more general axiom $\mathbf{P}_g$, applicable to both theories and bases? Suppose again that $\Upsilon \vdash_{\mathbf{CL}} \bot$ — this time, we need not assume that $\Upsilon$ is a **CL**-theory. By the same reasoning as in the previous paragraph, $\mathcal{W}^l_\sim \dashv\vdash_{\mathbf{CL}} \Upsilon$ and the finest splitting of $\Upsilon$ is $\mathbf{E} = \{\{A\} \mid A \in \mathcal{E}\}$. If $\mathbf{P}_g$ is obeyed, this means that for every $A \in \mathcal{W}^l_\sim$ such that $E(A) \cap E^*(\Psi) = \emptyset$, $A$ has to be in the **CL**-consequence set of $\Upsilon \oplus \Psi$. Hence it is required that $\Upsilon \oplus \Psi$ is inconsistent; but more importantly, it suffices to take *any* inconsistent set $\Upsilon'$, in order to obey the axiom $\mathbf{P}_g$. Arguably, this requirement is far too liberal to receive the status of a rationality postulate.

This does not mean that the intuition behind Parikh's relevance axiom is not applicable to inconsistent beliefs. Consider the belief base $\Upsilon_1 = \{p \wedge q, r, \sim r\}$, and suppose we have to contract this set by $q \vee r$. Even though one has to remove both $q$ and $r$, one can readily argue that $p$ is not relevant to this particular contraction. This has little to do with the fact that $\Upsilon_1$ is inconsistent.

How can we get more formal grip on this? Note that even a very weak paraconsistent logic will usually validate Simplification (from $A \wedge B$, infer $A$ and $B$).[3] Hence in the above example, every such logic will allow us to derive $p$ from $p \wedge q$, and hence consider $p$ as separable from $q$ in $\Upsilon_1$. More generally, a logic can be (fully) paraconsistent, yet still allow us to analyse our set of initial beliefs to some extent, and hence obey a certain degree of relevance. So it seems at least plausible that we can obtain a strong, but also non-trivializing relevance axiom, if we weaken our standard of deduction in such a way that inconsistencies do not cause us to believe anything.

[2]By "plain triviality" I mean that every $A \in \mathcal{W}_\sim$ is an element of $\Upsilon \oplus \Psi$, resp. $\Upsilon \ominus \Psi$. The triviality that **P** yields in the face of an inconsistent belief base is slightly weaker: for every $A \in \mathcal{W}_\sim$, if $E(A) \cap E^*(\Psi) = \emptyset$, then $A \in \Upsilon \oplus \Psi$, resp. $A \in \Upsilon \ominus \Psi$.

[3]For some examples of logics that do *not* validate Simplification, see [25, Chapter 8].

### F.2.2 Outline of a Solution

Before we move over to a concrete solution for the problem of relevance in an inconsistent setting, let me briefly spell out the basic ingredients we need, using **L** as a metavariable for any subclassical logic. We first generalize the definition of a splitting, obtaining the concept of an **L**-splitting:

**Definition F.1 (L-splitting)** *Let $\mathbb{E} = \{\Lambda_i\}_{i \in I}$ be a partition of $\mathcal{E}$. We say that $\mathbb{E}$ is a* **L**-splitting *of $\Gamma$ iff there is a $\Delta = \bigcup_{i \in I} \Delta_i$ such that each $E(\Delta_i) \subseteq \Lambda_i$ and $\Delta \dashv\vdash_{\mathbf{L}} \Gamma$.*

Also, to warrant that relevance is also independent of the syntax of the new information (see also Chapter 9), we need to generalize the definition of the least letter-set to **L**:

**Definition F.2** *$A^*$ is a least letter-set representation of $A$ in* **L** *iff (i) $A^* \dashv\vdash_{\mathbf{L}} A$ and (ii) for every $B$ such that $B \dashv\vdash_{\mathbf{L}} A$, $E(A^*) \subseteq E(B)$. Where $A^*$ is an arbitrary least letter-set representation of $A$ in* **L***, let $E^*_{\mathbf{L}}(A) =_{\mathsf{df}} E(A^*)$.*

*Likewise, $\Psi^*$ is a least letter-set representation of $\Psi$ in* **L** *iff (i) $\Psi^* \dashv\vdash_{\mathbf{L}} \Psi$ and (ii) for every $\Delta$ such that $\Delta \dashv\vdash_{\mathbf{L}} \Psi$, $E(\Psi^*) \subseteq E(\Delta)$. Where $\Psi^*$ is an arbitrary least letter-set representation of $\Psi$ in* **L***, let $E^*_{\mathbf{L}}(\Psi) =_{\mathsf{df}} E(\Psi^*)$.*

Suppose that for a specific logic $\mathbf{L} : \wp(\mathcal{W}_\sim) \to \wp(\mathcal{W}_\sim)$, we can prove that (i) every $\Gamma \subseteq \mathcal{W}_\sim$ has a finest **L**-splitting, and (ii) every $\Delta \subseteq \mathcal{W}_\sim$ has a least letter-set representation in **L**. Then we can use the notion of a finest splitting to define **L**-relevance, just as in the case where classical logic was the underlying logic:

**Definition F.3 (L-relevance)** *Let $\mathbb{E}$ be the finest* **L***-splitting of $\Upsilon$. We say that a formula $B$ is* **L***-irrelevant to the revision of $\Upsilon$ by $\Psi$ iff for every cell $\Lambda_i \in \mathbb{E}$: $\Lambda_i \cap E(B) = \emptyset$ or $\Lambda_i \cap E^*_{\mathbf{L}}(\Psi) = \emptyset$.*

Finally, this allows us to define an axiom of **L**-relevance:

**$\mathbf{P_L}$** If $B \in Cn_{\mathbf{L}}(\Upsilon)$ is **L**-irrelevant to the revision (contraction) of $\Upsilon$ by $\Psi$, then $B$ is an element of $Cn_{\mathbf{L}}(\Upsilon \oplus \Psi)$ ($Cn_{\mathbf{L}}(\Upsilon \ominus \Psi)$).

By these steps, we obtain a notion of subclassical relevance. As is clear from the above definitions, the crucial properties we need to arrive at this result are (i) and (ii). Of course, to allow for a sensible notion of relevance in the context of inconsistent beliefs, we have to use a *paraconsistent* logic **L**. As announced, I will first focus on one particular such logic. In the subsequent sections, I will move to a more general level, and show that the above strategy can be applied to a number of other subclassical systems, including intuitionistic logic.

## F.3 CLuNs-relevance and Local Consistency

### F.3.1 The Paraconsistent Logic CLuNs

To explain the idea behind a subclassical relevance axiom, I will use the paraconsistent logic **CLuNs**, as axiomatized in [28]. This choice is motivated by two properties of the logic: (i) **CLuNs** is maximally paraconsistent, which means that its analytic power is very close to that of **CL**; (ii) nevertheless, **CLuNs** is also fully paraconsistent, whence **CLuNs**-relevance will not trivialize any belief base $\Gamma \subseteq \mathcal{W}_{\sim}$. Each of these advantages will be illustrated below.

The propositional fragment of **CLuNs** is one of the three systems devised by Schütte in [132], the two others are **CLaNs** and **CLoNs** and will be presented in Section F.4. All three of these systems are particularly strong in that they allow us to drive the paraconsistent negation inwards; e.g. it is possible to derive $\sim A, \sim B$ from $\sim(A \vee B)$, and similariy to derive $A \wedge \sim B$ from $\sim(A \supset B)$. A distinctive feature of **CLuNs** is that it is paraconsistent but not paracomplete (unlike the other Schütte systems): it can model cases where both $A$ and $\sim A$ are true, but it cannot model cases in which both are false.

**CLuNs** is axiomatized by the rule **MP** (to infer $B$ from $A, A \supset B$), the positive fragment of **CL** (see Appendix B), the rule of excluded middle:

**EM** $\quad A \vee \sim A$

and the following axioms that drive negation inwards:

| | |
|---|---|
| **A**$\sim\sim$ | $\sim\sim A \equiv A$ |
| **A**$\sim\supset$ | $\sim(A \supset B) \equiv (A \wedge \sim B)$ |
| **A**$\sim\wedge$ | $\sim(A \wedge B) \equiv (\sim A \vee \sim B)$ |
| **A**$\sim\vee$ | $\sim(A \vee B) \equiv (\sim A \wedge \sim B)$ |
| **A**$\sim\equiv$ | $\sim(A \equiv B) \equiv \sim(A \supset B) \vee \sim(B \supset A))$ |

For reasons of space, I will not discuss the various semantic characterizations of **CLuNs** – see e.g. [28, 27, 25, 161]. Note that since $\supset$ and $\perp$ behave classically in **CLuNs**, it is possible to define a classical negation $\neg$ in this system by $\neg A =_{\mathsf{def}} (A \supset \perp)$.

To see how **CLuNs** behaves, consider $\Upsilon_2 = \{\sim(p \supset (q \vee r)), (\sim s \vee (t \wedge \sim\sim u)) \wedge p, \sim(\sim q \wedge p), v, \sim v \wedge \sim q\}$. Each of the following holds:

(1) $\Upsilon_2 \vdash_{\mathbf{CLuNs}} p \wedge \sim(q \vee r)$ (by **A**$\sim\supset$)
(2) $\Upsilon_2 \vdash_{\mathbf{CLuNs}} p, \sim q, \sim r$ (by (1) and **A**$\wedge$**1**, **A**$\wedge$**2**)
(3) $\Upsilon_2 \vdash_{\mathbf{CLuNs}} \sim s \vee (t \wedge u)$ (by **A**$\wedge$**1** and **A**$\sim\sim$)
(4) $\Upsilon_2 \vdash_{\mathbf{CLuNs}} \sim s \vee t, \sim s \vee u$ (by (3) and **A**$\wedge$**1**, **A**$\wedge$**2**)
(5) $\Upsilon_2 \vdash_{\mathbf{CLuNs}} \sim\sim q \vee \sim p$ (by **A**$\sim\wedge$)
(6) $\Upsilon_2 \vdash_{\mathbf{CLuNs}} q \vee \sim p$ (by (5) and $A\sim\sim$)
(7) $\Upsilon_2 \vdash_{\mathbf{CLuNs}} (p \wedge \sim p) \vee (q \wedge \sim q)$ (by (2), (6))

$\Upsilon_2$ is clearly inconsistent. Since **CLuNs** invalidates disjunctive syllogisme, it is not possible to **CLuNs**-derive e.g. $\sim p$ from $\sim q$ and $q \vee \sim p$. Hence $\Upsilon_2 \nvdash_{\mathbf{CLuNs}} \sim p$.

## F.3.2 CLuNs-relevance

I will now illustrate the three steps we need to obtain a subclassical relevance axiom, using **CLuNs** as an example: the definition of a finest **CLuNs**-splitting, the definition of **CLuNs**-relevance, and finally the axiom of **CLuNs**-relevance itself. The set $\Upsilon_2$ from Section F.3.1 will be used to illustrate each of these notions.

Consider $\Upsilon_2' = \{p, \sim q, \sim r, \sim s \vee t, \sim s \vee u, \sim p \vee q, v, \sim v\}$. In view of (1)-(7), it follows immediately that $\Upsilon_2 \vdash_{\mathbf{CLuNs}} \Upsilon_2'$. It can easily be verified that also $\Upsilon_2' \vdash_{\mathbf{CLuNs}} \Upsilon_2$. As a result, $\Upsilon_2$ and $\Upsilon_2'$ are **CLuNs**-equivalent.

Note that $\Upsilon_2'$ can be partitioned into three subsets: $\Delta_1 = \{p, \sim q, \sim p \vee q\}$, $\Delta_2 = \{\sim r\}$, $\Delta_3 = \{\sim s \vee t, \sim s \vee u\}$ and $\Delta_4 = \{v, \sim v\}$. Note also that the letter sets of the sets $\Delta_i$ are pairwise disjoint. Hence we can obtain a **CLuNs**-splitting of $\Upsilon_2$, i.e. $\mathbb{E}(\Upsilon_2) = \{\{p, q\}, \{r\}, \{s, t, u\}, \{v\}\}$.

In Section F.5.1, it is proven that every set $\Upsilon$ has a finest **CLuNs**-splitting, and in Section F.5.2 I show that one can obtain this splitting by the construction of a specific set $Min_{\mathbf{CluNs}}(\Upsilon) \subset Cn_{\mathbf{CLuNs}}(\Upsilon)$. In the current case, these results warrant that $\mathbb{E}(\Upsilon_2)$ is in fact the *finest* **CLuNs**-splitting of $\Upsilon_2$.

Suppose that we contract $\Upsilon_2$ by $(p \vee \sim s) \wedge (r \vee \sim r)$. Before we can determine which formulas are relevant to this contraction, we first have to find a least letter-set representation of the new information. In the current case, this is simple: we take $p \vee \sim s$. In Section F.5.3, it is shown that every formula and every set of formulas has a least-letter set representation with respect to **CLuNs**.

Note that $\{p, q\}$ and $\{s, t, u\}$ are the only sets $\Lambda$ in $\mathbb{E}(\Upsilon_2)$ for which $\Lambda \cap E(p \vee \sim s) \neq \emptyset$. The axiom of **CLuNs**-relevance tells us the following: a formula $A \in Cn_{\mathbf{CLuNs}}(\Upsilon_2)$ is relevant to the contraction of $\Upsilon_2$ by $p \vee \sim s$ iff $E(A) \cap \{p, q\} \neq \emptyset$ or $E(A) \cap \{s, t, u\} \neq \emptyset$. Hence the following **CLuNs**-consequences of $\Upsilon_2$ are *not* relevant to the contraction of $\Upsilon_2$ by $p$: $\sim r, v, \sim v$.

This immediately brings us to the axiom of relevance. In the current case, this axiom stipulates that the beliefs $\sim r, v, \sim v$ should be upheld. Note that this means that a contradiction has to be upheld, in order to obey $\mathbf{P_{CLuNs}}$ – I will return to this fact in Section F.3.3. However, the axiom does not require us to believe just anything: e.g. if we remove $p$ from $\Upsilon_2'$, we obtain a non-trivial yet fairly rich revision that does not **CLuNs**-entail $p \vee r$.

So, on the one hand, we are able to separate e.g. $\sim r$ from $p$, notwithstanding the fact that in the initial formulation of $\Upsilon_2$, these formulas are tied to each other. On the other hand, some beliefs are still considered relevant to the new information, and removing some of these results in a reasonable contraction set. In short, we obtain a non-trivial, yet also non-trivializing relevance axiom for inconsistent belief bases and sets.

### F.3.3 Local Change and CLuNs-relevance

Consider again the contraction operation of $\Upsilon_2$ by $(p \vee \sim s) \wedge (r \vee \sim r)$, as described in the preceding section. One could ask oneself: should the beliefs $v$ and $\sim v$ be upheld? If so, the resulting contraction set will remain inconsistent. But is this a sufficient reason to remove (either of) these beliefs from $\Upsilon_2$? Clearly, they have little to do with the formula by which we are contracting, no matter whether we consider $\Upsilon_2$ and the new information as they were initially formulated, or more analyzed versions of them, such as $\Upsilon_2'$ and $p \vee \sim s$.

According to the standard AGM approach, inconsistencies cannot occur in any contraction or revision set. This also applies to the more recent approaches in terms of belief bases: in both cases, it is required that $\Upsilon \ominus \Psi$ and $\Upsilon \oplus \Psi$ are consistent. Hence any inconsistency is removed from $\Upsilon$ whenever this set is contracted or revised.

At least some authors seem to suggest that it should be possible to leave certain inconsistencies in $\Upsilon$ undisturbed – to uphold the belief in these inconsistencies, while focusing on other problematic parts of $\Upsilon$. Such an approach may perhaps best be understood as a kind of "local consolidation" or "local revision", as described by Hansson and Wasserman in their [75, p. 51]:

> *Local Consolidation.* Inconsistencies are removed from some part of the belief base. The rest of the agent's beliefs may well be inconsistent. For instance, I can make my beliefs about biological evolution consistent, while retaining global inconsistency between biological and religious beliefs.
>
> *Local Revision.* A new belief is added to the belief base in such a way that a certain part of the resulting base is made (kept) consistent. If I see, for example, that it is a sunny day in Amsterdam, then this contradicts my belief that it is always raining in Holland, and leads to revision. This can be done without checking whether my beliefs about Brazilian politics are consistent with the new belief.

Similar ideas can be found in [127, p. 10], where it is argued that a relevantist approach to belief contraction allows us to model processes in which inconsistencies are removed one by one, such that the intermediary belief states remain inconsistent. The authors quote Fuhrmann, who writes the following in a section of his [54] titled "Local Inconsistency":

> [...] Thus, in the face of inconsistent theories we should want two things:
>
> (a) *localise inconsistencies* – an inconsistent theory should not be rendered totally corrupt just because some inconsistency has crept into the theory; and
>
> (b) *locally restore consistency* – we should be able to resolve one inconsistency at a time by contracting an inconsistent theory such that other inconsistencies, which we cannot yet resolve, may be carried over into the contraction theory.

In order to obtain (a) and (b), Fuhrmann recommends that "theories be generated from bases by means of a consequence operation induced by some parconsistent logic."[54, p. 187]

Recall that in our example, the axiom stipulated that the inconsistency $v, {\sim}v$ is upheld, since it is not relevant to the formula by which we contract. More generally, the axiom of **CLuNs**-relevance does not distinguish between formulas that behave consistently and those that behave inconsistently; all that matters is whether formulas are relevant to the revision or contraction. If an inconsistency is *not* relevant to this operation, then it is upheld.

In Furhmanns terms, the axiom of **CLuNs**-relevance requires that we should only locally restore consistency. Moreover, it does so in very clear and precise logical terms, and in a way that is perhaps much stronger than what Fuhrmann and the later advocates of local consolidation had in mind – recall the remark in Section 9.2 that relevance is invariant under different equivalent formulations of $\Upsilon$.

## F.4 Some Subclassical Logics

In this section, I define 9 subclassical logics, and list some generic properties of these systems. These properties make it possible to apply the strategy spelled out in Section F.2.2 to these logics. As a result, they provide a sufficient condition for the properties proven in Section F.5.

### F.4.1 The Logics

**The Schütte Logics.** Let us start with the two systems **CLoNs** and **CLaNs**, which were already mentioned in Section F.3.1. **CLoNs** is obtained by removing **EM** from **CLuNs**. As a result, the standard negation displays both gluts and gaps: $A$ and $\sim A$ can both be true, but they can also both be false. **CLaNs** is the counterpart of **CLuNs**, in that its negation displays gaps but not gluts. **CLaNs** and can be obtained by adding the rule of ex falso quodlibet to **CLoNs**:

**EFQ** $A \supset (\sim A \supset B)$

Note that, since **EM** is not valid in **CLaNs**, this logic invalidates the rule $(A \supset B) \wedge (\sim A \supset B) \vdash B$, which makes it somewhat similar to intuitionistic logic. However, Peirce's axiom **A⊃2** remains valid in **CLaNs**, unlike intuitionistic logic (see below).

In view of their axiomatization, it follows immediately that **CLuNs** is stronger than **CLoNs**, that **CLaNs** is stronger than **CLoNs** and that **CLuNs** and **CLaNs** are incommensurable.

**The Basic Paralogics.** The three basic paralogics **CLoN**, **CLuN** and **CLaN** are the weaker nephews of **CLoNs**, **CLuNs** and **CLaNs** respectively, in that they invalidate the axioms that drive negation inwards. Hence e.g. **CLaN** is obtained by closing the set of **CLaNs**-axioms, minus **A**$\sim\sim$, **A**$\sim\supset$, **A**$\sim\wedge$, **A**$\sim\vee$ and **A**$\sim\equiv$ under **MP**.

Alternatively, we may say that **CLuN**,**CLaN** and **CLoN** boil down to the full positive part of **CL** (see Appendix B), with gl**u**ts, g**a**ps, and respectively b**o**th gluts and gaps for the **N**egation. As before, we can easily infer that **CLuN** and **CLaN** are incommensurable, but that both are stronger than **CLoN**. Also, each of the basic logics is weaker than its respective Schütte-variant.

The advantage of the basic logics is that they maximally localize inconsistencies, or in the case of **CLaN**, $\sim$-incompleteness. For example, if $A$ is not itself contradictory and $A \neq B$, then $A \wedge \sim A \nvdash_{\mathbf{CLuN}} B \wedge \sim B$. This makes **CLuN** a very good candidate to serve as the lower limit logic of inconsistency-adaptive logics – see also Chapter 2.

**The Vasil'ev Logics.** A third class of logics are the Vasil'ev systems **CLoNv**, **CLuNv** and **CLaNv**. These are peculiarly strong, in that in them, the negation $\sim$ is assumed to behave classically in front of all complex formulas. Hence e.g. $\sim(A \wedge B)$ is equivalent to $(A \wedge B) \supset \perp$.

The Vasil'ev systems are obtained by adding the following axiom schema to **CLoN**, **CLuN** and **CLaN** respectively:

**A$\sim$V** Where $A \in \mathcal{W}_\sim - \mathcal{S}$: $\sim A \supset (A \supset \perp)$

Note that in view of this axiom schema, none of the Vasil'ev systems are fully paraconsistent. For example, $(p \vee q) \wedge \sim(p \vee q) \vdash_{\mathbf{CLuNv}} A$ for any $A \in \mathcal{W}_\sim$. As a result, these systems are only useful in a context where inconsistency or $\sim$-incompleteness is restricted to the level of propositional letters, as e.g. in $\Gamma = \{p, q \wedge \sim p, r \vee s, \sim r, \sim s\}$.

**Intuitionistic Logic.** The last system we will consider is of a rather different nature: intuitionistic logic. I will assume familiarity with this system and its properties – I merely present an axiomatization for the sake of self-containedness. The system **I** can be obtained as follows: (i) take the positive fragment of **CL**; (ii) remove axiom **A$\supset$2**, and (iii) add the following two axioms that characterize $\sim$ in **I**:

**A$\sim$I1** $(A \supset B) \supset ((A \supset \sim B) \supset \sim A)$
**A$\sim$I2** $\sim A \supset (A \supset B)$

## F.4.2 The Properties

Let in the remainder **L** be a metavariable for all logics axiomatized in the preceding sections plus **CL**. In view of my definition of $\Gamma \vdash_{\mathbf{L}} A$, the following is immediate:

**Theorem F.1** *$\Gamma \vdash_{\mathbf{L}} A$ iff there are $B_1, \ldots, B_n \in \Gamma$ such that $\{B_1, \ldots, B_n\} \vdash_{\mathbf{L}} A$. (Compactness)*

The next property on the list is the Deduction Theorem. For **I**, a proof can be found in [49, Chapter 4]. For the 9 paralogics from Section F.4.1, it follows immediately from the fact that $\supset$ behaves classically in each of these systems. For the current purposes, it is convenient to rewrite this property as follows:

**Theorem F.2** *$A \wedge B \vdash_{\mathbf{L}} C$ iff $A \vdash_{\mathbf{L}} B \supset C$ (Deduction Theorem)*

Note that this theorem follows from the regular Deduction Theorem for **L**, whenever $\wedge$ behaves classically in **L**, i.e. whenever $A \wedge B \dashv\vdash_{\mathbf{L}} \{A, B\}$.

In view of the Deduction Theorem and the definition of $\Gamma \vdash_{\mathbf{L}} A$, each of the logics **L** is reflexive, transitive and monotonic:

**Theorem F.3** *Each of the following holds:*

1. *$\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$ (Reflexivity)*
2. *if $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$, then $Cn_{\mathbf{L}}(\Gamma') \subseteq Cn_{\mathbf{L}}(\Gamma)$ (Transitivity)*

*3.* $Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$ *(Monotonicity)*

In view of the above properties, it can easily be shown that $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma')$ and $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$ iff $\Gamma \dashv\vdash_{\mathbf{L}} \Gamma'$. The last property I will need is (Standard) Interpolation:

**Theorem F.4** *If $\Gamma \vdash_{\mathbf{L}} A$, then there is a $B$ such that $\Gamma \vdash_{\mathbf{L}} B$, $B \vdash_{\mathbf{L}} A$, and $E(B) \subseteq E(\Gamma) \cap E(A)$. (Standard Interpolation)*

For all considered logics except **I** and **CL**, Standard Interpolation was proven in [28]. For **I**, I refer to [49, Chapter 4].

# F.5 Generic Results for L-relevance

In this section, I establish a number of theoretic results concerning **L**-splittings and **L**-relevance – recall that **L** is used as a metavariable for any of the logics axiomatized in the preceding sections, and **CL**. The proofs only rely on the properties mentioned in Section F.4.2, whence the current results may easily be generalized to a yet broader class of logics. Eventually, this yields a partial answer to the question posed in the concluding section of [87]:

> [...] how far can the results [of our paper] be established for sub-classical, (e.g. intuitionistic) consequence relations or supraclassical ones (e.g., preferential consequence relations or the relation of logical friendliness of Makinson [8])?

## F.5.1 Finest L-splitting

A crucial theorem for Kourousias and Makinson's finest splitting result in [87] is that of Parallel Interpolation for **CL**, which is a strengthening of Standard Interpolation. The proof for Theorem F.5 is readily obtained through a variation on the proof for Theorem 1.1 in [87].

**Theorem F.5** *Let $\Delta = \bigcup_{i\in I}\{\Delta_i\}$ where the letter sets $E(\Delta_i)$ are pairwise disjoint, and suppose $\Delta \vdash_{\mathbf{L}} A$. Then there are formulas $B_i$ such that (1) each $E(B_i) \subseteq E(\Delta_i) \cap E(A)$, (2) each $\Delta_i \vdash_{\mathbf{L}} B_i$, and (3) $\bigcup_{i\in I}\{B_i\} \vdash_{\mathbf{L}} A$. (Parallel Interpolation)*

*Proof.* Suppose the antecedent holds. By the compactness of **L**, there is a finite subfamily of finite subsets of the $\Delta_i$, the conjunction of whose elements implies $A$. Let these subsets be $\Delta'_{j_1}, \ldots, \Delta'_{j_n}$, and let for every $k \leq n$, $B_k$ be the conjunction of the members of $\Delta'_{j_k}$. It follows that $B_1 \wedge \ldots \wedge B_n \vdash_{\mathbf{L}} A$. By the Deduction Theorem, $B_1 \vdash_{\mathbf{L}} (B_2 \wedge \ldots \wedge B_n) \supset A$, which implies, by Standard Interpolation, that there is a formula $C_1$ such that (1) $B_1 \vdash_{\mathbf{L}} C_1$ and $C_1 \vdash_{\mathbf{L}} (B_2 \wedge \ldots \wedge B_n) \supset A$ and (2) $E(C_1) \subseteq E(B_1) \cap E((B_2 \wedge \ldots \wedge B_n) \supset A)$. Since the sets $E(B_i)$ are pairwise disjoint, (2) implies that $E(C_1) \subseteq E(B_1) \cap E(A)$.

By (1) and the Deduction Theorem, $C_1 \wedge B_2 \wedge \ldots \wedge B_n \vdash_{\mathbf{L}} A$ and the sets $E(C_1), E(B_2), \ldots, E(B_n)$ are pairwise disjoint. Hence we may repeat the procedure for $B_2$, obtaining a suitable interpolant $C_2$, and so on. After $n$ applications of Standard Interpolation, we have obtained $C_1, \ldots, C_n$, where each $E(C_i) \subseteq E(B_i) \cap E(A) \subseteq E(\Delta_{j_i}) \cap E(A)$ and $C_1 \wedge \ldots \wedge C_n \vdash_{\mathbf{L}} A$. ■

By a similar variation on the proofs for Lemma 2.3 and Theorem 2.4 of [87], we may derive the following:

**Theorem F.6** *Every* $\Gamma \subseteq \mathcal{W}_{\sim}$ *has a finest* **L**-*splitting.*

I will not provide the proof for this Theorem here. Just as the proof for Theorem F.5, it is almost identical to the proof in [87]. It suffices to merely replace the $\vdash$ in Lemma 2.3 and Theorem 2.4 from that paper by $\vdash_{\mathbf{L}}$. More importantly, in Section F.5.2 from the current chapter, it is explained how an alternative proof for Theorem F.6 can be obtained, relying on the notion of **L**-minimal formulas.

Once the notion of a finest splitting is generalized to a class of logics **L**, a question that immediately springs to mind is when and how the finest **L**-splitting relates to the finest **L**′-splitting, for two logics **L** and **L**′. In fact, the stronger a logic, the finer the associated finest splitting of Γ:

**Theorem F.7** *If* **L** *is at least as strong as* **L**′*, then the finest* **L**′*-splitting of* Γ *is a* **L**-*splitting of* Γ*.*

*Proof.* Suppose the antecedent holds and $\mathbb{E} = \{E_i\}_{i \in I}$ is the finest **L**′-splitting of Γ. Hence there is a $\Delta = \bigcup_{i \in I} \Delta_i$ such that $\Delta \dashv\vdash_{\mathbf{L}'} \Gamma$ and each $E(\Delta_i) \subseteq E_i$. It follows from the supposition that $\Delta \dashv\vdash_{\mathbf{L}} \Gamma$. But then $\mathbb{E}$ is a **L**-splitting of Γ. ■

Note that this proof works for any logic **L** and **L**′, on the assumption that every Γ has a finest **L**′-splitting. Trivial as its proof is, this is a noteworthy result. Recall that in order to avoid that relevance results in triviality, it was necessary to weaken the standard of deduction, hence to define a notion of subclassical finest splittings and an associated relevance criterion. Theorem F.7 indicates that the stronger the subclassical logic of our choice, the better we may approximate the finest **CL**-splitting without ending up with triviality in the case of an inconsistency.

By Theorem F.7, we may infer that e.g. the finest **CLuNs**-splitting of Υ is always at least as fine as the finest **CLuN**-splitting of Υ, and likewise that the finest **CLaNv**-splitting of Υ is always at least as fine as the finest **CLaN**-splitting. Also, we may infer that each of the subclassical splittings is further refined by the finest **CL**-splitting of Υ.

To illustrate this point, we may consider the following splittings of $\Upsilon_2$ from page 317:

$$\mathbb{E}_1(\Upsilon_2) = \{\{p, q, r\}, \{s, t, u\}, \{v\}\}$$
$$\mathbb{E}_2(\Upsilon_2) = \{\{p, q\}, \{r\}, \{s, t, u\}, \{v\}\}$$
$$\mathbb{E}_3(\Upsilon_2) = \{\{p\}, \{q\}, \{r\}, \{s\}, \{t\}, \{u\}, \{v\}\}$$

$\mathbb{E}_1(\Upsilon_2)$ is the finest **CLuN**-splitting of $\Upsilon_2$, $\mathbb{E}_2(\Upsilon_2)$ the finest **CLuNs**-splitting of $\Upsilon_2$ and $\mathbb{E}_3(\Upsilon_2)$ the finest **CL**-splitting of $\Upsilon_2$.

From the preceding observations, we can also infer that whenever $B \in Cn_{\mathbf{CL}}(\Gamma)$ is **L**-irrelevant to the revision (contraction) of $\Upsilon$ by $A$, then it is also **CL**-irrelevant to the revision (contraction) of $\Upsilon$ by $A$. In other words, there are just as many means to show that a formula is **CL**-irrelevant to a particular revision or contraction operation, as there are subclassical logics for which Theorem F.6 holds.

### F.5.2 The Set of L-minimal Formulas

In the current section, I define a unique set $Min_{\mathbf{L}}(\Gamma)$ for every $\Gamma$.[4] This set is used in subsequent sections, where it is shown that (i) $Min_{\mathbf{L}}(\Gamma)$ is a least letter-set representation of $\Gamma$ in **L**, and (ii) it is also an **L**-canonical form of $\Gamma$ – see below.

**Definition F.4** *$A$ is a minimal* **L**-*consequence of* $\Gamma$, $A \in Min_{\mathbf{L}}(\Gamma)$ *iff* $A \in Cn_{\mathbf{L}}(\Gamma)$ *and there is no* $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$ *such that (i)* $\Gamma' \vdash_{\mathbf{L}} A$ *and for every* $B \in \Gamma'$, $E(B) \subset E(A)$.

Intuitively, the set $Min_{\mathbf{L}}(\Gamma)$ corresponds to the maximal level of analysis (in terms of the separation of letters) the logic **L** allows us to perform.

**Lemma F.1** $Min_{\mathbf{L}}(\Gamma) \dashv\vdash_{\mathbf{L}} \Gamma$

*Proof.* In view of Definition F.4, it suffices to prove the left-right direction. Suppose $A \in \Gamma$, whence by the reflexivity of **L**, $A \in Cn_{\mathbf{L}}(\Gamma)$. I prove by an induction that $A \in Cn_{\mathbf{L}}(Min_{\mathbf{L}}(\Gamma))$. If $A \in Min_{\mathbf{L}}(\Gamma)$, then by the reflexivity of **L**, $A \in Cn_{\mathbf{L}}(Min_{\mathbf{L}}(\Gamma))$. If $A \notin Min_{\mathbf{L}}(\Gamma)$, then since $A \in Cn_{\mathbf{L}}(\Gamma)$ and by Definition F.4, there is a $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$, such that (i) $\Gamma' \vdash_{\mathbf{L}} A$ and (ii) for every $B \in \Gamma'$, $E(B) \subset E(A)$. For every $B \in \Gamma'$ such that $B \notin Min_{\mathbf{L}}(\Gamma)$, we repeat the same reasoning: since $B \in Cn_{\mathbf{L}}(\Gamma)$, there is a $\Gamma'' \subseteq Cn_{\mathbf{L}}(\Gamma)$ such that (i) for every $C \in \Gamma''$, $E(C) \subset E(B) \subset E(A)$ and (ii) $\Gamma'' \vdash_{\mathbf{L}} B$, whence by the transitivity and monotonicity of **L**, $(\Gamma' - \{B\}) \cup \Gamma'' \vdash_{\mathbf{L}} A$. Since $A$ contains finitely many letters, we will at a finite point arrive at a set $\Delta \subseteq Min_{\mathbf{L}}(\Gamma)$ such that $\Delta \vdash_{\mathbf{L}} A$. By the monotonicity of **L**, $Min_{\mathbf{L}}(\Gamma) \vdash_{\mathbf{L}} A$. ■

### F.5.3 The Least Letter-set Theorem

As explained in Chapter 9, the least letter-set theorem tells us that for every (possibly infinite) $\Gamma$, there is a unique least set of letters $\Delta \subseteq \mathcal{S}$, such that $\Gamma$ can be **CL**-equivalently expressed using only letters from $\Delta$.[5] Makinson makes the following remark on this property ([94, p. 378]):

> Intuitively, the least letter-set theorem is just what anyone would expect, but it needs proof. Getting minimal letter sets is trivial since every formula contains only finitely many letters. But getting a least one (which, by the antisymmetry of set-inclusion, will be unique) requires a bit more work.

[4]The precise formulation in the definition of $Min_{\mathbf{L}}(\Gamma)$ greatly benefited from a suggestion made by David Makinson (personal correspondence).

[5]For the finite case, the proof of the theorem is almost trivial whenever Standard Interpolation is available, as explained in the appendix of [93].

I refer to the same paper for some more background on this theorem, and to [93, Appendix] for Makinson's (semantic) proof. Both papers are restricted to the case where $\mathbf{L} = \mathbf{CL}$. I will prove here that this theorem holds whenever $\mathbf{L}$ is reflexive, transitive, monotonic and obeys Standard Interpolation.[6] However, we must be careful: the exact formulation of the theorem in [93] is slightly different from the one in [94], because it is applied in a different context.[7] My formulation is a variation on the one in [94]. The proof I will present is very short, thanks to the introduction of the concept of **L**-minimality in the preceding section.

**Theorem F.8** *For every $\Gamma \subseteq \mathcal{W}_\sim$, there is a $\Delta \subseteq \mathcal{E}$ such that (a) for every $\Gamma'$ that is* **L***-equivalent to $\Gamma$: $\Delta \subseteq E(\Gamma')$ and (b) for a $\Gamma''$ that is* **L***-equivalent to $\Gamma$, $\Delta = E(\Gamma'')$. (Least letter-set Theorem)*

*Proof.* Let $\Delta = E(Min_{\mathbf{L}}(\Gamma))$. (b) follows immediately by the construction and Lemma F.1; hence it suffices to prove (a). Assume that (1) $\Gamma' \dashv\vdash_{\mathbf{L}} \Gamma$, but $\Delta \not\subseteq E(\Gamma')$. Hence there is a $A \in Min_{\mathbf{L}}(\Gamma)$: $E(A) \not\subseteq E(\Gamma')$, whence also (2) $E(A) \cap E(\Gamma') \subset E(A)$. By (1) and Definition F.4, $\Gamma' \vdash_{\mathbf{L}} A$. By Interpolation, there is a $B$ such that (3) $\Gamma' \vdash_{\mathbf{L}} B$, (4) $B \vdash_{\mathbf{L}} A$ and (5) $E(B) \subseteq E(\Gamma') \cap E(A)$. By (1) and (3), it follows that $B \in Cn_{\mathbf{L}}(\Gamma)$, and by (2) and (5), it follows that $E(B) \subset E(A)$. But then by (3) and in view of Definition F.4, $A \notin Min_{\mathbf{L}}(\Gamma)$ — a contradiction. ■

Theorem F.8 states that every set $\Gamma \subseteq \mathcal{W}_\sim$ has a least letter-set representation $\Gamma^*$ in $\mathbf{L}$. To see that also every $A \in \mathcal{W}_\sim$ has a least letter-set representation $A^*$ in $\mathbf{L}$, it suffices to replace $\Gamma$ by $\{A\}$, and to let $A^* = \bigwedge \Gamma^*$. So for instance $p \wedge (q \vee {\sim} r)$ is a least letter-set representation of $(p \supset (q \vee {\sim} r) \wedge p \wedge (r \vee {\sim} r)$ in **CLuN**.

### F.5.4 L-canonical Forms

Recall the definition of a *canonical form* of the belief base $\Upsilon$ from Chapter 9: where $\mathbb{E} = \{\Lambda_i\}_{i \in I}$ is the finest **CL**-splitting of $\Upsilon$, $\Upsilon' = \bigcup_{i \in I} \Upsilon_i$ is a canonical form of $\Upsilon$ iff (i) it is **CL**-equivalent to $\Upsilon$, and (ii) each $E(\Upsilon_i) \subseteq E_i$. As I explained there, there may in fact be several canonical forms $\Upsilon'$ for one and the same $\Upsilon$, whence it is better to speak of the *set* of canonical forms of $\Upsilon$ instead of "the" canonical form of $\Upsilon$. If we generalize this notion in order to include subclassical logics, we obtain the following definition:

**Definition F.5 (Set of L-canonical Forms)** *Where $\mathbb{E} = \{\Lambda_i\}_{i \in I}$ is the finest* **L***-splitting of $\Upsilon$: $\mathbb{C}_{\mathbf{L}}(\Upsilon) = \{\Delta = \bigcup_{i \in I} \{\Delta_i\} \mid \Delta \dashv\vdash_{\mathbf{L}} \Upsilon$ and for every $i \in I$ : $E(\Delta_i) \subseteq \Lambda_i\}$.*

[6] As Makinson pointed out to me (personal correspondence), a weaker kind of Interpolation suffices to obtain the least letter-set theorem along the lines of my proof: if $\Gamma \vdash_{\mathbf{L}} A$, then there is a $\Gamma'$ such that (i) $\Gamma \vdash_{\mathbf{L}} \Gamma'$, (ii) $\Gamma' \vdash_{\mathbf{L}} A$ and (iii) $E(\Gamma') \subseteq E(\Gamma) \cap E(A)$ *(Non-compact Interpolation)*.

[7] In [93], a specific set $\Gamma^*$ is defined for every $\Gamma$, and it is shown that this set is a least letter-set representation of $\Gamma$. $\Gamma^*$ is defined in semantic terms, and the proof proceeds likewise. On the other hand, the formulation of the least letter-set theorem in [94] is a "bare statement of existence" (Makinson, personal correspondence), without reference to any specific least letter-set representation.

Kourousias and Makinson show that if $\Upsilon$ is consistent, then every partial meet contraction of a canonical form of $\Upsilon$ by $A$ obeys the axiom of relevance. To generalize the result of Kourousias and Makinson to the subclassical logics from Section F.4, we first have to define subclassical contraction and revision operations on belief bases.[8] For the sake of simplicity, I only consider contractions and revisions in view of a single formula in the current section.[9] Given this assumption, the operations of **L**-partial meet contraction and revision are obtained by a generalization of partial meet contraction and revision for bases, as defined in [73]. It suffices to replace all references to **CL** in the definition of partial meet contraction and revision by **L**:

**Definition F.6 (Set of L-remainders)** $\Upsilon \curlywedge_{\mathbf{L}} A$ *is the set of all* $\Delta \subseteq \Upsilon$ *such that:*

*(i)* $\Delta \nvdash_{\mathbf{L}} A$, *and*
*(ii) for no* $\Delta' \subseteq \Upsilon$: $\Delta \subset \Delta'$ *and* $\Delta' \nvdash_{\mathbf{L}} A$.

Let $\gamma$ be a choice function, such that $\emptyset \neq \gamma(\Upsilon \curlywedge_{\mathbf{L}} A) \subseteq (\Upsilon \curlywedge_{\mathbf{L}} A)$ whenever $\Upsilon \curlywedge_{\mathbf{L}} A$ is non-empty, and $\gamma(\Upsilon \curlywedge_{\mathbf{L}} A) = \Upsilon$ otherwise.

**Definition F.7 (L-partial meet contraction)** $\Upsilon \ominus^{\gamma}_{\mathbf{L}} A = \bigcap \gamma(\Upsilon \curlywedge_{\mathbf{L}} A)$.

**Definition F.8 (L-partial meet revision)** $\Upsilon \oplus^{\gamma}_{\mathbf{L}} A = \Upsilon \ominus_{\mathbf{L}} {\sim}A \cup \{A\}$.

Note that when $A \dashv\vdash_{\mathbf{L}} A'$, it follows that $\Upsilon \curlywedge_{\mathbf{L}} A = \Upsilon \curlywedge_{\mathbf{L}} A'$, whence also $\Upsilon \ominus^{\gamma}_{\mathbf{L}} A = \Upsilon \ominus^{\gamma}_{\mathbf{L}} A'$ and $\Upsilon \oplus^{\gamma}_{\mathbf{L}} A = \Upsilon \oplus^{\gamma}_{\mathbf{L}} A'$. Below, I show that any operation of **L**-partial meet contraction and revision of an **L**-canonical form by a formula $A$ warrants that **L**-relevance is obeyed. I use $\ominus_{\mathbf{L}}$ and $\oplus_{\mathbf{L}}$ as metavariables for operations of **L**-partial meet contraction, resp. revision.

We first need to prove two lemmas:

**Lemma F.2** *Where* $\Upsilon$ *is not* **L***-trivial and* $\Upsilon' \in \mathbb{C}_{\mathbf{L}}(\Upsilon)$*: if* $B \in \Upsilon'$ *is not* **L***-relevant to the contraction of* $\Upsilon$ *by* $A$*, then* $B \in \Upsilon' \ominus_{\mathbf{L}} A$.

*Proof.* Suppose $B \in \Upsilon'$ but $B \notin \Upsilon' \ominus_{\mathbf{L}} A$, whereas $B$ is not **L**-relevant to the contraction of $\Upsilon$ by $A$ — I derive a contradiction. Let $\mathbb{E}_{\mathbf{L}}(\Upsilon) = \{E_i\}_{i \in I}$ be the finest **L**-splitting of $\Upsilon$, such that $\Upsilon' = \bigcup_{i \in I} \Upsilon_i$ and for each $i \in I$, $E(\Upsilon_i) \subseteq E_i$. Let $A^*$ be a least letter-set representation of $A$ in **L**. Let $\{E_j\}_{j \in J}$ be the subfamily of cells in $\mathbb{E}_{\mathbf{L}}(\Upsilon)$ that share some elementary letter with $E(A^*) = E^*_{\mathbf{L}}(A)$. By the irrelevance, $\bigcup_{j \in J}\{E_j\} \cap E(B) = \emptyset$.

Note that $\Upsilon' \ominus_{\mathbf{L}} A^* = \Upsilon' \ominus_{\mathbf{L}} A$ by the **L**-equivalence of $A$ and $A^*$. It follows that $A \notin \Upsilon' \ominus_{\mathbf{L}} A^*$. Hence by Definition F.7, there is a $\Delta \in \Upsilon' \curlywedge_{\mathbf{L}} A^*$ such

[8] Where $\Upsilon$ is a theory, we may obtain the contracted resp. revised theory by closing the result of the contraction (revision) operation defined here under **L**.

[9] According to Fuhrmann and Hansson, we can interpret the contraction of $\Upsilon$ by $\Psi$ (also called "multiple contraction") in several ways: either we want to obtain a $\Upsilon' \subseteq \Upsilon$ such that it does not entail *any* member of $\Psi$, or we want to obtain a $\Upsilon' \subseteq \Upsilon$ such that it does not entail *all* members of $\Psi$ ($\Upsilon' \nvdash_{\mathbf{L}} \Upsilon$). Which of the two readings is most suitable, depends on the goal of the contraction: if it is a preparatory step before a revision, we need the first interpretation; if the contraction is a goal in itself, then the second interpretation is more apt.

that $B \notin \Delta$. By Definition F.6, $\Delta \cup \{B\} \vdash_{\mathbf{L}} A^*$. Put $\Upsilon_a = \bigcup_{j\in J} \Upsilon_j$ and $\Upsilon_b = \bigcup_{i\in I-J} \Upsilon_i$. Then since $\Delta \subseteq \Upsilon' = \Upsilon_a \cup \Upsilon_b$, we have $(\Delta \cap \Upsilon_a) \cup (\Delta \cap \Upsilon_b) \cup \{B\} = \Delta \cap (\Upsilon_a \cup \Upsilon_b) \cup \{B\} = \Delta \cup \{B\} \vdash_{\mathbf{L}} A^*$. Hence by compactness, $\{C_1, \ldots, C_n\} \cup \{D_1, \ldots, D_m\} \cup \{B\} \vdash_{\mathbf{L}} A$, where $C_1, \ldots, C_n$ are elements of $\Upsilon_a$ and $D_1, \ldots, D_m$ are elements of $\Upsilon_b$.

By the Deduction Theorem: $\{D_1, \ldots, D_m\} \cup \{B\} \vdash_{\mathbf{L}} (C_1 \wedge \ldots \wedge C_n) \supset A^*$. In view of the construction, the formulas on the left side and those on the right side have no letters in common. But this means that either $\{D_1, \ldots, D_m\} \cup \{B\}$ is **L**-trivial, or $(C_1 \wedge \ldots \wedge C_n) \supset A^*$ is an **L**-theorem. In the former case, since $D_1, \ldots, D_m, B \in \Upsilon'$, it follows that $\Upsilon'$ is **L**-trivial whence also $\Upsilon$ is **L**-trivial — a contradiction. In the latter case, since $C_1, \ldots, C_n \in \Delta$, by the Deduction Theorem, $\Delta \vdash_{\mathbf{L}} A^*$, which contradicts the fact that $\Delta \in \Upsilon' \curlywedge_{\mathbf{L}} A^*$. ■

The following lemma is easily derivable from the Parallel Interpolation theorem. It provides the link between the weak relevance from the preceding lemma, and the strong relevance as defined in Section 9.2.

**Lemma F.3** *If $\Delta = \bigcup_{i\in I} \Delta_i \vdash_{\mathbf{L}} A$, $\Delta$ is not* **L**-*trivial, and the letter sets $E(\Delta_i)$ are pairwise disjoint, then $\bigcup_{i\in I}\{\Delta_i \mid E(\Delta_i) \cap E(A) \neq \emptyset\} \vdash_{\mathbf{L}} A$.*

*Proof.* Suppose the antecedent holds. By Parallel Interpolation, there are $B_i$ (with $i \in I$) such that (1) each $\Delta_i \vdash_{\mathbf{L}} B_i$, (2) each $E(B_i) \subset E(\Delta_i) \cap E(A)$ and (3) $\{B_i\}_{i\in I} \vdash_{\mathbf{L}} A$. Suppose that for an $i \in I$, $E(\Delta_i) \cap E(A) = \emptyset$. By (3), $E(B_i) = \emptyset$, whence (4) $B_i \dashv\vdash_{\mathbf{L}} \bot$ or (5) $B_i \dashv\vdash_{\mathbf{L}} \top$. In view of the supposition and (1), (4) is false. In view of (5), it follows that $\{B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n\} \vdash_{\mathbf{L}} A$.

Hence for every $i$ such that $E(\Delta_i) \cap E(A) = \emptyset$, we may remove the formula $B_i$ from the set $\{B_i\}_{i\in I}$, without losing $A$ as a **L**-consequence. Hence $\{B_i \mid E(B_i) \cap E(A) \neq \emptyset\} \vdash_{\mathbf{L}} A$. By (1), (3) and the transitivity of **L**, $\bigcup_{i\in I}\{\Delta_i \mid E(\Delta_i) \cap E(A) \neq \emptyset\} \vdash_{\mathbf{L}} A$. ■

**Theorem F.9** *Where $\Upsilon$ is not* **L**-*trivial and $\Upsilon' \in \mathbb{C}_{\mathbf{L}}(\Upsilon)$: if $B \in Cn_{\mathbf{L}}(\Upsilon)$ and $B$ is not* **L**-*relevant to the contraction of $\Upsilon$ by $A$, then $B \in Cn_{\mathbf{L}}(\Upsilon' \ominus_{\mathbf{L}} A)$.*

*Proof.* Suppose the antecedent holds. Note that since $\Upsilon' \dashv\vdash_{\mathbf{L}} \Upsilon$, $\Upsilon' \vdash_{\mathbf{L}} B$. Let $\mathbb{E}_{\mathbf{L}}(\Upsilon) = \{E_i\}_{i\in I}$ be the finest **L**-splitting of $\Upsilon$, such that $\Upsilon' = \bigcup_{i\in I} \Upsilon_i$ and for each $i \in I$, $E(\Upsilon_i) \subseteq E_i$. By Lemma F.3, it follows that (†) $\bigcup_{i\in I}\{\Upsilon_i \mid E(\Upsilon_i) \cap E(B) \neq \emptyset\} \vdash_{\mathbf{L}} B$.

Let $A^*$ be a least letter-set representation of $A$ in **L**. Note that for every $i \in I$ such that $E_i \cap E(B) \neq \emptyset$, $E_i \cap E(A^*) = \emptyset$, in view of the supposition and Definition 9.3. Hence for every $C \in \Upsilon_i$ such that $E(\Upsilon_i) \cap E(B) = \emptyset$, $C$ is not relevant to the contraction of $\Upsilon$ by $A^*$. By Lemma F.2, $C \in \Upsilon' \ominus_{\mathbf{L}} A^*$. Hence, $\{\Upsilon_i \mid E(\Upsilon_i) \cap E(B) \neq \emptyset\} \subseteq \Upsilon' \ominus_{\mathbf{L}} A^*$. By (†) and the monotonicity of **L**, $B \in Cn_{\mathbf{L}}(\Upsilon' \ominus_{\mathbf{L}} A^*)$. Since $A$ and $A^*$ are **L**-equivalent, $B \in Cn_{\mathbf{L}}(\Upsilon' \ominus_{\mathbf{L}} A)$. ■

**Theorem F.10** *Where $\Upsilon$ is not* **L**-*trivial and $\Upsilon' \in \mathbb{C}_{\mathbf{L}}(\Upsilon)$: if $B \in Cn_{\mathbf{L}}(\Upsilon)$ and $B$ is not* **L**-*relevant to the revision of $\Upsilon$ by $A$, then $B \in Cn_{\mathbf{L}}(\Upsilon' \oplus_{\mathbf{L}} A^*)$.*

*Proof.* Immediate in view of (i) the fact that relevance to a revision by $A$ is equivalent to relevance to a contraction by $\sim A$, (ii) Definition F.8, (iii) Theorem F.9, and (iv) the monotonicity of **L**. ■

### F.5.5 $Min_{\mathbf{L}}(\Gamma)$ is an L-canonical form of $\Gamma$

In this section, I will prove the following:

**Theorem F.11** *$Min_{\mathbf{L}}(\Gamma)$ is an* **L***-canonical form of* $\Gamma$

As the reader will note, the proof of Theorem F.11 is obtained by a rather straightforward generalization of the proof for Theorem 9.3 from Chapter 9. In view of Theorem F.9 and Theorem F.10, we may use $Min_{\mathbf{L}}(\Gamma)$ in order to obtain a relevant belief contraction or revision. Moreover, the result below forms the basis of a proof for Theorem F.6: it is shown how we may obtain the finest splitting of $\Upsilon$ from the set $Min_{\mathbf{L}}(\Gamma)$.

In view of Lemma F.1, it suffices to prove that there is a splitting $\mathbb{E} = \{E_i\}_{i\in I}$ of $\mathcal{E}$, and a partition $\{\Delta_i\}_{i\in I}$ of $Min_{\mathbf{L}}(\Gamma)$ such that (1) each $E(\Delta_i) \subseteq E_i$ and (2) $\mathbb{E}$ is the finest splitting of $\Gamma$. In the remainder of this section, I will define an $\mathbb{E}$ for which it is quite easy to show that it fulfills requirement (1); by a slightly longer proof, I will arrive at (2), as stated in Theorem F.12.

As in Chapter 9, we may define a relation $\sim_\Delta$ over the members of $\Delta$, for every $A, B \in \mathcal{W}_\sim$ and $\Delta \subseteq \mathcal{W}_\sim$:

**Definition F.9** *Let $A$ is path-relevant to $B$ modulo $\Delta$ ($A \sim_\Delta B$) iff there are $C_1, \ldots, C_n \in \Delta$ such that $E(A) \cap E(C_1) \neq \emptyset$, $E(C_1) \cap E(C_2) \neq \emptyset$, …, and $E(C_n) \cap E(B) \neq \emptyset$.*

It will be convenient to rely on the following property specific to $\sim_\Delta$ defined only over the members of $\Delta$:

**Fact F.1** *$\sim_\Delta$ is transitive, reflexive and symmetric with respect to all $A, B, C \in \Delta$, whence $\sim_\Delta$ is an equivalence relation on the members of $\Delta$.*

**Definition F.10** *$\mathbb{M}_{\mathbf{L}}(\Gamma)$ is the quotient set of $Min_{\mathbf{L}}(\Gamma)$ by $\sim_{Min_{\mathbf{L}}(\Gamma)}$.*[10] *Where $\mathbb{M}_{\mathbf{L}}(\Gamma) = \{\Delta_i\}_{i\in I}$, $\mathbb{E}_{\mathbf{L}}(\Gamma) = \{E(\Delta_i)\}_{i\in I} \cup \{\{A\} \mid A \in \mathcal{E} - E(Min_{\mathbf{L}}(\Gamma))\}$.*

Since $\sim_{Min_{\mathbf{L}}(\Gamma)}$ is an equivalence relation on $Min_{\mathbf{L}}(\Gamma)$, $\mathbb{M}_{\mathbf{L}}(\Gamma)$ is a partition of $Min_{\mathbf{L}}(\Gamma)$. Also, note that for no $\Delta_i \in \mathbb{M}_{\mathbf{L}}(\Gamma) : \Delta_i = \emptyset$, whence also for no $E_i \in \mathbb{E}_{\mathbf{L}}(\Gamma)$: $E_i = \emptyset$. It remains prove that $\mathbb{E}_{\mathbf{L}}(\Gamma)$ is the finest **L**-splitting of $\Gamma$.

I first prove that $\mathbb{E}_{\mathbf{L}}(\Gamma)$ is a partition of $\mathcal{E}$. This follows immediately from (1) the fact that every $E_i$ is non-empty, (2) the fact that $\bigcup \mathbb{E}_{\mathbf{L}}(\Gamma) = \mathcal{E}$, and the following lemma:

**Lemma F.4** *For every $E_i, E_j \in \mathbb{E}_{\mathbf{L}}(\Gamma)$: $E_i \neq E_j$ iff $E_i \cap E_j = \emptyset$.*

*Proof.* Let $E_i, E_j \in \mathbb{E}_{\mathbf{L}}(\Gamma)$. The right-left direction is obvious since no $E_i \in \mathbb{E}_{\mathbf{L}}(\Gamma)$ is empty. For the left-right direction, suppose that for $E_i, E_j \in \mathbb{E}_{\mathbf{L}}(\Gamma)$, $E_i \cap E_j \neq \emptyset$. I only consider the case where $E_i = E(\Delta_i)$ and $E_j = E(\Delta_j)$ for $\Delta_i, \Delta_j \in \mathbb{M}_{\mathbf{L}}(\Gamma)$ – in the other case, it follows immediately that $E_i \cap E_j = \emptyset$. Suppose that $E(\Delta_i) \cap E(\Delta_j) \neq \emptyset$. This implies that there are $A \in \Delta_i, B \in \Delta_j$: $E(A) \cap E(B) \neq \emptyset$, whence $A \sim_{Min_{\mathbf{L}}(\Gamma)} B$, hence $A$ and $B$ are in the same equivalence class. As a result, $\Delta_i = \Delta_j$, whence $E_i = E_j$. ■

[10] This is the set of all equivalence sets of $Min_{\mathbf{L}}(\Gamma)$, given the equivalence relation $\sim_{Min_{\mathbf{L}}(\Gamma)}$ on $Min_{\mathbf{L}}(\Gamma)$.

**Theorem F.12** $\mathbb{E}_{\mathbf{L}}(\Gamma)$ *is the finest* **L**-*splitting of* $\Gamma$.

*Proof.* Suppose there is a splitting $\mathbb{E} = \{E_j\}_{j \in J}$ of $\Gamma$, such that $\mathbb{E}$ is finer than $\mathbb{E}_{\mathbf{L}}(\Gamma)$. Hence for some $E_k \in \mathbb{E}_{\mathbf{L}}(\Gamma)$, there is a $j \in J$: $\emptyset \subset E_j \subset E_k$. This means that $E_k$ cannot be a singleton, whence $E_k = E(\Delta_k)$ for some $\Delta_k \in \mathbb{M}_{\mathbf{L}}(\Gamma)$. So we have:

(†) For a $\Delta_k \in \mathbb{M}_{\mathbf{L}}(\Gamma)$, there is a $j \in J$: $\emptyset \subset E_j \subset E(\Delta_k)$

I will first prove that (‡) there is a $D \in \Delta_k$, for which $E(D) \cap E_j \neq \emptyset$, $E(D) \not\subseteq E_j$.

Suppose that for every $A \in \Delta_k, E(A) \subseteq E_j$. In that case, $E(\Delta_k) \subseteq E_j$, which contradicts (†). Hence there is a $A \in \Delta_k : E(A) \not\subseteq E_j$. Suppose that for every $B \in \Delta_k, E(B) \cap E_j = \emptyset$. In that case, $E(\Delta_k) \cap E_j = \emptyset$, which also contradicts (†). Hence there is a $B \in \Delta_k : E(B) \cap E_j \neq \emptyset$.

Since $A, B \in \Delta_k$, $A \sim_{Min_{\mathbf{L}}(\Gamma)} B$. Hence there are $C_1, \ldots, C_n \in \Delta_k$ such that $E(A) \cap E(C_1) \neq \emptyset, E(C_1) \cap E(C_2) \neq \emptyset, \ldots, E(C_n) \cap E(B) \neq \emptyset$. If $E(A) \cap E_j \neq \emptyset$, take $D = A$. If $E(A) \cap E_j = \emptyset$, we can infer that $E(C_1) \not\subseteq E_j$ from the fact that $E(A) \cap E(C_1) \neq \emptyset$ and that $E(C_1)$ is non-empty. We now start up recursive procedure, relying on the same reasoning:

If $E(C_l) \cap E_j \neq \emptyset$, then let $D = C_l$
If $E(C_l) \cap E_j = \emptyset$, then $E(C_{l+1}) \not\subseteq E_j$

This means that sooner or later, and to the latest at $B$, we arrive at a $D \in \Delta_k$, for which it holds that $E(D) \cap E_j \neq \emptyset$, $E(D) \not\subseteq E_j$.

I will now derive a contradiction from (‡). Note that since $\mathbb{E}$ is a splitting of $\Gamma$, $\mathbb{E}' = \{E_j, \bigcup \mathbb{E} - E_j\}$ is also a splitting of $\Gamma$. Hence there are $\Theta_j, \Theta$ such that $\Theta_j \cup \Theta \dashv\vdash_{\mathbf{L}} \Gamma$, $E(\Theta_j) \subseteq E_j$ and $E(\Theta) \subseteq \bigcup \mathbb{E} - E_j$. It follows that (‡) $E(\Theta_j) \cap E(\Theta) = \emptyset$. Moreover, since $\Gamma \vdash_{\mathbf{L}} D$, also $\Theta_j \cup \Theta \vdash_{\mathbf{L}} D$, whence by Parallel Interpolation, there are two formulae $F_j$ and $F$ such that (1) $E(F_j) \subseteq E(\Theta_j) \cap E(D)$, (2) $E(F) \subseteq (\Theta) \cap E(B)$ and (3) $\{F_j, F\} \vdash_{\mathbf{L}} D$.

Note that since $\Theta \cup \Theta_j \dashv\vdash_{\mathbf{L}} \Gamma$, also $\Gamma \vdash_{\mathbf{L}} F_j$ and $\Gamma \vdash_{\mathbf{L}} F$. By (‡), (1) and (2), $E(F_i) \subset E(D)$ and $E(F) \subset E(D)$. Hence by (3), $D \notin Min_{\mathbf{L}}(\Gamma)$ — a contradiction. ■

## F.6 Further Research

Note that we can easily obtain prioritized axioms of **L**-relevance, along the lines spelled out in Section 10.4 of Chapter 10. It suffices to relativize the notions of finest splitting, relevance and least letter-set representation to the subclassical logic under consideration, in exactly the same way as it happened for "flat" relevance in the current chapter. This way, we can obtain sensible relevance axioms for prioritized belief bases $\mathbf{\Upsilon} = \langle \Upsilon_1, \Upsilon_2, \ldots \rangle$, where the sets $\Upsilon_i$ can be inconsistent in themselves.

In view of the general set-up of the current chapter and Chapters 9-10, the following topics for further research require little explanation:

- Is it possible to further generalize the results from this chapter to other subclassical logics such as e.g. Priest's paraconsistent logic **LP**, Brazilian anti-intuitionistic logic or the very powerful system $\mathbf{CL}^-$ from [159]?[11]

- Is it possible to develop adaptive logics for **L**-relevant belief revision or contraction, in a similar vein as the adaptive logics for **CL**-relevant belief revision from Chapter 9, but where **L** is a paraconsistent or paracomplete logic?

Another interesting topic would be the possibility of a non-monotonic (corrective) finest splitting. For instance, quite a few adaptive logics have been developed that allow one to interpret a set of beliefs "as consistently as possible", without trivializing inconsistent beliefs. The logic $\mathbf{CLuN^m}$, introduced in Chapter 2 is only one of them.[12] Some of these systems are equivalent to **CL** whenever the belief base is consistent, and most of them are usually much stronger than the monotonic systems from the current chapter. It would hence be worthwhile to see whether such non-monotonic logics also yield a finest splitting for every belief base – in that case, the associated relevance axiom would be very strong, but it would still not trivialize inconsistent belief bases.

[11] To the best of my knowledge, no Interpolation results are available for these systems, which makes them tougher candidates to prove the finest splitting theorem for.

[12] I refer to [25, Chapters 7] for an introduction to and overview of other inconsistency-adaptive logics.